1. Given: $\frac{dy}{dx} = xy^2$ and $x = 1$ when $y = 1$. Which of the following is the solution?

a. $y = x^2$

b. $y = \frac{-2}{x^2 - 3}$

c. $y = x^2 - 3$

d. $y = \frac{2}{x^2 + 1}$

e. $y = \frac{x^2 - 3}{2}$

$\frac{dy}{y^2} = x\,dx$

$\frac{-1}{y} = \frac{x^2}{2} + C$

$(1,1)$

$-1 = \frac{1}{2} + c$

$-\frac{3}{2} = c$

So $-\frac{1}{y} = \frac{x^2}{2} - \frac{3}{2} \rightarrow \frac{-1}{y} = \frac{x^2-3}{2} \rightarrow y = \frac{-2}{x^2-3}$

2. At each point (x, y) on a certain curve, the slope of the curve is given by $4xy$. The curve contains the point (0, 4). The equation of the curve is:

a. $y = e^{2x^2} + 4$

b. $y = e^{2x^2} + 3$

c. $y = 4e^{2x^2}$

d. $y^2 = 2x^2 + 4$

e. $y = 2x^2 + 4$

$\frac{dy}{dx} = 4xy$

$\frac{dy}{y} = 4x\,dx$

$\ln y = 2x^2 + c$

$y = Ce^{2x^2}$ $(0,4)$ $4 = Ce^0$ So

$4 = C$ $y = 4e^{2x^2}$

3. Given: $\frac{dy}{dx} = 2xy$ and $y = 2$ when $x = 0$. The value of x when $y = e$ is:

a. 0.307
b. 0.554
c. 0.693
d. 1.000
e. 2.718

$\frac{dy}{y} = 2x\,dx$

$\ln y = x^2 + c$

$y = Ce^{x^2}$ $(0,2) \rightarrow y = 2e^{x^2}$ when $y = e$

$e = 2e^{x^2}$

$\ln\left(\frac{e}{2}\right) = x^2$

$\sqrt{\ln\left(\frac{e}{2}\right)} = x$

$.5539.. \approx x$

4. If $\frac{dy}{dx} = \frac{x}{y}$ and $y(3) = 4$ then

a. $x^2 - y^2 = -7$

b. $x^2 + y^2 = 7^2$

c. $x^2 - y^2 = 7$

d. $x^2 - y^2 = 5$

e. $x^2 - y^2 = 7^2$

$y\,dy = x\,dx$

$\frac{1}{2}y^2 = \frac{1}{2}x^2 + c$ $\leftarrow$ $\frac{1}{2}y^2 = \frac{1}{2}x^2 + 3.5$

$\frac{1}{2}(16) = \frac{1}{2}(9) + c$ $y^2 = x^2 + 7$

$-7 = x^2 - y^2$

$c = 3.5$

5. If $\frac{dy}{dx} = y\cos x$ and $y = 3$ when $x = 0$, then

a. $y = e^{\sin x} + 2$

b. $y = e^{\sin x} + 3$

c. $y = \sin x + 3$

d. $y = \sin x + 3e^x$

e. $y = 3e^{\sin x}$

$$\frac{dy}{y} = \cos x \, dx$$
$$\ln y = \sin x + C$$
$$y = Ce^{\sin x} \qquad 3 = Ce^0 \qquad 3 = C$$

6. A function whose derivative is a constant multiple of itself must be

a. periodic

b. linear

c. exponential

d. quadratic

e. logarithmic

$$\frac{dy}{dx} = k \cdot y$$
$$\frac{dy}{y} = k\,dx$$
$$\ln y = kx + C \qquad y = Ce^{kx}$$

7. At any time $t > 0$, the rate of growth of a bacteria population is given by $y' = ky$, where y is the number of bacteria present and k is a constant. The initial population is 1,500 and the population quadrupled during the first 2 days. By what factor will the population have increased during the first three days?

a. 4

b. 5

c. 6

d. 8

e. 10

t	P
0	1500
2	6000
3	?

$$y = Ce^{kt}$$
$$(0, 1500) \quad 1500 = Ce^0 \quad \text{so } C = 1500$$
$$y = 1500e^{kt}$$
$$(2, 6000) \quad 6000 = 1500e^{2k}$$
$$\frac{\ln(4)}{2} = k$$
$$\text{at } t = 3 \quad y = 1500\left[e^{\frac{\ln 4}{2}(3)}\right]$$
$$y = 1500(8)$$
8 times.

