Related Rate Word Problems

1. Draw a diagram and label the quantities that don't change with their respective values and quantities that do change with respect to time as variables.
2. Mathematically specify the rate of change that you are looking for and record all other given information.
3. Find an equation involving the variables whose rates of change you are looking for and that you have been given.
4. Implicitly differentiate the equation found in Step 3 with respect to time.
5. State the final answer in a coherent form being sure to specify the units of the answer and being sure the original question is answered not just something simply related to the answer.

Hints:

- *Always use units to double check your algebra.*
- *DO NOT plug in any constants for changing quantities until after you have differentiated.*
- *Know what the units are when you differentiate a quantity.*

1. A conical paper cup 3 inches across the top and 4 inches deep is full of water. The cup springs a leak at the bottom and loses water at the rate of 2 cubic inches per minute. How fast is the water level dropping at the instant when the water is exactly 3 inches deep? Express the answer in inches per minute.
[Note: The formula for the volume of a cone of base radius r and height h is $\frac{\pi}{3}r^2h$.]

2. Air is being pumped into a spherical balloon at the rate of 7 cubic centimeters per second. What is the rate of change of the radius at the instant the volume equals 36π? The volume of a sphere of radius r is $\frac{4\pi}{3}r^3$.

3. A kite 100 feet above the ground is being blown away from the person holding its string in a direction parallel to the ground at the rate of 10 feet per second. At what rate must the string be let out when the length of the string already let out is 200 feet?

4. A kite is flying at an angle of elevation of $\frac{\pi}{3}$. The kite string is being taken in at the rate of 1 foot per second. If the angle of elevation does not change, how fast is the kite losing altitude?

5. Sand is dumped off a conveyor belt into a pile at the rate of 2 cubic feet per minute. The sand pile is shaped like a cone whose height and base diameter are always equal. At what rate is the height of the pile growing when the pile is 5 feet high? (The volume of a cone is $\frac{\pi}{3}r^2h$ where r is the radius of the base and h is the height.)

6. A camera is located 50 feet from a straight road along which a car is traveling at 100 feet per second. The camera turns so that it is pointed at the car at all times. In radians per second, how fast is the camera turning as the car passes closest to the camera?

7. A balloon leaves the ground 500 feet away from an observer and rises vertically at the rate of 140 feet per minute. At what rate is the angle of inclination of the observer's line of sight increasing at the instant when the balloon is exactly 500 feet above the ground?

8. A ladder 13 feet long is leaning against the side of a building. If the foot of the ladder is pulled away from the building at a constant rate of 2 inches per second, how fast is the angle formed by the ladder and the ground changing (in radians per second) at the instant when the top of the ladder is 12 feet above the ground?

9. A ladder 13 feet long is leaning against the side of a building. If the foot of the ladder is pulled away from the building at a constant rate of 8 inches per second, how fast is the area of the triangle formed by the ladder, the building and the ground changing (in feet squared per second) at the instant when the top of the ladder is 12 feet above the ground?

10. A particle is moving around the ellipse $4x^2 + 16y^2 = 64$. At any time t, its $x-$ and $y-$coordinates are given by $x(t) = 4\cos t$ and $y(t) = 2\sin t$. At what rate is the particle's distance to the point (2,0) changing at any time t? At what rate is the distance changing when $t = \frac{\pi}{4}$?

1

$\boxed{1}$

3 in (top width)
4 in (height)
r, h labeled

①
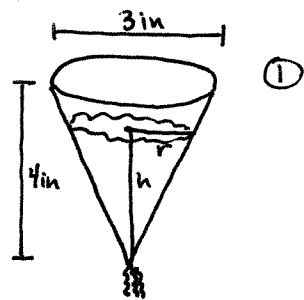
② Want $\dfrac{dh}{dt}\Big|_{h=3in}$
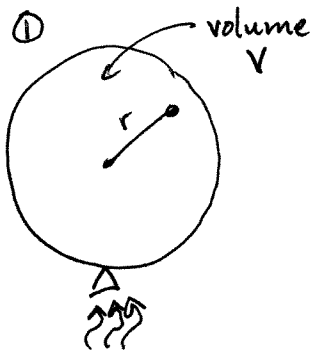
Given $\dfrac{dV}{dt} = -2\,\dfrac{in^3}{min}$

