

## Chapter 9 Discrete Mathematics

### Section 9.1 Basic Combinatorics

#### Exploration 1

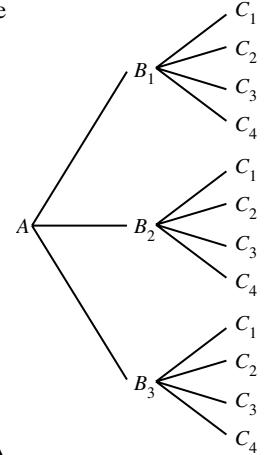
- Six:  $ABC, ACB, BAC, BCA, CAB, CBA$ .
- Approximately 1 person out of 6, which would mean 10 people out of 60.
- No. If they all looked the same, we would expect approximately 10 people to get the order right simply by chance. The fact that this did not happen leads us to reject the “look-alike” conclusion.
- It is likely that the salesman rigged the test to mislead the office workers. He might have put the copy from the more expensive machine on high-quality bond paper to make it look more like an original, or he might have put a tiny ink smudge on the original to make it look like a copy. You can offer your own alternate scenarios.

#### Quick Review 9.1

- 52
- 13
- 6
- 11
- 10
- 4
- 11
- 4
- 64
- 13

#### Section 9.1 Exercises

- There are three possibilities for who stands on the left, and then two remaining possibilities for who stands in the middle, and then one remaining possibility for who stands on the right:  $3 \cdot 2 \cdot 1 = 6$ .
- Any of the four jobs could be ranked most important, and then any of the remaining three jobs could be ranked second, and so on:  $4 \cdot 3 \cdot 2 \cdot 1 = 24$ .
- Any of the five books could be placed on the left, and then any of the four remaining books could be placed next to it, and so on:  $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ .
- Any of the five dogs could be awarded 1st place, and then any of the remaining four dogs could be awarded 2nd place, and so on:  $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ .
- There are  $3 \cdot 4 = 12$  possible pairs:  $K_1Q_1, K_1Q_2, K_1Q_3, K_1Q_4, K_2Q_1, K_2Q_2, K_2Q_3, K_2Q_4, K_3Q_1, K_3Q_2, K_3Q_3,$  and  $K_3Q_4$ .
- There are  $3 \cdot 4 = 12$  possible routes. In the tree diagram,  $B_1$  represents the first road from town  $A$  to town  $B$ , etc.
- $9! = 362,880$   
(ALGORITHM)
- $22 \cdot 21 \cdot 20 = 9240$
- There are 11 letters, where  $S$  and  $I$  each appear 4 times and  $P$  appears 2 times. The number of distinguishable permutations is  $\frac{11!}{4!4!2!} = 34,650$ .
- There are 11 letters, where  $A$  appears 3 times and  $O$  and  $T$  each appear 2 times. The number of distinguishable permutations is  $\frac{11!}{3!2!2!} = 1,663,200$ .
- The number of ways to fill 3 distinguishable offices from a pool of 13 candidates is  ${}_{13}P_3 = \frac{13!}{10!} = 1716$ .
- The number of ways to select and prioritize 6 out of 12 projects is  ${}_{12}P_6 = \frac{12!}{6!} = 665,280$ .
- $4 \cdot 3 \cdot 2 \cdot 1 = 24$
- $(3 \cdot 2 \cdot 1)(1) = 6$
- $\frac{6!}{(6-2)!} = \frac{6 \cdot 5 \cdot 4!}{4!} = 30$
- $\frac{9!}{(9-2)!} = \frac{9 \cdot 8 \cdot 7!}{7!} = 72$
- $\frac{10!}{7!(10-7)!} = \frac{10 \cdot 9 \cdot 8 \cdot 7!}{7! \cdot 3 \cdot 2 \cdot 1} = 120$
- $\frac{10!}{3!(10-3)!} = \frac{10 \cdot 9 \cdot 8 \cdot 7!}{3! \cdot 7!} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$
- combinations
- permutations
- combinations
- permutations (different roles)
- There are 10 choices for the first character, 9 for the second, 26 for the third, then 25, then 8, then 7, then 6:  $10 \cdot 9 \cdot 26 \cdot 25 \cdot 8 \cdot 7 \cdot 6 = 19,656,000$ .
- There are 36 choices for each character:  $36^5 = 60,466,176$ .
- There are 6 possibilities for the red die, and 6 for the green die:  $6 \cdot 6 = 36$ .
- There are 2 possibilities for each flip:  $2^{10} = 1024$ .



27.  ${}_{25}C_3 = \frac{25!}{3!(25-3)!} = \frac{25!}{3!22!} = 2300$
28.  ${}_{52}C_5 = \frac{52!}{5!(52-5)!} = \frac{52!}{5!47!} = 2,598,960$
29.  ${}_{48}C_3 = \frac{48!}{3!(48-3)!} = \frac{48!}{3!45!} = 17,296$
30. Choose 7 positions from the 20:  
 ${}_{20}C_7 = \frac{20!}{7!(20-7)!} = \frac{20!}{7!13!} = 77,520$
31. Choose A♠ and K♠, and 11 cards from the other 50:  
 ${}_{2}C_2 \cdot {}_{50}C_{11} = 1 \cdot {}_{50}C_{11} = \frac{50!}{11!(50-11)!} = \frac{50!}{11!39!}$   
 $= 37,353,738,800$
32.  ${}_{8}C_3 = \frac{8!}{3!(8-3)!} = \frac{8!}{3!5!} = 56$
33. We either have 3, 2, or 1 student(s) nominated:  
 ${}_6C_3 + {}_6C_2 + {}_6C_1 = 20 + 15 + 6 = 41$
34. We either have 3, 2, or 1 appetizer(s) represented:  
 ${}_5C_3 + {}_5C_2 + {}_5C_1 = 10 + 10 + 5 = 25$
35. Each of the 5 dice have 6 possible outcomes:  $6^5 = 7776$
36.  ${}_{20}C_8 = \frac{20!}{8!(20-8)!} = \frac{20!}{8!12!} = 125,970$
37.  $2^9 - 1 = 511$
38.  $3 \times 4 \times 3 \times 2^6 = 2304$
39. Since each topping can be included or left off, the total number of possibilities with  $n$  toppings is  $2^n$ . Since  $2^{11} = 2048$  is less than 4000 but  $2^{12} = 4096$  is greater than 4000, Luigi offers at least 12 toppings.
40. There are  $2^n$  subsets, of which  $2^n - 2$  are proper subsets.
41.  $2^{10} = 1024$
42.  $5^{10} = 9,765,625$
43. True.  $\binom{n}{a} = \frac{n!}{a!(n-a)!} = \frac{n!}{a!b!} = \frac{n!}{(n-b)!b!} = \binom{n}{b}$
44. False. For example,  $\binom{5}{2} = 10$  is greater than  $\binom{5}{4} = 5$ .
45. There are  $\binom{6}{2} = 15$  different combinations of vegetables. The total number of entrée-vegetable-dessert variations is  $4 \cdot 15 \cdot 6 = 360$ . The answer is D.
46.  ${}_{10}P_5 = 30,240$ . The answer is D.
47.  ${}_nP_n = \frac{n!}{(n-n)!} = n!$  The answer is B.
48. There are as many ways to vote as there are subsets of a set with 5 members. That is, there are  $2^5$  ways to fill out the ballot. The answer is C.
49. Answers will vary. Here are some possible answers:
- (a) Number of 3-card hands that can be dealt from a deck of 52 cards
  - (b) Number of ways to choose 3 chocolates from a box of 12 chocolates
  - (c) Number of ways to choose a starting soccer team from a roster of 25 players (where position matters)

- (d) Number of 5-digit numbers that can be formed using only the digits 1 and 2
- (e) Number of possible pizzas that can be ordered at a place that offers 3 different sizes and up to 10 different toppings.

50. Counting the number of ways to choose the 2 eggs you are going to have for breakfast is equivalent to counting the number of ways to choose the ten eggs you are *not* going to have for breakfast.

51. (a) Twelve
- (b) Every 0 represents a factor of 10, or a factor of 5 multiplied by a factor of 2. In the product  $50 \cdot 49 \cdot 48 \cdot \dots \cdot 2 \cdot 1$ , the factors 5, 10, 15, 20, 30, 35, 40, and 45 each contain 5 as a factor once, and 25 and 50 each contain 5 twice, for a total of twelve occurrences. Since there are 47 factors of 2 to pair up with the twelve factors of 5, 10 is a factor of  $50!$  twelve times.

52. (a) Each combination of the  $n$  vertices taken 2 at a time determines a segment that is either an edge or a diagonal. There are  ${}_nC_2$  such combinations.

(b) Subtracting the  $n$  edges from the answers in (a), we find that  ${}_nC_2 - n = \frac{n!}{2!(n-2)!} - n$   
 $= \frac{n(n-1)}{2} - \frac{2n}{2} = \frac{n^2 - 3n}{2}$ .

53. In the  $n$ th week,  $5^n$  copies of the letter are sent. In the last week of the year, that's  $5^{52} \approx 2.22 \times 10^{36}$  copies of the letter. This exceeds the population of the world, which is about  $6 \times 10^9$ , so someone (several people, actually) has had to receive a second copy of the letter.

54. Six. No matter where the first person sits, there are the  $3! = 6$  ways to sit the others in different positions relative to the first person.

55. Three. This is equivalent to the round table problem (Exercise 42), except that the necklace can be *turned upside-down*. Thus, each different necklace accounts for two of the six different orderings.

56. The chart on the left is more reasonable. Each pair of actresses will require about the same amount of time to interview. If we make a chart showing  $n$  (the number of actresses) and  ${}_nC_2$  (the number of pairings), we can see that chart 1 allows approximately 3 minutes per pair throughout, while chart 2 allows less and less time per pair as  $n$  gets larger.

Number $n$	Number of pairs ${}_nC_2$	Time per Pair Chart1	Time per Pair Chart 2
3	3	3.33	3.33
6	15	3	2
9	36	3.06	1.67
12	66	3.03	1.52
15	105	3.05	1.43

57. There are  ${}_{52}C_{13} = 635,013,559,600$  distinct bridge hands. Every day has  $60 \cdot 60 \cdot 24 = 86,400$  seconds; a year has 365.24 days, which is 31,556,736 seconds. Therefore it will take about  $\frac{635,013,559,600}{31,556,736} \approx 20,123$  years. (Using 365 days per year, the computation gives about 20,136 years.)