8. Solve the differential equation: $\frac{dy}{dx} = 2x\cos^2 y$; $y(3) = 0$

$$\frac{dy}{\cos^2 y} = 2x\,dx \rightarrow \int \sec^2 y\,dy = \int 2x\,dx$$
$$\tan y = x^2 + C$$
$$y = \tan^{-1}(x^2 + C)$$
$$(3, 0)$$
$$0 = \tan^{-1}(9 + C)$$
$$\tan 0 = 9 + C$$
$$-9 = C$$
Answer: $y = \tan^{-1}(x^2 - 9)$

9. A population is growing at a rate that is directly proportional to the size of the population. If the population is 2000 initially, and has grown to 5000 by the end of the second day, how long will it take to grow to 10,000?

$$\frac{dP}{dt} = kP \rightarrow P = Ce^{kt}$$
$$C = 2000$$
$$5000 = 2000e^{k \cdot 2} \quad \left(\tfrac{1}{2}\ln\tfrac{5}{2} = k\right)$$
$$10000 = 2000e^{\left(\frac{1}{2}\ln\frac{5}{2}\right)t}$$
$$\ln 5 = \tfrac{1}{2}\ln\left(\tfrac{5}{2}\right)t$$
$$t = 3.512 \text{ days}$$

10. A population is growing at a rate that is three times the size of the population. If the population is 12 after three hours, what was the size of the population initially?

t	Population
0	P_0
3	12

$$\frac{dP}{dt} = 3P$$
$$P = P_0 e^{3t}$$
$$12 = P_0 e^9$$
$$P_0 = \frac{12}{e^9}$$

AP Calculus - BC
Exponential Growth/the Logistic Growth Model

1. A function whose derivative is a constant multiple of itself must be

   a. periodic
   b. linear
   c. exponential
   d. quadratic
   e. logarithmic

2. At any time $t > 0$, the rate of growth of a bacteria population is given by $y' = ky$, where y is the number of bacteria present and k is a constant. The initial population is 1,500 and the population quadrupled during the first 2 days. By what factor will the population have increased during the first three days?

   eight

3. A population of rabbits is given by the formula below where t is the number of months after a few rabbits are released:

$$P(t) = \frac{1000}{1 + e^{4.8-0.7t}}$$

   a. Find $P(0)$ and explain its meaning in the problem.

   $P(0) = \frac{1000}{1+e^{4.8}} \doteq 8.163$ is the population size at $t = 0$

   b. Find $\lim_{t\to\infty} P(t)$ and explain its meaning in the problem.

   $\lim_{t\to\infty} \frac{1000}{1+e^{4.8-.7t}}$ 1000 is the carrying capacity

   as $t \to \infty$, exponent gets very neg. $e^{\text{very neg}}$ is very small so denom $\to 1$. Limit is 1000.

4. The number of students infected by measles in a certain school is given by the formula below where t is the number of days after students are first exposed to an infected student:

$$P(t) = \frac{200}{1 + e^{5.3-t}}$$

   a. Find $P(0)$ and explain its meaning in the problem.

   $P(0) = \frac{200}{1+e^{5.3}}$ initial population of infected students $\approx$ 1 student

   b. Find $\lim_{t\to\infty} P(t)$ and explain its meaning in the problem.

   limit is 200 infected students

5. A certain wild animal preserve can support no more than 250 lowland gorillas. Twenty-eight gorillas were known to be in the preserve in 1970. Assume that the rate of growth of the population is given the differential equation below where t is measured in years:

$.0004 = \frac{k}{250}$

$$\frac{dP}{dt} = 0.0004P(250-P) = \frac{.1}{250}P(250-P)$$

(Logistic Growth)

$M = 250, \quad k = .1$

Find a formula for the gorilla population in terms of t and use it to determine how long it will take for the population to reach the carrying capacity of the preserve:

$$P = \frac{250}{1+Ce^{-.1t}}$$

	t	P
(1970)	0	28
	?	250

$$28 = \frac{250}{1+Ce^{0}}$$

$$28 + C = 250$$

$$C = \frac{222}{28}$$

$$P = \frac{250}{1+\frac{222}{28}e^{-.1t}}$$

$$249.5 = \frac{250}{1+\frac{222}{28}e^{-.1t}}$$

Theoretically, P never reaches 250, but since gorillas come in whole #s, we will consider P = 249.5 to round up to 250.

$$t = 82.831 \text{ yrs.}$$

AP Calculus BC *Name*________________________________
Mixed Chapter Five Review

No calculator.