$\dfrac{r}{\frac{3}{2}in} = \dfrac{h}{4in} \Longleftrightarrow r = \dfrac{3}{8}h$

③ $V = \dfrac{\pi}{3}r^2 h$

$= \dfrac{\pi}{3}\left(\dfrac{3}{8}h\right)^2 h$

$= \dfrac{3\pi}{64}h^3$

④ $\dfrac{d}{dt}(V) = \dfrac{d}{dt}\left(\dfrac{3\pi}{64}h^3\right) \Longleftrightarrow \dfrac{dV}{dt} = \dfrac{9\pi}{64}h^2 \cdot \dfrac{dh}{dt}$

⑤ $\dfrac{dh}{dt}\Big|_{h=3in} = \dfrac{\frac{dV}{dt}}{\frac{9\pi}{64}h^2}\Bigg|_{h=3in} = \dfrac{-2\,\frac{in^3}{min}}{\frac{9\pi}{64}\cdot 9\,in^2} = \boxed{\dfrac{-128}{81\pi}\,\dfrac{in}{min}}$

2

$\boxed{2}$

①

volume V
r

② Want $\dfrac{dr}{dt}\Big|_{V=36\pi\,cm^3}$

Given $\dfrac{dV}{dt} = 7\,\dfrac{cm^3}{sec}$

If $V = 36\pi\,cm^3$

then $36\pi\,cm^3 = \dfrac{4}{3}\pi r^3$

$\Longleftrightarrow 27\,cm^3 = r^3$

$\Longleftrightarrow r = 27\,cm$

③ $V = \dfrac{4}{3}\pi r^3$

④ $\dfrac{d}{dt}(V) = \dfrac{d}{dt}\left(\dfrac{4}{3}\pi r^3\right) \Longleftrightarrow \dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$

⑤ $\dfrac{dr}{dt}\Big|_{V=36\pi\,cm^3} = \dfrac{dr}{dt}\Big|_{r=3cm} = \dfrac{\frac{dV}{dt}}{4\pi r^2}\Bigg|_{r=3cm} = \dfrac{7\,\frac{cm^3}{sec}}{4\pi(9)\,cm^2} = \boxed{\dfrac{7}{36\pi}\,\dfrac{cm}{sec}}$

3

①

100 ft

x
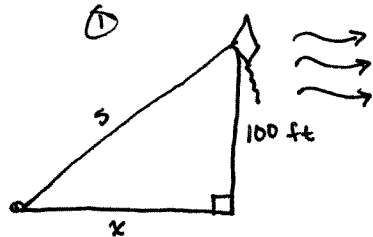
② Want $\left.\dfrac{ds}{dt}\right|_{s=200\,ft}$

Given $\dfrac{dx}{dt} = 10\,\frac{ft}{sec}$

~~(illegible scribble)~~

③ $x^2 + 100^2\,ft^2 = s^2$

④ $\dfrac{d}{dt}(x^2 + 100^2\,ft^2) = \dfrac{d}{dt}(s^2) \iff 2x\,\dfrac{dx}{dt} = 2s\,\dfrac{ds}{dt}$

⑤ $\left.\dfrac{ds}{dt}\right|_{s=200\,ft} = \left.\dfrac{x}{s}\dfrac{dx}{dt}\right|_{s=200\,ft}$ If $s = 200\,ft$ then $x^2 = -100^2\,ft^2 + s^2$

$\iff x^2 = -100^2\,ft^2 + 200^2\,ft^2 = 30000\,ft^2$

$= (100\sqrt{3}\,ft)^2$

So $\left.\dfrac{ds}{dt}\right|_{s=200\,ft} = \left.\dfrac{x}{s}\dfrac{dx}{dt}\right|_{\substack{s=200ft\\x=100\sqrt{3}ft}} = \dfrac{100\sqrt{3}\,ft}{200\,ft}\cdot \overset{5}{\cancel{10}}\,\frac{ft}{sec} = \boxed{5\sqrt{3}\,\frac{ft}{sec}}$

4

①

s

h

$\frac{\pi}{3}$

x
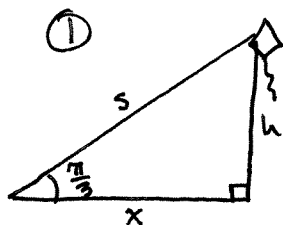
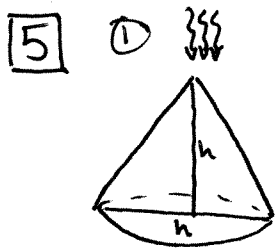② Want $\left.\dfrac{dh}{dt}\right|_{any\,h}$