58. Each team can choose 5 players in  ${}_{12}C_5 = 792$  ways, so there are  $792^2 = 627,264$  ways total.

## Section 9.2 The Binomial Theorem

### Exploration 1

- ${}_3C_0 = \frac{3!}{0!3!} = 1$ ,  ${}_3C_1 = \frac{3!}{1!2!} = 3$ ,  
 ${}_3C_2 = \frac{3!}{2!1!} = 3$ ,  ${}_3C_3 = \frac{3!}{3!0!} = 1$ . These are (in order) the coefficients in the expansion of  $(a + b)^3$ .
- {1 4 6 4 1}. These are (in order) the coefficients in the expansion of  $(a + b)^4$ .
- {1 5 10 10 5 1}. These are (in order) the coefficients in the expansion of  $(a + b)^5$ .

### Quick Review 9.2

- $x^2 + 2xy + y^2$
- $a^2 + 2ab + b^2$
- $25x^2 - 10xy + y^2$
- $a^2 - 6ab + 9b^2$
- $9s^2 + 12st + 4t^2$
- $9p^2 - 24pq + 16q^2$
- $u^3 + 3u^2v + 3uv^2 + v^3$
- $b^3 - 3b^2c + 3bc^2 - c^3$
- $8x^3 - 36x^2y + 54xy^2 - 27y^3$
- $64m^3 + 144m^2n + 108mn^2 + 27n^3$

### Section 9.2 Exercises

- $(a + b)^4 = \binom{4}{0}a^4b^0 + \binom{4}{1}a^3b^1 + \binom{4}{2}a^2b^2 + \binom{4}{3}a^1b^3 + \binom{4}{4}a^0b^4$   
 $= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
- $(a + b)^6 = \binom{6}{0}a^6b^0 + \binom{6}{1}a^5b^1 + \binom{6}{2}a^4b^2 + \binom{6}{3}a^3b^3 + \binom{6}{4}a^2b^4 + \binom{6}{5}a^1b^5 + \binom{6}{6}a^0b^6$   
 $= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$
- $(x + y)^7 = \binom{7}{0}x^7y^0 + \binom{7}{1}x^6y^1 + \binom{7}{2}x^5y^2 + \binom{7}{3}x^4y^3 + \binom{7}{4}x^3y^4 + \binom{7}{5}x^2y^5 + \binom{7}{6}x^1y^6 + \binom{7}{7}x^0y^7$   
 $= x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$

$$\begin{aligned} 4. (x + y)^{10} &= \binom{10}{0}x^{10}y^0 + \binom{10}{1}x^9y^1 + \binom{10}{2}x^8y^2 \\ &\quad + \binom{10}{3}x^7y^3 + \binom{10}{4}x^6y^4 \\ &\quad + \binom{10}{5}x^5y^5 + \binom{10}{6}x^4y^6 \\ &\quad + \binom{10}{7}x^3y^7 + \binom{10}{8}x^2y^8 \\ &\quad + \binom{10}{9}x^1y^9 + \binom{10}{10}x^0y^{10} \\ &= x^{10} + 10x^9y + 45x^8y^2 + 120x^7y^3 + 210x^6y^4 \\ &\quad + 252x^5y^5 + 210x^4y^6 + 120x^3y^7 \\ &\quad + 45x^2y^8 + 10xy^9 + y^{10} \end{aligned}$$

- Use the entries in row 3 as coefficients:  
 $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
- Use the entries in row 5 as coefficients:  
 $(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$
- Use the entries in row 8 as coefficients:  
 $(p + q)^8 = p^8 + 8p^7q + 28p^6q^2 + 56p^5q^3 + 70p^4q^4 + 56p^3q^5 + 28p^2q^6 + 8pq^7 + q^8$
- Use the entries in row 9 as coefficients:  
 $(p + q)^9 = p^9 + 9p^8q + 36p^7q^2 + 84p^6q^3 + 126p^5q^4 + 126p^4q^5 + 84p^3q^6 + 36p^2q^7 + 9pq^8 + q^9$
- $\binom{9}{2} = \frac{9!}{2!7!} = \frac{9 \cdot 8}{2 \cdot 1} = 36$
- $\binom{15}{11} = \frac{15!}{11!4!} = \frac{15 \cdot 14 \cdot 13 \cdot 12}{4 \cdot 3 \cdot 2 \cdot 1} = 1365$
- $\binom{166}{166} = \frac{166!}{166!0!} = 1$
- $\binom{166}{0} = \frac{166!}{0!166!} = 1$
- $\binom{14}{3} = \binom{14}{11} = 364$
- $\binom{13}{8} = \binom{13}{5} = 1287$
- $(-2)^8 \binom{12}{8} = (-2)^8 \binom{12}{4} = 126,720$
- $(-3)^4 \binom{11}{4} = (-3)^4 \binom{11}{7} = 26,730$
- $f(x) = (x - 2)^5$   
 $= x^5 + 5x^4(-2) + 10x^3(-2)^2 + 10x^2(-2)^3 + 5x(-2)^4 + (-2)^5$   
 $= x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$
- $g(x) = (x + 3)^6$   
 $= x^6 + 6x^5 \cdot 3 + 15x^4 \cdot 3^2 + 20x^3 \cdot 3^3 + 15x^2 \cdot 3^4 + 6x \cdot 3^5 + 3^6$   
 $= x^6 + 18x^5 + 135x^4 + 540x^3 + 1215x^2 + 1458x + 729$
- $h(x) = (2x - 1)^7$   
 $= (2x)^7 + 7(2x)^6(-1) + 21(2x)^5(-1)^2 + 35(2x)^4(-1)^3 + 35(2x)^3(-1)^4 + 21(2x)^2(-1)^5 + 7(2x)(-1)^6 + (-1)^7$   
 $= 128x^7 - 448x^6 + 672x^5 - 560x^4 + 280x^3 - 84x^2 + 14x - 1$

$$\begin{aligned}
20. f(x) &= (3x + 4)^5 \\
&= (3x)^5 + 5(3x)^4 \cdot 4 + 10(3x)^3 \cdot 4^2 \\
&\quad + 10(3x)^2 \cdot 4^3 + 5(3x) \cdot 4^4 + 4^5 \\
&= 243x^5 + 1620x^4 + 4320x^3 + 5760x^2 + 3840x \\
&\quad + 1024
\end{aligned}$$

$$\begin{aligned}
21. (2x + y)^4 &= (2x)^4 + 4(2x)^3y + 6(2x)^2y^2 + 4(2x)y^3 + y^4 \\
&= 16x^4 + 32x^3y + 24x^2y^2 + 8xy^3 + y^4
\end{aligned}$$

$$\begin{aligned}
22. (2y - 3x)^5 &= (2y)^5 + 5(2y)^4(-3x) + 10(2y)^3(-3x)^2 \\
&\quad + 10(2y)^2(-3x)^3 + 5(2y)(-3x)^4 \\
&\quad + (-3x)^5 \\
&= 32y^5 - 240y^4x + 720y^3x^2 - 1080y^2x^3 \\
&\quad + 810yx^4 - 243x^5
\end{aligned}$$

$$\begin{aligned}
23. (\sqrt{x} - \sqrt{y})^6 &= (\sqrt{x})^6 + 6(\sqrt{x})^5(-\sqrt{y}) + 15(\sqrt{x})^4 \\
&\quad \cdot (-\sqrt{y})^2 + 20(\sqrt{x})^3(-\sqrt{y})^3 \\
&\quad + 15(\sqrt{x})^2(-\sqrt{y})^4 \\
&\quad + 6(\sqrt{x})(-\sqrt{y})^5 + (-\sqrt{y})^6 \\
&= x^3 - 6x^{5/2}y^{1/2} + 15x^2y - 20x^{3/2}y^{3/2} \\
&\quad + 15xy^2 - 6x^{1/2}y^{5/2} + y^3
\end{aligned}$$

$$\begin{aligned}
24. (\sqrt{x} + \sqrt{3})^4 &= (\sqrt{x})^4 + 4(\sqrt{x})^3(\sqrt{3}) + 6(\sqrt{x})^2 \\
&\quad \cdot (\sqrt{3})^2 + 4(\sqrt{x})(\sqrt{3})^3 + (\sqrt{3})^4 \\
&= x^2 + 4x\sqrt{3x} + 18x + 12\sqrt{3x} + 9
\end{aligned}$$

$$\begin{aligned}
25. (x^{-2} + 3)^5 &= (x^{-2})^5 + 5(x^{-2})^4 \cdot 3 + 10(x^{-2})^3 \cdot 3^2 \\
&\quad + 10(x^{-2})^2 \cdot 3^3 + 5(x^{-2}) \cdot 3^4 + 3^5 \\
&= x^{-10} + 15x^{-8} + 90x^{-6} + 270x^{-4} + 405x^{-2} \\
&\quad + 243
\end{aligned}$$

$$\begin{aligned}
26. (a - b^{-3})^7 &= a^7 + 7a^6(-b^{-3}) + 21a^5(-b^{-3})^2 \\
&\quad + 35a^4(-b^{-3})^3 + 35a^3(-b^{-3})^4 \\
&\quad + 21a^2(-b^{-3})^5 + 7a(-b^{-3})^6 + (-b^{-3})^7 \\
&= a^7 - 7a^6b^{-3} + 21a^5b^{-6} - 35a^4b^{-9} \\
&\quad + 35a^3b^{-12} - 21a^2b^{-15} + 7ab^{-18} - b^{-21}
\end{aligned}$$

27. Answers will vary.

28. Answers will vary.

$$\begin{aligned}
29. \binom{n}{1} &= \frac{n!}{1!(n-1)!} = n = \frac{n!}{(n-1)!1!} \\
&= \frac{n!}{(n-1)![n-(n-1)]!} = \binom{n}{n-1}
\end{aligned}$$

$$\begin{aligned}
30. \binom{n}{r} &= \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!r!} \\
&= \frac{n!}{(n-r)![n-(n-r)]!} = \binom{n}{n-r}
\end{aligned}$$

$$\begin{aligned}
31. \binom{n-1}{r-1} + \binom{n-1}{r} &= \frac{(n-1)!}{(r-1)![n-1-(r-1)]!} + \frac{(n-1)!}{r!(n-1-r)!} \\
&= \frac{(n-1)!}{r(n-1)!} + \frac{(n-1)!(n-r)}{r!(n-r)(n-r-1)!} \\
&= \frac{r(n-1)!}{r!(n-r)!} + \frac{(n-r)(n-1)!}{r!(n-r)!} \\
&= \frac{(r+n-r)(n-1)!}{r!(n-r)!} \\
&= \frac{n!}{r!(n-r)!} \\
&= \binom{n}{r}
\end{aligned}$$

32. (a) Any pair  $(n, m)$  of nonnegative integers — except for  $(1, 1)$  — provides a counterexample. For example,  $n = 2$  and  $m = 3$ :  $(2 + 3)! = 5! = 120$ , but  $2! + 3! = 2 + 6 = 8$ .