1. Given $m(6)=6,\ n(6)=\pi,\ m'(6)=2,\ n'(6)=7$, evaluate:

a. $\left(\dfrac{m}{n}\right)'\Big|_{x=6} = \dfrac{n(m') - m(n')}{n^2} = \dfrac{\pi(2) - 6(7)}{\pi^2} = \dfrac{2\pi - 42}{\pi^2}$

b. $\left(n(m(x))\right)'\Big|_{x=6} = n'(m(6)) \cdot m'(6) = n'(6) \cdot 2 = 7 \cdot 2 = 14$

c. $\left(\tan(n(x))\right)'\Big|_{x=6} = \sec^2(n) \cdot n' = \sec^2(\pi) \cdot 7 = 7$

2. Find the equation of all tangents to the graph of $y=\sin(2x)$ for $x\in[0,\ \pi]$ where the slope of the graph is -2.

$\dfrac{dy}{dx} = 2\cos(2x) = -2$

$\cos(2x) = -1$

$2x = \pi, 3\pi, 5\pi, \ldots$

$x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \ldots$

pt. $\left(\frac{\pi}{2}, \sin\pi\right) = \left(\frac{\pi}{2}, 0\right)$

slope $= -2$

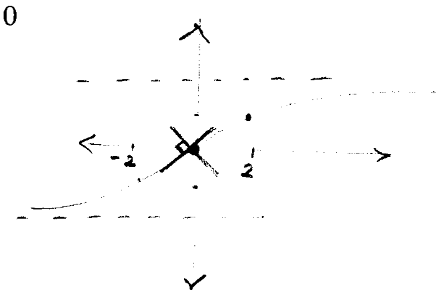
$\boxed{y-0 = -2\left(x - \frac{\pi}{2}\right)}$

3. Given $\dfrac{dx}{dt}=3$ and $y=x^2$, find the value of $\dfrac{dy}{dt}$ when $x=5$

$\dfrac{dy}{dx} = 2x$

$\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt} = 2x \cdot 3 \Big|_{x=5} = 30$

4. Sketch a graph of $g(x)=\tan^{-1}\left(\dfrac{x}{2}\right)$. Graph and find the equation of the normal to the graph of $g(x)$ where $x=0$



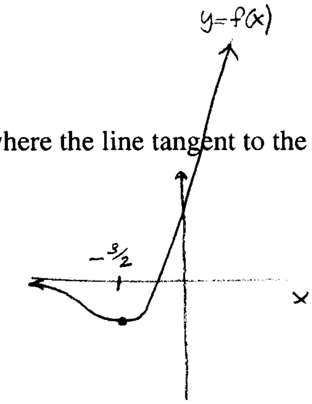
x	$g(x)$
0	0
2	$\frac{\pi}{4}$
-2	$-\frac{\pi}{4}$

slope: $g'(x) = \dfrac{1}{1+\left(\frac{x}{2}\right)^2} \cdot \dfrac{1}{2}$

$g'(0) = \frac{1}{2}$

Normal: $y - 0 = -2(x-0)$

5. Find the coordinates of each point on the graph of $f(x) = e^{2x}(x+1)$ where the line tangent to the graph is horizontal.



$$f' = e^{2x}(1) + (x+1) \cdot 2e^{2x} = 0$$

$$e^{2x}\left[1 + 2(x+1)\right] = 0$$

$e^{2x} = 0$ never or $1 + 2x + 2 = 0$, $x = \frac{-3}{2}$

6. Find the x and y-intercepts of the line tangent to the graph of $f(x) = x^2 \ln x$ at the point where $x = 1$

$f(1) = 0$ (pt)

Slope: $f' = x^2 \cdot \frac{1}{x} + \ln x \cdot 2x \Big|_{x=1}$

$1 + 0 = 1$

tangent line: $y - 0 = 1(x-1)$

$y = x - 1$

x-int $= 1$

y-int $= -1$

7. Find the x-coordinate of each point on the graph of $y = \log_3(2x)$ where the normal line is parallel to $2x + 6y + k = 0$. (k is a constant)

$\frac{dy}{dx} = \frac{1}{\cancel{2}x \ln 3} \cdot \cancel{2} = \frac{1}{x \ln 3}$

$6y = -2x - k$

$y = -\frac{2}{6}x - \frac{k}{6}$

$m = -\frac{1}{3}$

normal: $3 = m$

$\frac{1}{x \ln 3} = 3$

$3x \ln 3 = 1$

$x = \frac{1}{3 \ln 3}$

8. Given two points moving along a line such that their positions on the line are given by $s_1(t) = 9^{0.5t}$ and $s_2(t) = \ln t$. Determine what time(s), if any, that the particles are moving along the line with the same velocity. You may use a calculator.

$$\left.\begin{array}{l} V_1 = 9^{.5t} \cdot \frac{1}{2} \cdot \ln 9 \\ V_2 = \frac{1}{t} \end{array}\right\}$$

$9^{.5t} \cdot \frac{\ln 9}{2} = \frac{1}{t}$

by calculator:

$t = .516$