Given $\dfrac{ds}{dt} = -1\,\frac{ft}{sec}$

③ $\sin\left(\frac{\pi}{3}\right) = \dfrac{h}{s}$

$\iff s\cdot\sin\left(\frac{\pi}{3}\right) = h$

$\iff \dfrac{\sqrt{3}}{2}\,s = h$

④ $\dfrac{d}{dt}\left(\dfrac{\sqrt{3}}{2}s\right) = \dfrac{d}{dt}(h) \iff \dfrac{\sqrt{3}}{2}\dfrac{ds}{dt} = \dfrac{dh}{dt}$

⑤ $\dfrac{dh}{dt} = \dfrac{\sqrt{3}}{2}\cdot\left(-1\,\frac{ft}{sec}\right) = \boxed{-\dfrac{\sqrt{3}}{2}\,\frac{ft}{sec}}$

$\boxed{5}$ ①

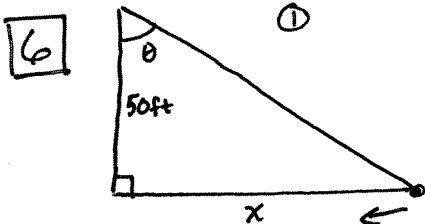② Want $\left.\dfrac{dh}{dt}\right|_{h=5ft}$

Given $\dfrac{dV}{dt} = 2\ \dfrac{ft^3}{min}$

③ $V = \dfrac{\pi}{3}r^2 h$

$= \dfrac{\pi}{3}\left(\dfrac{1}{2}h\right)^2 h$

$= \dfrac{\pi}{12}h^3$

④ $\dfrac{d}{dt}(V) = \dfrac{d}{dt}\left(\dfrac{\pi}{12}h^3\right) \iff \dfrac{dV}{dt} = \dfrac{\pi}{4}h^2 \cdot \dfrac{dh}{dt}$

⑤ $\left.\dfrac{dh}{dt}\right|_{h=5ft} = \left.\dfrac{\frac{dV}{dt}}{\frac{\pi}{4}h^2}\right|_{h=5ft} = \dfrac{2\ \frac{ft^3}{min}}{\frac{\pi}{4}\cdot 25\ ft^2} = \boxed{\dfrac{8}{25\pi}\ \dfrac{ft}{min}}$

$\boxed{6}$

①

② Want $\left.\dfrac{d\theta}{dt}\right|_{\theta=0}$

Given $\dfrac{dx}{dt} = -100\ \dfrac{ft}{sec}$
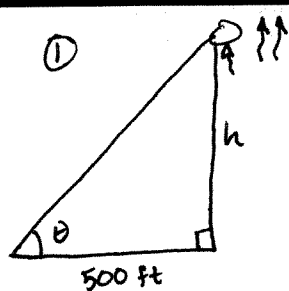
③ $\tan(\theta) = \dfrac{x}{50\ ft}$

④ $\dfrac{d}{dt}(\tan(\theta)) = \dfrac{d}{dt}\left(\dfrac{x}{50ft}\right) \iff \sec^2(\theta)\cdot\dfrac{d\theta}{dt} = \dfrac{1}{50ft}\cdot\dfrac{dx}{dt}$

⑤ $\left.\dfrac{d\theta}{dt}\right|_{\theta=0} = \left.\dfrac{\frac{1}{50ft}\cdot\frac{dx}{dt}}{\sec^2(\theta)}\right|_{\theta=0} = \dfrac{\frac{1}{50ft}\cdot -100\frac{ft}{sec}}{\sec^2(0)}$

$= -2\dfrac{1}{sec}$

$= \boxed{-2\ \dfrac{rad}{sec}}$

remember that radians are unitless
so getting units of $\frac{1}{sec}$ is the same
as $\frac{rad}{sec}$.