(b) Any pair  $(n, m)$  of nonnegative integers — except for  $(0, 0)$  or any pair  $(1, m)$  or  $(n, 1)$  — provides a counterexample. For example,  $n = 2$  and  $m = 3$ :  $(2 \cdot 3)! = 6! = 720$ , but,  $2! \cdot 3! = 2 \cdot 6 = 12$ .

$$\begin{aligned}
33. \binom{n}{2} + \binom{n+1}{2} &= \frac{n!}{2!(n-2)!} + \frac{(n+1)!}{2!(n-1)!} \\
&= \frac{n(n-1)}{2} + \frac{(n+1)n}{2} \\
&= \frac{n^2 - n + n^2 + n}{2} = n^2
\end{aligned}$$

$$\begin{aligned}
34. \binom{n}{n-2} + \binom{n+1}{n-1} &= \frac{n!}{(n-2)![n-(n-2)]!} \\
&\quad + \frac{(n+1)!}{(n-1)![n+1-(n-1)]!} \\
&= \frac{n!}{(n-2)!2!} + \frac{(n+1)!}{(n-1)!2!} \\
&= \frac{n(n-1)}{2} + \frac{(n+1)n}{2} \\
&= \frac{n^2 - n + n^2 + n}{2} = n^2
\end{aligned}$$

35. True. The signs of the coefficients are determined by the powers of the  $(-y)$  terms, which alternate between odd and even.

36. True. In fact, the sum of every row is a power of 2.

37. The fifth term of the expansion is

$$\binom{8}{4}(2x)^4(1)^4 = 1120x^4. \text{ The answer is C.}$$

38. The two smallest numbers in row 10 are 1 and 10. The answer is B.

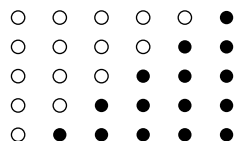
39. The sum of the coefficients of  $(3x - 2y)^{10}$  is the same as the value of  $(3x - 2y)^{10}$  when  $x = 1$  and  $y = 1$ . The answer is A.

40. The even-numbered terms in the two expressions are opposite-signed and cancel out, while the odd-numbered terms are identical and add together. The answer is D.

41. (a) 1, 3, 6, 10, 15, 21, 28, 36, 45, 55.

(b) They appear diagonally down the triangle, starting with either of the 1's in row 2.

(c) Since  $n$  and  $n + 1$  represent the sides of the given rectangle, then  $n(n + 1)$  represents its area. The triangular number is  $1/2$  of the given area. Therefore, the triangular number is  $\frac{n(n+1)}{2}$ .



(d) From part (c), we observe that the  $n$ th triangular number can be written as  $\frac{n(n+1)}{2}$ . We know that

binomial coefficients are the values of  $\binom{n}{r}$  for

$r = 0, 1, 2, 3, \dots, n$ . We can show that

$$\frac{n(n+1)}{2} = \binom{n+1}{2} \text{ as follows:}$$

$$\frac{n(n+1)}{2} = \frac{(n+1)n(n-1)!}{2(n-1)!}$$

$$= \frac{(n+1)!}{2!(n-1)!}$$

$$= \frac{(n+1)!}{2!((n+1)-2)!}$$

$$= \binom{n+1}{2}$$

So, to find the fourth triangular number, for example,

$$\text{compute } \binom{4+1}{2} = \binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5 \cdot 4 \cdot 3!}{2!3!}$$

$$= \frac{5 \cdot 4}{2} = 10.$$

42. (a) 2. (Every other number appears at least twice.)  
 (b) 1  
 (c) No. (They all appear in order down the second diagonal.)  
 (d) 0. (See Exercise 44 for a proof.)  
 (e) All are divisible by  $p$ .  
 (f) Rows that are powers of 2: 2, 4, 8, 16, etc.  
 (g) Rows that are 1 less than a power of 2: 1, 3, 7, 15, etc.  
 (h) For any prime numbered row, or row where the first element is a prime number, all the numbers in that row (excluding the 1's) are divisible by the prime. For example, in the seventh row (1 7 21 35 35 21 7 1) 7, 21, and 35 are all divisible by 7.

43. The sum of the entries in the  $n$ th row equals the sum of the coefficients in the expansion of  $(x+y)^n$ . But this sum, in turn, is equal to the value of  $(x+y)^n$  when  $x=1$  and  $y=1$ :

$$\begin{aligned} 2^n &= (1+1)^n \\ &= \binom{n}{0}1^n1^0 + \binom{n}{1}1^{n-1}1^1 + \binom{n}{2}1^{n-2}1^2 \\ &\quad + \dots + \binom{n}{n}1^01^n \\ &= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} \end{aligned}$$

44.  $0 = (1-1)^n$

$$\begin{aligned} &= \binom{n}{0}1^n + \binom{n}{1}1^{n-1}(-1) + \binom{n}{2}1^{n-2}(-1)^2 \\ &\quad + \dots + \binom{n}{n}(-1)^n \\ &= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \dots + (-1)^n \binom{n}{n} \end{aligned}$$

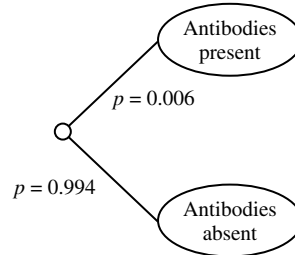
45.  $3^n = (1+2)^n$

$$\begin{aligned} &= \binom{n}{0}1^n + \binom{n}{1}1^{n-1}2 + \binom{n}{2}1^{n-2}2^2 + \dots + \binom{n}{n}2^n \\ &= \binom{n}{0} + 2\binom{n}{1} + 4\binom{n}{2} + \dots + 2^n\binom{n}{n} \end{aligned}$$

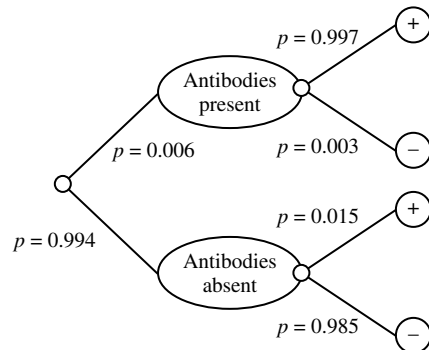
## Section 9.3 Probability

### Exploration 1

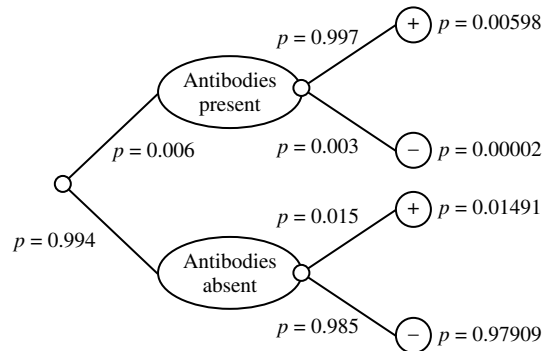
1.



2.



3.



4.  $P(+)=0.00598+0.01491=0.02089$   
 5.  $P(\text{antibody present} | +) = \frac{P(\text{antibody present and } +)}{P(+)}$   
 $\frac{0.00598}{0.02089} \approx 0.286$  (A little more than 1 chance in 4.)

### Quick Review 9.3

1. 2  
 2. 6  
 3.  $2^3 = 8$   
 4.  $6^3 = 216$   
 5.  ${}_{52}C_5 = 2,598,960$   
 6.  ${}_{10}C_2 = 45$   
 7.  $5! = 120$

$$8. {}_5P_3 = 60$$

$$9. \frac{{}_5C_3}{{}_{10}C_3} = \frac{\frac{5!}{3!2!}}{\frac{10!}{3!7!}} = \frac{1}{12}$$

$$10. \frac{{}_5C_2}{{}_{10}C_2} = \frac{\frac{5!}{2!3!}}{\frac{10!}{2!8!}} = \frac{2}{9}$$

**Section 9.3 Exercises**

For #1–8, consider ordered pairs  $(a, b)$  where  $a$  is the value of the red die and  $b$  is the value of the green die.

1.  $E = \{(3, 6), (4, 5), (5, 4), (6, 3)\}; P(E) = \frac{4}{36} = \frac{1}{9}$
2.  $E = \{\text{both dice even}\}, \{\text{both dice odd}\};$   
 $P(E) = \frac{3 \cdot 3 + 3 \cdot 3}{36} = \frac{18}{36} = \frac{1}{2}$
3.  $E = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3), (5, 1), (5, 2), (5, 3), (5, 4), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5)\};$   
 $P(E) = \frac{15}{36} = \frac{5}{12}$
4.  $E = \{(1, 1), (1, 2), \dots, (6, 2), (6, 3)\}$   
 $P(E) = \frac{30}{36} = \frac{5}{6}$
5.  $P(E) = \frac{3 \cdot 3}{36} = \frac{1}{4}$
6.  $P(E) = \frac{3 \cdot 3}{36} = \frac{1}{4}$
7.  $E = \{(1, 1), (1, 2), (1, 4), (1, 6), (2, 1), (2, 3), (2, 5), (3, 2), (3, 4), (4, 1), (4, 3), (5, 2), (5, 6), (6, 1), (6, 5)\}$   
 $P(E) = \frac{15}{36} = \frac{5}{12}$
8.  $E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (5, 6), (6, 1), (6, 5)\};$   
 $P(E) = \frac{8}{36} = \frac{2}{9}$

9. (a) No.  $0.25 + 0.20 + 0.35 + 0.30 = 1.1$ . The numbers do not add up to 1.

(b) There is a problem with Alrik’s reasoning. Since the gerbil must always be in exactly one of the four rooms, the proportions must add up to 1, just like a probability function.

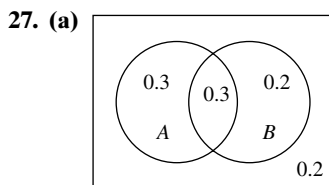
10. Since  $4 + 3 + 2 + 1 = 10$ , we can divide each number in the ratio by 10 and get the proportions relative to the whole. The table then becomes

Compartment	A	B	C	D
Proportion	0.4	0.3	0.2	0.1

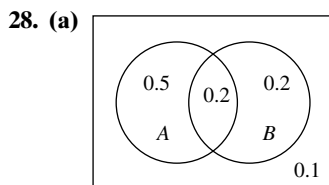
Yes, this is a valid probability function.