7

①

② Want $\dfrac{d\theta}{dt}\Big|_{h=500\,ft}$

③ $\tan(\theta) = \dfrac{h}{500\,ft}$

Given $\dfrac{dh}{dt} = 140\,\dfrac{ft}{sec}$

When $h = 500\,ft,\ \theta = \dfrac{\pi}{4}$

④ $\dfrac{d}{dt}\left(\tan(\theta)\right) = \dfrac{d}{dt}\left(\dfrac{h}{500\,ft}\right)$ $\Longleftrightarrow$ $\sec^2(\theta)\cdot\dfrac{d\theta}{dt} = \dfrac{1}{500\,ft}\cdot\dfrac{dh}{dt}$

⑤ $\dfrac{d\theta}{dt}\Big|_{h=500\,ft} = \dfrac{\dfrac{1}{500\,ft}\dfrac{dh}{dt}}{\sec^2(\theta)}\Bigg|_{\substack{\theta=\frac{\pi}{4} \\ h=500\,ft}} = \dfrac{\dfrac{1}{25\,\cancel{500}\,\cancel{ft}}\cdot\cancel{140}^{\,7}\,\frac{\cancel{ft}}{sec}}{\sec^2\left(\frac{\pi}{4}\right)}$ $\boxed{\begin{array}{l} = \dfrac{7}{50}\,\dfrac{1}{sec} \\[4pt] = \dfrac{7}{50}\,\dfrac{rad}{sec}\end{array}}$

8

①
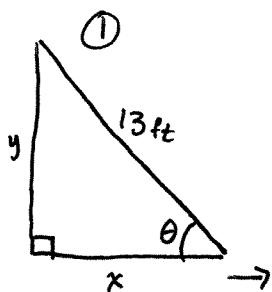
② Want $\dfrac{d\theta}{dt}\Big|_{y=12\,ft}$

③ $\cos(\theta) = \dfrac{x}{13\,ft}$

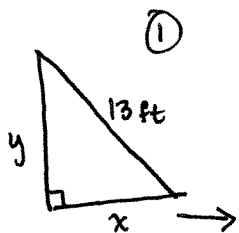Given $\dfrac{dx}{dt} = \dfrac{1}{6}\,\dfrac{ft}{sec}$

④ $\dfrac{d}{dt}\left(\cos(\theta)\right) = \dfrac{d}{dt}\left(\dfrac{x}{13\,ft}\right)$ $\Longleftrightarrow$ $-\sin(\theta)\cdot\dfrac{d\theta}{dt} = \dfrac{1}{13\,ft}\cdot\dfrac{dx}{dt}$

⑤ $\dfrac{d\theta}{dt}\Big|_{y=12\,ft} = \dfrac{\dfrac{1}{13\,ft}\dfrac{dx}{dt}}{-\sin(\theta)}\Bigg|_{y=12\,ft} = \dfrac{\dfrac{1}{13\,\cancel{ft}}\cdot\dfrac{1}{6}\,\frac{\cancel{ft}}{sec}}{-\dfrac{12}{13}}$ $\boxed{\begin{array}{l} = -\dfrac{1}{72}\,\dfrac{1}{sec} \\[4pt] = -\dfrac{1}{72}\,\dfrac{rad}{sec}\end{array}}$

Notice that when $y = 12\,ft$

then $\sin(\theta) = \dfrac{12\,ft}{13\,ft} = \dfrac{12}{13}$

9

① ② Want $\dfrac{dA}{dt}\Big|_{y=12ft}$ ③ $A = \dfrac{1}{2}xy$

Given $\dfrac{dx}{dt} = \dfrac{2}{3}\dfrac{ft}{sec}$

④ $\dfrac{d}{dt}(A) = \dfrac{d}{dt}\left(\dfrac{1}{2}xy\right) \iff \dfrac{dA}{dt} = \dfrac{1}{2}x\dfrac{dy}{dt} + \dfrac{1}{2}y\dfrac{dx}{dt}$

⑤ In order to find $\dfrac{dA}{dt}\Big|_{y=12ft}$, we need to find

$\dfrac{dy}{dt}\Big|_{y=13ft}$. We can do this by using the

relationship $x^2 + y^2 = 13^2\,ft^2 \Rightarrow 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0$

$\Rightarrow \dfrac{dy}{dt} = -\dfrac{x}{y}\dfrac{dx}{dt} \Rightarrow \dfrac{dy}{dt}\Big|_{y=12ft} = -\dfrac{5\,ft}{\cancel{12ft}_{6}}\cdot \dfrac{\overset{1}{\cancel{2}}}{3}\dfrac{ft}{sec} = \cancel{\text{(scribbled)}}$