11.  $P(B \text{ or } T) = P(B) + P(T) = 0.3 + 0.1 = 0.4$
12.  $P(R \text{ or } G \text{ or } O) = P(R) + P(G) + P(O)$   
 $= 0.2 + 0.1 + 0.1 = 0.4$
13.  $P(R) = 0.2$
14.  $P(\text{not } R) = 1 - P(R) = 1 - 0.2 = 0.8$
15.  $P[\text{not } (O \text{ or } Y)] = 1 - P(O \text{ or } Y)$   
 $= 1 - (0.2 + 0.1) = 0.7$

16.  $P[\text{not } (B \text{ or } T)] = 1 - P(B \text{ or } T) = 1 - (0.3 + 0.1) = 0.6$
17.  $P(B_1 \text{ and } B_2) = P(B_1) \cdot P(B_2) = (0.3)(0.3) = 0.09$
18.  $P(O_1 \text{ and } O_2) = P(O_1) \cdot P(O_2) = (0.1)(0.1) = 0.01$
19.  $P[(R_1 \text{ and } G_2) \text{ or } (G_1 \text{ and } R_2)] = P(R_1) \cdot P(G_2)$   
 $+ P(G_1) \cdot P(R_2) = (0.2)(0.2) + (0.2)(0.2) = 0.08$
20.  $P(B_1 \text{ and } Y_2) = P(B_1) \cdot P(Y_2) = (0.3)(0.2) = 0.06$
21.  $P(\text{neither is yellow}) = P(\text{not } Y_1 \text{ and not } Y_2)$   
 $= P(\text{not } Y_1) \cdot P(\text{not } Y_2) = (0.8)(0.8) = 0.64$
22.  $P(\text{not } R_1 \text{ and not } O_2) = P(\text{not } R_1) \cdot P(\text{not } O_2)$   
 $= (0.8)(0.9) = 0.72$
23. There are  ${}_{24}C_6 = 134,596$  possible hands; of these, only 1 consists of all spades, so the probability is  $\frac{1}{134,596}$ .
24. Of the  ${}_{24}C_6 = 134,596$  possible hands, one consists of all spades, one consists of all clubs, one consists of all hearts, and one consists of all diamonds, so the probability is  $\frac{4}{134,596} = \frac{1}{33,649}$ .
25. Of the  ${}_{24}C_6 = 134,596$  possible hands, there are  ${}_4C_4 \cdot {}_{20}C_2 = 190$  hands with all the aces, so the probability is  $\frac{190}{134,596} = \frac{5}{3542}$ .
26. There are  ${}_2C_2 \cdot {}_{22}C_4 = 7315$  ways to get both black jacks and 4 “other” cards. Similarly, there are 7315 ways to get both red jacks. These two numbers together count twice the  ${}_2C_2 \cdot {}_{20}C_2 = 190$  ways to get all four jacks. Therefore, altogether we have  $2 \cdot 7315 - 190 = 14,440$  distinct ways to have both bowers, so the probability is  $\frac{14,440}{134,596} = \frac{190}{1771}$ .



- (b) 0.3
- (c) 0.2
- (d) 0.2
- (e) Yes.  $P(A|B) = \frac{P(A \text{ and } B)}{P(B)} = \frac{0.3}{0.5} = 0.6 = P(A)$



- (b) 0.5
- (c) 0.2
- (d) 0.1
- (e) No.  $P(A|B) = \frac{P(A \text{ and } B)}{P(B)} = \frac{0.2}{0.4} = 0.5 \neq P(A)$
29.  $P(\text{John will practice}) = (0.6)(0.8) + (0.4)(0.4) = 0.64$

$$30. \text{(a)} P(\text{meatloaf is served}) = \frac{1}{5} = 0.20$$

$$\text{(b)} P(\text{meatloaf and peas are served}) = (0.20)(0.70) = 0.14$$

$$\text{(c)} P(\text{peas are served}) = (0.20)(0.70) + (0.80)(0.30) = 0.38$$

31. If all precalculus students were put on a single list and a name then randomly chosen, the probability  $P$ (from Mr. Abel's class | girl) would be  $12|22 = 6|11$ . But when one of the two classes is selected at random, and then a student from this class is selected,

$$\begin{aligned} P(\text{from Mr. Abel's class | girl}) &= \frac{P(\text{girl from Mr. Abel's class})}{P(\text{girl})} \\ &= \frac{\left(\frac{1}{2}\right)\left(\frac{12}{20}\right)}{\left(\frac{1}{2}\right)\left(\frac{12}{20}\right) + \left(\frac{1}{2}\right)\left(\frac{10}{25}\right)} \\ &= \frac{3}{5} \end{aligned}$$

32. Within each box, any of the coins is equally likely to be chosen and either side is equally likely to be shown. But a head in the 2-coin box is more likely to be displayed than a head in the 3-coin box.

$$\begin{aligned} P(\text{from 2-coin box | H}) &= \frac{P(\text{H from 2-coin box})}{P(\text{H})} \\ &= \frac{\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)}{\left(\frac{1}{2}\right)\left(\frac{3}{4}\right) + \left(\frac{1}{2}\right)\left(\frac{3}{6}\right)} \\ &= \frac{3}{5} \end{aligned}$$

$$33. \frac{{}_{20}C_2}{{}_{25}C_2} = \frac{190}{300} = \frac{19}{30}$$

$$34. P(\text{none are defective}) = {}_4C_0 \cdot (0.037)^0(0.963)^4 = (0.963)^4 \approx 0.860$$

$$35. \text{(a)} P(\text{cardiovascular disease or cancer}) = 0.45 + 0.22 = 0.67$$

$$\text{(b)} P(\text{other cause of death}) = 1 - 0.67 = 0.33$$

$$36. P(\text{Yahtzee}) = 6 \cdot \left(\frac{1}{6}\right)^5 = \frac{1}{1296}$$

37. The sum of the probabilities is greater than 1 – an impossibility, since the events are mutually exclusive.

$$38. P(\text{at least one false positive}) = 1 - P(\text{no false positives}) = 1 - (0.993)^{60} \approx 0.344.$$

$$39. \text{(a)} P(\text{a woman}) = \frac{172}{254} = \frac{86}{127}$$

$$\text{(b)} P(\text{went to graduate school}) = \frac{124 + 58}{254} = \frac{91}{127}$$

$$\begin{aligned} \text{(c)} P(\text{a woman who went to graduate school}) &= \frac{124}{254} = \frac{62}{127} \end{aligned}$$

$$40. \text{(a)} \frac{{}_{14}C_8}{{}_{20}C_8} = \frac{3003}{125,970} = \frac{77}{3230}$$

$$\text{(b)} \frac{{}_{14}C_5 \cdot {}_6C_3}{{}_{20}C_8} = \frac{40,040}{125,970} = \frac{308}{969}$$

$$\begin{aligned} \text{(c)} \frac{{}_{14}C_6 \cdot {}_6C_2 + {}_{14}C_7 \cdot {}_6C_1 + {}_{14}C_8 \cdot {}_6C_0}{{}_{20}C_8} &= \frac{45,045 + 20,592 + 3003}{125,970} = \frac{68,640}{125,970} = \frac{176}{323} \end{aligned}$$

$$41. \frac{1}{{}_9C_2} = \frac{1}{36}$$

42. This cannot be true. Let  $A$  be the event that it is cloudy all day,  $B$  be the event that there is at least 1 hour of sunshine, and  $C$  be the event that there is some sunshine, but less than 1 hour. Then  $A$ ,  $B$ , and  $C$  are mutually exclusive events, so  $P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C) = 0.22 + 0.78 + P(C) = 1 + P(C)$ . Then it must be the case that  $P(C) = 0$ . This is absurd; there must be some probability of having more than 0 but less than 1 hour of sunshine.

$$43. P(\text{HTTTTTTTTTT}) = \left(\frac{1}{2}\right)^{10} = \frac{1}{2^{10}} = \frac{1}{1024}$$

$$44. P(\text{THTTTTHTTT}) = \left(\frac{1}{2}\right)^{10} = \frac{1}{1024}$$

$$45. P(\text{HHHHHHHHHH}) = \left(\frac{1}{2}\right)^{10} = \frac{1}{1024}$$

$$46. P(9 \text{ H and } 1 \text{ T}) = {}_{10}C_1 \cdot \left(\frac{1}{2}\right)^{10} = \frac{10}{1024} = \frac{5}{512}$$

$$47. P(2 \text{ H and } 8 \text{ T}) = {}_{10}C_2 \cdot \left(\frac{1}{2}\right)^{10} = \frac{45}{1024}$$

$$48. P(3 \text{ H and } 7 \text{ T}) = {}_{10}C_3 \cdot \left(\frac{1}{2}\right)^{10} = \frac{120}{1024} = \frac{15}{128}$$

$$\begin{aligned} 49. P(\text{at least one H}) &= 1 - P(0 \text{ H}) = 1 - {}_{10}C_0 \cdot \left(\frac{1}{2}\right)^{10} \\ &= 1 - \frac{1}{1024} = \frac{1023}{1024} \end{aligned}$$

$$\begin{aligned} 50. P(\text{at least two H}) &= 1 - P(0 \text{ H}) - P(1 \text{ H}) = \\ &= 1 - {}_{10}C_0 \cdot \left(\frac{1}{2}\right)^{10} - {}_{10}C_1 \cdot \left(\frac{1}{2}\right)^{10} = 1 - \frac{11}{1024} = \frac{1013}{1024} \end{aligned}$$

51. False. A sample space consists of outcomes, which are not necessarily equally likely.

52. False. All probabilities are between 0 and 1, inclusive.

53. Of the 36 different, equally-likely ways the dice can land, 4 ways have a total of 5. So the probability is  $4/36 = 1/9$ . The answer is D.

54. A probability must always be between 0 and 1, inclusive. The answer is E.

55.  $P(B \text{ and } A) = P(B)P(A|B)$ , and for independent events,  $P(B \text{ and } A) = P(B)P(A)$ . It follows that the answer is A.

56. A specific sequence of one “heads” and two “tails” has probability  $(1/2)^3 = 1/8$ . There are three such sequences. The answer is C.

57. (a)

Type of Bagel	Probability
Plain	0.37
Onion	0.12
Rye	0.11
Cinnamon Raisin	0.25
Sourdough	0.15

- (b)  $(0.37)(0.37)(0.37) \approx 0.051$   
 (c) No. They are more apt to share bagel preferences if they arrive at the store together.

58. (a)  $P(\text{at least one king}) = 1 - P(\text{no kings})$

$$= 1 - \frac{{}_{48}C_5}{{}_{52}C_5} = \frac{18,472}{54,145} \approx 0.34$$

- (b) The number of ways to choose, e.g., 3 fives and 2 jacks is  ${}_4C_3 \cdot {}_4C_2$ . There are  ${}_{13}P_2 = 13 \cdot 12$  different combinations of cards that can make up the full house, so

$$P(\text{full house}) = \frac{13 \cdot 12 \cdot {}_4C_3 \cdot {}_4C_2}{{}_{52}C_5} = \frac{6}{4165} \approx 0.0014$$

59. (a) We expect  $(8)(0.23) = 1.84$  (about 2) to be married.

- (b) Yes, this would be an unusual sample.  
 (c)  $P(5 \text{ or more are married}) = {}_8C_5 \cdot (0.23)^5(0.77)^3$   
 $+ {}_8C_6 \cdot (0.23)^6(0.77)^2 + {}_8C_7 \cdot (0.23)^7(0.77)^1$   
 $+ {}_8C_8 \cdot (0.23)^8(0.77)^0 \approx 0.01913 = 1.913\%$

60. (a)  ${}_{32}C_{17} \cdot (0.75)^{17}(0.25)^{15} \approx 0.00396 = 0.396\%$

(b)  $\sum_{k=0}^{17} {}_{32}C_k \cdot (0.75)^k(0.25)^{15-k} \approx 0.00596 = 0.596\%$

- (c) The university's graduation rate seems to be exaggerated; at least, this particular class did not fare as well as the university claims.

61. (a) \$1.50

(b)  $3 \cdot \frac{2}{6} + (-1) \cdot \frac{4}{6} = \frac{6}{6} - \frac{4}{6} = \frac{1}{3}$

62. (a)  $\frac{10}{9,366,819} \approx 0.0000010676$

(b)

Value	Probability
-10	$\frac{9,366,809}{9,366,819}$
+3,999,990	$\frac{10}{9,366,819}$

(c)  $3,999,990 \cdot \frac{10}{9,366,819} + (-10)$

$$\cdot \left(1 - \frac{10}{9,366,819}\right) \approx -5.73$$

- (d) In the long run, Gladys is losing \$5.73 every time she buys the ten tickets. Given the low probability of a positive payoff, she stands to lose a lot of money if she does this often.

3.  $5 \cdot 4^2 = 80$

4.  $\frac{5(1 - 4^3)}{(1 - 4)} = \frac{-315}{-3} = 105$

5.  $a_{10} = \frac{10}{11}$

6.  $a_{10} = 5 + (10 - 1)3 = 32$

7.  $a_{10} = 5 \cdot 2^9 = 2560$

8.  $a_{10} = \left(\frac{4}{3}\right)\left(\frac{1}{2}\right)^9 = \left(\frac{4}{3}\right)\left(\frac{1}{512}\right) = \frac{1}{384}$

9.  $a_{10} = 32 - 17 = 15$

10.  $a_{10} = \frac{10^2}{2^{10}} = \frac{100}{1024} = \frac{25}{256}$

### Section 9.4 Exercises

For #1–4, substitute  $n = 1, n = 2, \dots, n = 6$ , and  $n = 100$ .

1.  $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \frac{7}{6}, \frac{101}{100}$

2.  $\frac{4}{3}, 1, \frac{4}{5}, \frac{2}{3}, \frac{4}{7}, \frac{1}{2}, \frac{2}{51}$

3. 0, 6, 24, 60, 120, 210; 999,900

4. -4, -6, -6, -4, 0, 6; 9500

For #5–10, use previously computed values of the sequence to find the next term in the sequence.

5. 8, 4, 0, -4; -20

6. -3, 7, 17, 27; 67

7. 2, 6, 18, 54; 4374

8. 0.75, -1.5, 3, -6; -96

9. 2, -1, 1, 0; 3

10. -2, 3, 1, 4; 23

11.  $\lim_{n \rightarrow \infty} n^2 = \infty$ , so the sequence diverges.

12.  $\lim_{n \rightarrow \infty} \frac{1}{2^n} = 0$ , so the sequence converges to 0.

13.  $\frac{1}{1}, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \dots, \frac{1}{n^2}, \dots$

$\lim_{n \rightarrow \infty} \frac{1}{n^2} = 0$ , so the sequence converges to 0.

14.  $\lim_{n \rightarrow \infty} (3n - 1) = \infty$ , so the sequence diverges.

15. Since the degree of the numerator is the same as the degree of the denominator, the limit is the ratio of the leading coefficients. Thus  $\lim_{n \rightarrow \infty} \frac{3n - 1}{2 - 3n} = -1$ . The sequence converges to -1.

16. Since the degree of the numerator is the same as the degree of the denominator, the limit is the ratio of the leading coefficients. Thus  $\lim_{n \rightarrow \infty} \frac{2n - 1}{n + 1} = 2$ . The sequence converges to 2.

17.  $\lim_{n \rightarrow \infty} (0.5)^n = \lim_{n \rightarrow \infty} \left(\frac{1}{2}\right)^n = 0$ , so the sequence converges to 0.

18.  $\lim_{n \rightarrow \infty} (1.5)^n = \lim_{n \rightarrow \infty} \left(\frac{3}{2}\right)^n = \infty$ , so the sequence diverges.

## Section 9.4 Sequences

### Quick Review 9.4

1.  $3 + (5 - 1)4 = 3 + 16 = 19$

2.  $\frac{5}{2}[(6 + (5 - 1)4)] = \frac{5}{2}(22) = 55$



19.  $a_1 = 1$  and  $a_{n+1} = a_n + 3$  for  $n \geq 1$  yields 1, 4, 7,  $\dots$ ,  $(3n - 2)$ ,  $\dots$   
 $\lim_{n \rightarrow \infty} (3n - 2) = \infty$ , so the sequence diverges.

20.  $u_1 = 1$  and  $u_{n+1} = \frac{u_n}{3}$  for  $n \geq 1$  yields  $1, \frac{1}{3}, \frac{1}{9}, \dots, \frac{1}{3^{n-1}}, \dots$   
 $\lim_{n \rightarrow \infty} \frac{1}{3^{n-1}} = 0$ , so the sequence converges to 0.

For #21–24, subtract the first term from the second to find the common difference  $d$ . Use the formula  $a_n = a_1 + (n - 1)d$  with  $n = 10$  to find the tenth term. The recursive rule for the  $n$ th term is  $a_n = a_{n-1} + d$ , and the explicit rule is the one given above.

21. (a)  $d = 4$

(b)  $a_{10} = 6 + 9(4) = 42$

(c) Recursive rule:  $a_1 = 6; a_n = a_{n-1} + 4$  for  $n \geq 2$

(d) Explicit rule:  $a_n = 6 + 4(n - 1)$

22. (a)  $d = 5$

(b)  $a_{10} = -4 + 9(5) = 41$

(c) Recursive rule:  $a_1 = -4; a_n = a_{n-1} + 5$  for  $n \geq 2$

(d) Explicit rule:  $a_n = -4 + 5(n - 1)$

23. (a)  $d = 3$

(b)  $a_{10} = -5 + 9(3) = 22$

(c) Recursive rule:  $a_1 = -5; a_n = a_{n-1} + 3$  for  $n \geq 2$

(d) Explicit rule:  $a_n = -5 + 3(n - 1)$

24. (a)  $d = 11$

(b)  $a_{10} = -7 + 9(11) = 92$

(c) Recursive rule:  $a_1 = -7; a_n = a_{n-1} + 11$  for  $n \geq 2$

(d) Explicit rule:  $a_n = -7 + 11(n - 1)$

For #25–28, divide the second term by the first to find the common ratio  $r$ . Use the formula  $a_n = a_1 \cdot r^{n-1}$  with  $n = 8$  to find the eighth term. The recursive rule for the  $n$ th term is  $a_n = a_{n-1} \cdot r$ , and the explicit rule is the one given above.

25. (a)  $r = 3$

(b)  $a_8 = 2 \cdot 3^7 = 4374$

(c) Recursive rule:  $a_1 = 2; a_n = 3a_{n-1}$  for  $n \geq 2$

(d) Explicit rule:  $a_n = 2 \cdot 3^{n-1}$

26. (a)  $r = 2$

(b)  $a_8 = 3 \cdot 2^7 = 384$

(c) Recursive rule:  $a_1 = 3; a_n = 2a_{n-1}$  for  $n \geq 2$

(d) Explicit rule:  $a_n = 3 \cdot 2^{n-1}$

27. (a)  $r = -2$

(b)  $a_8 = (-2)^7 = -128$

(c) Recursive rule:  $a_1 = 1; a_n = -2a_{n-1}$  for  $n \geq 2$

(d) Explicit rule:  $a_n = (-2)^{n-1}$

28. (a)  $r = -1$

(b)  $a_8 = -2 \cdot (-1)^7 = 2$

(c) Recursive rule:  $a_1 = -2; a_n = -1a_{n-1} = -a_{n-1}$  for  $n \geq 2$

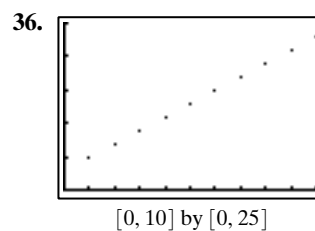
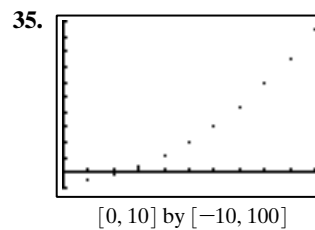
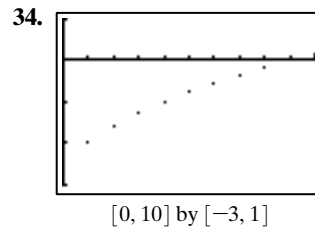
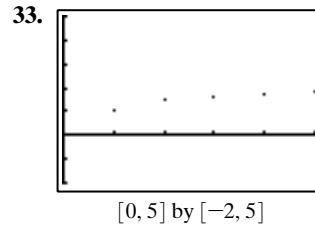
(d) Explicit rule:  $a_n = -2 \cdot (-1)^{n-1} = 2 \cdot (-1)^n$

29.  $a_4 = -8 = a_1 + 3d$  and  $a_7 = 4 = a_1 + 6d$ , so  $a_7 - a_4 = 12 = 3d$ . Therefore  $d = 4$ , so  $a_1 = -8 - 3d = -20$  and  $a_n = a_{n-1} + 4$  for  $n \geq 2$ .

30.  $a_5 = -5 = a_1 + 4d$  and  $a_9 = -17 = a_1 + 8d$ , so  $a_9 - a_5 = -12 = 4d$ . Therefore  $d = -3$ , so  $a_1 = -5 - 4d = 7$  and  $a_n = a_{n-1} - 3$  for  $n \geq 2$ .