$-\dfrac{5}{18}\dfrac{ft}{sec}$

So now we can find $\dfrac{dA}{dt}\Big|_{y=12ft}$
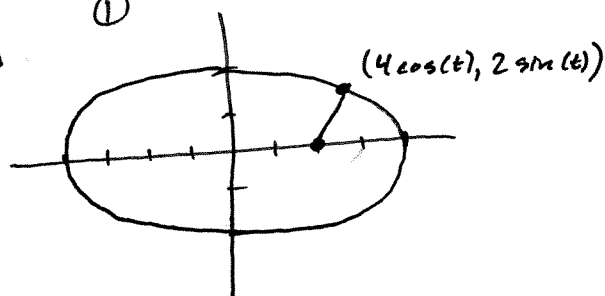
$\dfrac{dA}{dt}\Big|_{y=12ft} = \left(\dfrac{1}{2}x\dfrac{dy}{dt} + \dfrac{1}{2}y\dfrac{dx}{dt}\right)\Big|_{y=12ft}$

$= \dfrac{1}{2}(5\,ft)\left(\dfrac{-5}{18}\dfrac{ft}{sec}\right) + \dfrac{1}{2}(12\,ft)\left(\dfrac{2}{3}\dfrac{ft}{sec}\right)$

$= -\dfrac{25}{36}\dfrac{ft^2}{sec} + \dfrac{144}{36}\dfrac{ft^2}{sec} = \boxed{\dfrac{119}{36}\dfrac{ft^2}{sec}}$

10

①

$(4\cos(t), 2\sin(t))$

② Want $\left.\dfrac{ds}{dt}\right|_{t=\frac{\pi}{4}}$

Since $x(t) = 4\cos(t)$ then

$$\frac{dx}{dt} = -4\sin(t)$$

Since $y(t) = 2\sin(t)$ then

$$\frac{dy}{dt} = 2\cos(t)$$

③ $s^2 = (x-2)^2 + y^2 = x^2 - 4x + 4 + y^2$

So $\dfrac{ds}{dt}$ can be found by

$$\frac{d}{dt}(s^2) = \frac{d}{dt}(x^2 - 4x + 4 + y^2) \iff 2s\frac{ds}{dt} = 2x\frac{dx}{dt} - 4\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\Rightarrow \frac{ds}{dt} = \frac{x}{s}\frac{dx}{dt} - \frac{2}{s}\frac{dx}{dt} + \frac{y}{s}\frac{dy}{dt}$$

④ $\left.\dfrac{ds}{dt}\right|_{t=\frac{\pi}{4}} = \left.\left(\dfrac{x}{s}\dfrac{dx}{dt} - \dfrac{2}{s}\dfrac{dx}{dt} + \dfrac{y}{s}\dfrac{dy}{dt}\right)\right|_{t=\frac{\pi}{4}}$

When $t = \frac{\pi}{4}$, $\quad x = 4\cos\left(\frac{\pi}{4}\right) = 2\sqrt{2} \qquad \left.\dfrac{dx}{dt}\right|_{t=\frac{\pi}{4}} = -4\sin\left(\frac{\pi}{4}\right) = 2\sqrt{2}$

$y = 2\sin\left(\frac{\pi}{4}\right) = \sqrt{2} \qquad \left.\dfrac{dy}{dt}\right|_{t=\frac{\pi}{4}} = 2\cos\left(\frac{\pi}{4}\right) = \sqrt{2}$

and $s = \sqrt{(2\sqrt{2})^2 - 4(2\sqrt{2}) + 4 + (\sqrt{2})^2}$

$= \sqrt{8 - 8\sqrt{2} + 4 + 2} = \sqrt{14 - 8\sqrt{2}}$

So $\left.\dfrac{ds}{dt}\right|_{t=\frac{\pi}{4}} = \left(\dfrac{2\sqrt{2}}{\sqrt{14-8\sqrt{2}}}\cdot(-2\sqrt{2}) - \dfrac{2}{\sqrt{14-8\sqrt{2}}}(-2\sqrt{2}) + \dfrac{\sqrt{2}}{\sqrt{14-8\sqrt{2}}}(\sqrt{2})\right)$

$= \boxed{\dfrac{4\sqrt{2} - 6}{\sqrt{14-8\sqrt{2}}}}$

There are not units because there were no units given in the question.