31.  $a_2 = 3 = a_1 \cdot r$  and  $a_8 = 192 = a_1 \cdot r^7$ , so  $a_8/a_2 = 64 = r^6$ . Therefore  $r = \pm 2$ , so  $a_1 = 3/(\pm 2) = \pm \frac{3}{2}$  and  $a_n = \pm \frac{3}{2} \cdot (-2)^{n-1} = 3 \cdot (-2)^{n-2}$  or  $a_n = \frac{3}{2} \cdot 2^{n-1} = 3 \cdot 2^{n-2}$ .

32.  $a_3 = -75 = a_1 \cdot r^2$  and  $a_6 = -9375 = a_1 \cdot r^5$ , so  $a_6/a_3 = 125 = r^3$ . Therefore  $r = 5$ , so  $a_1 = -75/5^2 = -3$  and  $a_n = -3 \cdot 5^{n-1}$ .



37. The height (in cm) will be an arithmetic sequence with common difference  $d = 2.3$  cm, so the height in week  $n$  is  $700 + 2.3(n - 1)$ : 700, 702.3, 704.6, 706.9,  $\dots$ , 815, 817.3.

38. The first column is an arithmetic sequence with common difference  $d = 14$ . The second column is a geometric sequence with common ratio  $r = \frac{1}{2}$ .

Time (billions of years)	Mass (g)
0	16
14	8
28	4
42	2
56	1

39. The numbers of seats in each row form a finite arithmetic sequence with  $a_1 = 7, d = 2$ , and  $n = 25$ . The total number of seats is

$$\frac{25}{2} [2(7) + (25 - 1)(2)] = 775.$$

40. The numbers of tiles in each row form a finite arithmetic sequence with  $a_1 = 15, a_n = 30$ , and  $n = 16$ . The total number of tiles is

$$16 \left( \frac{15 + 30}{2} \right) = 360.$$

41. The ten-digit numbers will vary; thus the sequences will vary. The end result will, however, be the same. Each limit will be 9. One example is:

Five random digits: 1, 4, 6, 8, 9

Five random digits: 2, 3, 4, 5, 6

List: 1, 2, 3, 4, 4, 5, 6, 6, 8, 9

Ten-digit number: 2, 416, 345, 689

Ten-digit number: 9, 643, 128, 564

$a_1 =$  positive difference of the ten-digit numbers  
 $= 7, 226, 782, 875$

$a_{n+1} =$  sum of the digits of  $a_n$ , so

$a_2 =$  sum of the digits of  $a_1 = 54$

$a_3 =$  sum of the digits of  $a_2 = 9$

All successive sums of digits will be 9, so the sequence converges and the limit is 9.

42. Everyone should end up at the word “all”.  
 43. True. Since two successive terms are negative, the common ratio  $r$  must be positive, and so the sign of the first term determines the sign of every number in the sequence.  
 44. False. For example, consider the sequence 5, 1, -3, -7, . . .

45.  $a_1 = 2$  and  $a_2 = 8$  implies  $d = 8 - 2 = 6$   
 $c = a_1 - d$  so  $c = 2 - 6 = -4$   
 $a_4 = 6 \cdot 4 + (-4) = 20$   
 The answer is A.

46.  $\lim_{n \rightarrow \infty} \sqrt{n} = \lim_{n \rightarrow \infty} n^{1/2} = \infty$ , so the sequence diverges.  
 The answer is B.

47.  $r = \frac{a_2}{a_1} = \frac{6}{2} = 3$   
 $a_6 = a_1 r^5 = 2 \cdot 3^5 = 486$  and  $a_2 = 6$ , so  
 $\frac{a_6}{a_2} = \frac{486}{6} = 81$ .  
 The answer is E.

48. The geometric sequence will be defined by  $a_{n+1} = a_n \div 3$  for  $n \geq 1$  and  $a_1 \neq 0$ .

$$a_2 = \frac{a_1}{3}$$

$$a_3 = \frac{a_2}{3} = \frac{a_1/3}{3} = \frac{a_1}{9}$$

$$a_4 = \frac{a_3}{3} = \frac{a_1/9}{3} = \frac{a_1}{27}$$

$a_n = \frac{a_1}{3^{n-1}}$ , which represents a geometric sequence.

The answer is C.

49. (a)  $a_1 = 1$  because there is initially one male-female pair (this is the number of pairs after 0 months).  $a_2 = 1$  because after one month, the original pair has only just become fertile.  $a_3 = 2$  because after two months, the original pair produces a new male-female pair.

- (b) Notice that after  $n - 2$  months, there are  $a_{n-1}$  pairs, of which  $a_{n-2}$  (the number of pairs present one month earlier) are fertile. Therefore, after  $n - 1$  months, the number of pairs will be  $a_n = a_{n-1} + a_{n-2}$ : to last month's total, we add the number of new pairs born. Thus  $a_4 = 3, a_5 = 5, a_6 = 8, a_7 = 13, a_8 = 21, a_9 = 34, a_{10} = 55, a_{11} = 89, a_{12} = 144, a_{13} = 233$ .

- (c) Since  $a_1$  is the initial number of pairs, and  $a_2$  is the number of pairs after one month, we see that  $a_{13}$  is the number of pairs after 12 months.

50. Use a calculator:  $a_1 = 1, a_2 = 1, a_3 = 2, a_4 = 3, a_5 = 5, a_6 = 8, a_7 = 13$ . These are the first seven terms of the Fibonacci sequence.

51. (a) For a polygon with  $n$  sides, let  $A$  be the vertex in quadrant I at the top of the vertical segment, and let  $B$  be the point on the  $x$ -axis directly below  $A$ . Together with  $(0, 0)$ , these two points form a right triangle; the acute angle at the origin has measure

$$\theta = \frac{2\pi}{2n} = \frac{\pi}{n},$$

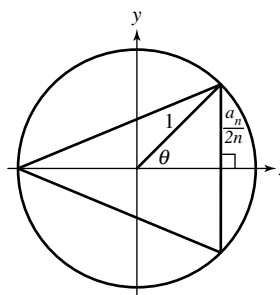
since there are  $2n$  such triangles making up the polygon. The length of the side opposite this angle is

$$\sin \theta = \sin \frac{\pi}{n},$$

and there are  $2n$  such sides making up the perimeter of the polygon, so  $\sin \frac{\pi}{n} = \frac{a_n}{2n}$ , or

$$a_n = 2n \sin(\pi/n).$$

- (b)  $a_{10} \approx 6.1803, a_{100} \approx 6.2822, a_{1000} \approx 6.2832, a_{10,000} \approx 6.2832 \approx 2\pi$ . It appears that  $a_n \rightarrow 2\pi$  as  $n \rightarrow \infty$ , which makes sense since the perimeter of the polygon should approach the circumference of the circle.



52.  $P_1 = 525,000; P_n = 1.0175P_{n-1}, n \geq 2$

53. The difference of successive terms in  $\{\log(a_n)\}$  will be of the form  $\log(a_{n+1}) - \log(a_n) = \log\left(\frac{a_{n+1}}{a_n}\right)$ . Since  $\{a_n\}$  is geometric,  $\frac{a_{n+1}}{a_n}$  is constant. This makes  $\log\left(\frac{a_{n+1}}{a_n}\right)$  constant, so  $\{\log(a_n)\}$  is a sequence with a constant difference (arithmetic).

54. The ratios of successive terms in  $\{10^{b_n}\}$  will be of the form  $10^{b_{n+1}} \div 10^{b_n} = 10^{b_{n+1} - b_n}$ . Since  $\{b_n\}$  is arithmetic,  $b_{n+1} - b_n$  is constant. This makes  $10^{b_{n+1} - b_n}$  constant, so  $\{10^{b_n}\}$  is a sequence with a common ratio (geometric).

55.  $a_1 = [1 \ 1], a_2 = [1 \ 2], a_3 = [2 \ 3], a_4 = [3 \ 5],$   
 $a_5 = [5 \ 8], a_6 = [8 \ 13], a_7 = [13 \ 21]$ . The entries in  
 the terms of this sequence are successive pairs of terms  
 from the Fibonacci sequence.

56.  $a_1 = [1 \ a] \quad r = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$   
 $a_2 = a_1 \cdot r = [1 \ a + d]$   
 $a_3 = a_2 \cdot r = [1 \ d + a + d] = [1 \ a + 2d]$   
 $a_4 = a_3 \cdot r = [1 \ d + a + 2d] = [1 \ a + 3d]$   
 $a_n = [1 \ a + (n - 1)d]$ .  
 So, the second entries of this geometric sequence of  
 matrices form an arithmetic sequence with the first term  $a$   
 and common difference  $d$ .

## Section 9.5 Series

### Exploration 1

- $3 + 6 + 9 + 12 + 15 = 45$
- $5^2 + 6^2 + 7^2 + 8^2 = 25 + 36 + 49 + 64 = 174$
- $\cos(0) + \cos(\pi) + \dots + \cos(11\pi) + \cos(12\pi)$   
 $= 1 - 1 + 1 + \dots - 1 + 1 = 1$
- $\sin(0) + \sin(\pi) + \dots + \sin(k\pi)$   
 $+ \dots = 0 + 0 + \dots + 0 + \dots = 0$
- $\frac{3}{10} + \frac{3}{100} + \frac{3}{1,000} + \dots + \frac{3}{1,000,000} + \dots = \frac{1}{3}$

### Exploration 2

- $1 + 2 + 3 + \dots + 99 + 100$
- $100 + 99 + 98 + \dots + 2 + 1$
- $101 + 101 + 101 + \dots + 101 + 101$
- $100(101) = 10,100$
- The sum in 4 involves two copies of the same progression,  
 so it doubles the sum of the progression. The answer that  
 Gauss gave was 5050.

### Quick Review 9.5

- $a_1 = 4; d = 2$  so  $a_{10} = a_1 + (n - 1)d$   
 $a_{10} = 4 + (10 - 1)2 = 4 + 18 = 22$   
 $a_{10} = 22$
- $a_1 = 3; a_2 = 1$  so  $d = 1 - 3 = -2$   
 $a_{10} = a_1 + (n - 1)d$   
 $a_{10} = 3 + (10 - 1)(-2) = 3 - 18 = -15$   
 $a_{10} = -15$
- $a_3 = 6$  and  $a_8 = 21$   
 $a_3 = a_1 + 2d$  and  $a_8 = a_1 + 7d$   
 $(a_1 + 7d) - (a_1 + 2d) = 21 - 6$  so  $5d = 15 \Rightarrow d = 3$ .  
 $6 = a_1 + 2(3)$  so  $a_1 = 0$   
 $a_{10} = 0 + 9(3) = 27$   
 $a_{10} = 27$
- $a_5 = 3$ , and  $a_{n+1} = a_n + 5$  for  $n \geq 1 \Rightarrow$   
 $a_6 = 3 + 5 = 8$   
 $a_5 = 3$  and  $a_6 = 8 \Rightarrow d = 5$   
 $a_5 = a_1 + 4d$  so  $3 = a_1 + 4(5) \Rightarrow a_1 = -17$   
 $a_{10} = -17 + 9(5) = 28$   
 $a_{10} = 28$

5.  $a_1 = 1$  and  $a_2 = 2$  yields  $r = \frac{2}{1} = 2$

$$a_{10} = 1 \cdot 2^9 = 512$$

$$a_{10} = 512$$

6.  $a_4 = 1$  and  $a_4 = a_1 \cdot r^3$ ;  $a_6 = 2$  and  $a_6 = a_1 \cdot r^5$

$$\frac{a_1 \cdot r^5}{a_1 \cdot r^3} = \frac{2}{1}$$

$$r^2 = 2 \Rightarrow r = \sqrt{2}$$

$$1 = a_1(\sqrt{2})^3; \quad a_1 = \frac{1}{(\sqrt{2})^3} = \frac{1}{2\sqrt{2}}$$

$$a_{10} = \frac{1}{2\sqrt{2}}(\sqrt{2})^9 = \frac{16\sqrt{2}}{2\sqrt{2}} = 8$$

$$a_{10} = 8$$

7.  $a_7 = 5$  and  $r = -2 \Rightarrow 5 = a_1(-2)^6$

$$a_1 = \frac{5}{64}; \quad a_{10} = \frac{5}{24}(-2)^9 = \frac{-2560}{24} = -40$$

$$a_{10} = -40$$

8.  $a_8 = 10$  and  $a_8 = a_1 \cdot r^7$ ;  $a_{12} = 40 \Rightarrow a_{12} = a_1 \cdot r^{11}$

$$\frac{a_1 \cdot r^{11}}{a_1 \cdot r^7} = \frac{40}{10}$$

$$r^4 = 4; \text{ so } r = (4)^{1/4}$$

$$10 = a_1((4)^{1/4})^7; \quad a_1 = \frac{10}{4^{7/4}}$$

$$a_{10} = \frac{10}{4^{7/4}}(4^{1/4})^9 = \frac{10(4^{9/4})}{4^{7/4}} = 10(4^{2/4}) = 10 \cdot 2 = 20$$

$$a_{10} = 20$$

9.  $\sum_{n=1}^5 n^2 = 1 + 4 + 9 + 16 + 25 = 55$

10.  $\sum_{n=1}^5 (2n - 1) = 1 + 3 + 5 + 7 + 9 = 25$

### Section 9.5 Exercises

1.  $\sum_{k=1}^{11} (6k - 13)$

2.  $\sum_{k=1}^{10} (3k - 1)$

3.  $\sum_{k=1}^{n+1} k^2$

4.  $\sum_{k=1}^{n+1} k^3$

5.  $\sum_{k=0}^{\infty} 6(-2)^k$

6.  $\sum_{k=0}^{\infty} 5(-3)^k$

For #7–12, use one of the formulas  $S_n = n\left(\frac{a_1 + a_n}{2}\right)$  or

$S_n = \frac{n}{2}[2a_1 + (n - 1)d]$ . In most cases, the first of these is  
 easier (since the last term  $a_n$  is given); note that

$$n = \frac{a_n - a_1}{d} + 1.$$

7.  $6 \cdot \left(\frac{-7 + 13}{2}\right) = 6 \cdot 3 = 18$

8.  $6 \cdot \left(\frac{-8 + 27}{2}\right) = 3 \cdot 19 = 57$

9.  $80 \cdot \left(\frac{1 + 80}{2}\right) = 40 \cdot 81 = 3240$

10.  $35 \cdot \left(\frac{2 + 70}{2}\right) = 35 \cdot 36 = 1260$

11.  $13 \cdot \left(\frac{117 + 33}{2}\right) = 13 \cdot 75 = 975$

12.  $\left(\frac{111 + 27}{2}\right) = 29 \cdot 69 = 2001$

For #13–16, use the formula  $S_n = \frac{a_1(1 - r^n)}{1 - r}$ . Note that

$$n = 1 + \log_{|r|} \left| \frac{a_n}{a_1} \right| = 1 + \frac{\ln |a_n/a_1|}{\ln |r|}.$$

13.  $\frac{3(1 - 2^{13})}{1 - 2} = 24,573$

14.  $\frac{5(1 - 3^{10})}{1 - 3} = 147,620$

15.  $\frac{42[1 - (1/6)^9]}{1 - (1/6)} = 50.4(1 - 6^{-9}) \approx 50.4$

16.  $\frac{42[1 - (-1/6)^{10}]}{1 - (-1/6)} = 36(1 - 6^{-10}) = 36 - 6^{-8} \approx 36$

For #17–22, use one of the formulas  $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$

or  $S_n = \frac{a_1(1 - r^n)}{1 - r}$ .

17. Arithmetic with  $d = 3$ :  $\frac{10}{2}[2 \cdot 3 + (10 - 1)(3)]$   
 $= 5 \cdot 31 = 155$

18. Arithmetic with  $d = -6$ :  $\frac{9}{2}[2 \cdot 14 + (9 - 1)(-6)]$   
 $= 9 \cdot (-10) = -90$

19. Geometric with  $r = -\frac{1}{2}$ :  $\frac{4[1 - (-1/2)^{12}]}{1 - (-1/2)}$   
 $= \frac{8}{3} \cdot (1 - 2^{-12}) \approx 2.666$

20. Geometric with  $r = -\frac{1}{2}$ :  $\frac{6[1 - (-1/2)^{11}]}{1 - (-1/2)}$   
 $= 4 \cdot (1 + 2^{-11}) \approx 4.002$

21. Geometric with  $r = -11$ :  $\frac{-1[1 - (-11)^9]}{1 - (-11)}$   
 $= -\frac{1}{12} \cdot (1 + 11^9) = -196,495,641$

22. Geometric with  $r = -12$ :  $\frac{-2[1 - (-12)^8]}{1 - (-12)}$   
 $= -\frac{2}{13} \cdot (1 - 12^8) = 66,151,030$

23. (a) The first six partial sums are  $\{0.3, 0.33, 0.333, 0.3333, 0.33333, 0.333333\}$ . The numbers appear to be approaching a limit of  $0.\bar{3} = 1/3$ . The series is convergent.

(b) The first six partial sums are  $\{1, -1, 2, -2, 3, -3\}$ . The numbers approach no limit. The series is divergent.

24. (a) The first six partial sums are  $\{-2, 0, -2, 0, -2, 0\}$ . The numbers approach no limit. The series is divergent.

(b) The first six partial sums are  $\{1, 0.3, 0.23, 0.223, 0.2223, 0.22223\}$ . The numbers appear to be approaching a limit of  $0.\bar{2} = 2/9$ . The series is convergent.

25.  $r = \frac{1}{2}$ , so it converges to  $S = \frac{6}{1 - (1/2)} = 12$ .

26.  $r = \frac{1}{3}$ , so it converges to  $S = \frac{4}{1 - (1/3)} = 6$ .

27.  $r = 2$ , so it diverges.

28.  $r = 3$ , so it diverges.

29.  $r = \frac{1}{4}$ , so it converges to  $S = \frac{3/4}{1 - (1/4)} = 1$ .

30.  $r = \frac{2}{3}$ , so it converges to  $S = \frac{10/3}{1 - (2/3)} = 10$ .

31.  $7 + \frac{14}{99} = \frac{693}{99} + \frac{14}{99} = \frac{707}{99}$

32.  $5 + \frac{93}{99} = 5 + \frac{31}{33} = \frac{196}{33}$

33.  $-17 - \frac{268}{999} = -\frac{17,251}{999}$

34.  $-12 - \frac{876}{999} = -12 - \frac{292}{333} = -\frac{4288}{333}$

35. (a) The ratio of any two successive account balances is  $r = 1.1$ . That is,

$$\frac{\$22,000}{\$20,000} = \frac{\$24,200}{\$22,000} = \frac{\$26,620}{\$24,200} = \frac{\$29,282}{\$26,620} = 1.1.$$

(b) Each year, the balance is 1.1 times as large as the year before. So,  $n$  years after the balance is \$20,000, it will be  $\$20,000(1.1)^n$ .

(c) The sum of the eleven terms of the geometric sequence is  $\frac{\$20,000(1 - 1.1^{11})}{1 - 1.1} = \$370,623.34$

36. (a) The difference of any two successive account balances is  $d = \$2016$ . That is  $\$20,016 - \$18,000 = \$2,016$   
 $= \$22,032 - \$20,016 = \$2,016$   
 $= \$24,048 - \$22,032 = \$2,016$   
 $= \$26,064 - \$24,048 = \$2,016$ .

(b) Each year, the balance is \$2016 more than the year before. So,  $n$  years after the balance is \$18,000, it will be  $\$18,000 + \$2016n$ .

(c) The sum of the eleven terms of the arithmetic sequence is  $\frac{11}{2}[2(\$18,000) + (10)(\$2016)] = \$308,880$

37. (a) The first term,  $120(1 + 0.07/12)^0$ , simplifies to 120. The common ratio of terms,  $r$ , equals  $1 + 0.07/12$ .

(b) The sum of the 120 terms is  $\frac{120[1 - (1 + 0.07/12)^{120}]}{1 - (1 + 0.07/12)} = \$20,770.18$

38. (a) The first term,  $100(1 + 0.08/12)^0$ , simplifies to 100. The common ratio of terms,  $r$ , equals  $1 + 0.08/12$ .

(b) The sum of the 120 terms is  $\frac{100[1 - (1 + 0.08/12)^{120}]}{1 - (1 + 0.08/12)} = \$18,294.60$ .

39.  $2 + 2 \cdot [2(0.9) + (2(0.9)^2 + 2(0.9)^3 + \dots + 2(0.9)^9]$   
 $= 2 + \sum_{k=1}^9 4(0.9)^k = 2 + \frac{3.6(1 - 0.9^9)}{1 - 0.9} =$

$38 - 36(0.9^9) \approx 24.05$  m: The ball travels down 2 m. Between the first and second bounces, it travels  $2(0.9)$  m up, then the same distance back down. Between the second and third bounces, it travels  $2(0.9)^2$  m up, then the same distance back down, etc.

40.  $f(x)$  does have a horizontal asymptote to the left (since as  $x \rightarrow -\infty, f(x) \rightarrow -40$ ), but as  $x \rightarrow \infty, f(x) \rightarrow \infty$ . The given series is geometric with  $r = 1.05$ ; the sum of the first  $n$  terms in the series is  $f(n)$ . Observing that  $f(x) \rightarrow \infty$  as  $x \rightarrow \infty$  is confirmation that the geometric series diverges.

41. False. The series might diverge. For example, examine the series  $1 + 2 + 3 + 4 + 5$  where all of the terms are positive. Consider the limit of the sequence of partial sums. The first five partial sums are  $\{1, 3, 6, 10, 15\}$ . These numbers increase without bound and do not approach a limit. Therefore, the series diverges and has no sum.

42. False. Justifications will vary. One example is to examine  $\sum_{n=1}^{\infty} n$  and  $\sum_{n=1}^{\infty} (-n)$ . Both of these diverge, but  $\sum_{n=1}^{\infty} (n + (-n)) = \sum_{n=1}^{\infty} 0 = 0$ . So the sum of the two divergent series converges.

47. (a) Heartland: 19,237,759 people.  
 Southeast: 42,614,977 people.

(b) Heartland: 517,825 mi<sup>2</sup>.  
 Southeast: 348,999 mi<sup>2</sup>.

(c) Heartland:  $\frac{19,237,759}{517,825} \approx 37.15$  people/mi<sup>2</sup>.  
 Southeast:  $\frac{42,614,977}{348,999} \approx 122.11$  people/mi<sup>2</sup>.

(d) The table is shown on the right; the answer differs because the overall population density

$\frac{\sum \text{population}}{\sum \text{area}}$  is generally not the same as the average of the population densities,  $\frac{1}{n} \sum \left( \frac{\text{population}}{\text{area}} \right)$ . The larger states

within each group have a greater effect on the overall mean density. In a similar way, if a student's grades are based on a 100-point test and four 10-point quizzes, her overall average grade depends more on the test grade than on the four quiz grades.

43.  $3^{-1} + 3^{-2} + 3^{-3} + \dots + 3^{-n} + \dots =$   
 $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \dots + \frac{1}{3^n} + \dots$

The first five partial sums are  $\left\{ \frac{1}{3}, \frac{4}{9}, \frac{13}{27}, \frac{40}{81}, \frac{121}{243} \right\}$ . These appear to be approaching a limit of  $1/2$ , which would suggest that the series converges to  $1/2$ . The answer is A.

44. If  $\sum_{n=1}^{\infty} x^n = 4$ , then  $x = 0.8$ .

$$\sum_{n=1}^{\infty} 0.8^n = 0.8 + 0.64 + 0.512 + 0.4096 + 0.32768 + 0.262144 + \dots$$

The first six partial sums are  $\{0.8, 1.44, 1.952, 2.3616, 2.68928, 2.951424\}$ . It appears from this sequence of partial sums that the series is converging. If the sequence of partial sums were extended to the 40th partial sum, you would see that the series converges to 4. The answer is D.

45. The common ratio is  $0.75/3 = 0.25$ , so the sum of the infinite series is  $3/(1 - 0.25) = 4$ . The answer is C.

46. The sum is an infinite geometric series with  $|r| = 5/3 > 1$ . The answer is E.

Heartland:		Southeast:	
Iowa	$\approx 52.00$	Alabama	$\approx 86.01$
Kansas	$\approx 32.68$	Arkansas	$\approx 50.26$
Minnesota	$\approx 58.29$	Florida	$\approx 272.53$
Missouri	$\approx 80.28$	Georgia	$\approx 138.97$
Nebraska	$\approx 22.12$	Louisiana	$\approx 93.59$
N. Dakota	$\approx 9.08$	Mississippi	$\approx 59.65$
S. Dakota	$\approx 9.79$	S. Carolina	$\approx 128.95$
Total	$\approx 264.24$	Total	$\approx 829.96$
Average	$\approx 37.75$	Average	$\approx 118.57$

48.  $\sum_{k=1}^8 (k^2 - 2)$

49. The table suggests that  $S_n = \sum_{k=1}^n F_k = F_{n+2} - 1$ .

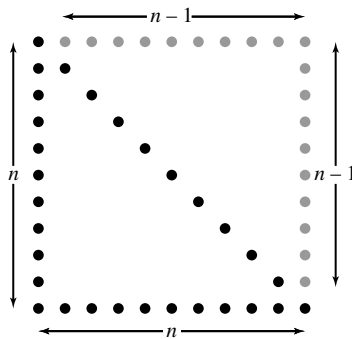
$n$	$F_n$	$S_n$	$F_{n+2} - 1$
1	1	1	1
2	1	2	2
3	2	4	4
4	3	7	7
5	5	12	12
6	8	20	20
7	13	33	33
8	21	54	54
9	34	88	88

50. The  $n$ th triangular number is simply the sum of the first  $n$  consecutive positive integers:

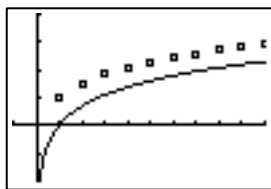
$$1 + 2 + 3 + \dots + n = n \left( \frac{1+n}{2} \right) = \frac{n(n+1)}{2}$$

51. Algebraically:  $T_{n-1} + T_n = \frac{(n-1)n}{2} + \frac{n(n+1)}{2}$   
 $= \frac{n^2 - n + n^2 + n}{2} = n^2$ .

Geometrically: the array of black dots in the figure represents  $T_n = 1 + 2 + 3 + \dots + n$  (that is, there are  $T_n$  dots in the array). The array of gray dots represents  $T_{n-1} = 1 + 2 + 3 + \dots + (n-1)$ . The two triangular arrays fit together to form an  $n \times n$  square array, which has  $n^2$  dots.



52. If  $\sum_{k=1}^n \frac{1}{k} \geq \ln n$  for all  $n$ , then the sum diverges since as  $n \rightarrow \infty, \ln n \rightarrow \infty$ .



$[-1, 10]$  by  $[-2, 4]$

## Section 9.6 Mathematical Induction

### Exploration 1

1. Start with the rightmost peg if  $n$  is odd and the middle peg if  $n$  is even. From that point on, the first move for moving any smaller stack to a destination peg should be directly to the destination peg if the smaller stack's size  $n$  is odd and to the other available peg if  $n$  is even. The fact that the winning strategy follows such predictable rules is what makes it so interesting to students of computer programming.

### Exploration 2

1. 43, 47, 53, 61, 71, 83, 97, 113, 131, 151. Yes.
2. 173, 197, 223, 251, 281, 313, 347, 383, 421, 461. Yes.
3. 503, 547, 593, 641, 691, 743, 797, 853, 911, 971. Yes. Inductive thinking might lead to the conjecture that  $n^2 + n + 41$  is prime for all  $n$ , but we have no proof as yet!
4. The next 9 numbers are all prime, but  $40^2 + 40 + 41$  is not. Quite obviously, neither is the number  $41^2 + 41 + 41$ .

### Quick Review 9.6

1.  $n^2 + 5n$
2.  $n^2 - n - 6$
3.  $k^3 + 3k^2 + 2k$
4.  $(n+3)(n-1)$
5.  $(k+1)^3$
6.  $(n-1)^3$
7.  $f(1) = 1 + 4 = 5, f(t) = t + 4,$   
 $f(t+1) = t + 1 + 4 = t + 5$
8.  $f(1) = \frac{1}{1+1} = \frac{1}{2}, f(k) = \frac{k}{k+1},$   
 $f(k+1) = \frac{k+1}{k+1+1} = \frac{k+1}{k+2}$
9.  $P(1) = \frac{2 \cdot 1}{3 \cdot 1 + 1} = \frac{1}{2},$   
 $P(k) = \frac{2k}{3k+1} = \frac{2(k+1)}{3(k+1)+1} = \frac{2k+2}{3k+4}$
10.  $P(1) = 2(1)^2 - 1 - 3 = -2, P(k) = 2k^2 - k - 3,$   
 $P(k+1) = 2(k+1)^2 - (k+1) - 3 = 2k^2 + 3k - 2$

### Section 9.6 Exercises

1.  $P_n: 2 + 4 + 6 + \dots + 2n = n^2 + n.$   
 $P_1$  is true:  $2(1) = 1^2 + 1.$   
 Now assume  $P_k$  is true:  $2 + 4 + 6 + \dots + 2k = k^2 + k.$  Add  $2(k+1)$  to both sides:  
 $2 + 4 + 6 + \dots + 2k + 2(k+1) = k^2 + k + 2(k+1)$   
 $= k^2 + k + 2k + 2 = k^2 + 3k + 2$   
 $= k^2 + 2k + 1 + k + 1 = (k+1)^2 + (k+1),$  so  $P_{k+1}$  is true. Therefore,  $P_n$  is true for all  $n \geq 1.$

2.  $P_n$ :  $8 + 10 + 12 + \dots + (2n + 6) = n^2 + 7n$ .

$P_1$  is true:  $2(1) + 6 = 1^2 + 7 \cdot 1$ .

Now assume  $P_k$  is true:

$$8 + 10 + 12 + \dots + (2k + 6) = k^2 + 7k.$$

Add  $2(k + 1) + 6 = 2k + 8$  to both sides:

$$\begin{aligned} 8 + 10 + 12 + \dots + (2k + 6) + [2(k + 1) + 6] \\ = k^2 + 7k + 2k + 8 = (k^2 + 2k + 1) + 7k + 7 \\ = (k + 1)^2 + 7(k + 1), \text{ so } P_{k+1} \text{ is true.} \end{aligned}$$

Therefore,  $P_n$  is true for all  $n \geq 1$ .

3.  $P_n$ :  $6 + 10 + 14 + \dots + (4n + 2) = n(2n + 4)$ .

$P_1$  is true:  $4(1) + 2 = 1(2 + 4)$ .

Now assume  $P_k$  is true:

$$6 + 10 + 14 + \dots + (4k + 2) = k(2k + 4).$$

Add  $4(k + 1) + 2 = 4k + 6$  to both sides:

$$\begin{aligned} 6 + 10 + 14 + \dots + (4k + 2) + [4(k + 1) + 2] \\ = k(2k + 4) + 4k + 6 = 2k^2 + 8k + 6 \\ = (k + 1)(2k + 6) = (k + 1)[2(k + 1) + 4], \text{ so } P_{k+1} \\ \text{is true. Therefore, } P_n \text{ is true for all } n \geq 1. \end{aligned}$$

4.  $P_n$ :  $14 + 18 + 22 + \dots + (4n + 10) = 2n(n + 6)$ .

$P_1$  is true:  $4(1) + 10 = 2 \cdot 1(1 + 6)$ . Now assume  $P_k$  is true:

$$14 + 18 + 22 + \dots + (4k + 10) = 2k(k + 6).$$

Add  $4(k + 1) + 10 = 4k + 14$  to both sides:

$$\begin{aligned} 14 + 18 + 22 + \dots + (4k + 10) + [4(k + 1) + 10] \\ = 2k(k + 6) + (4k + 14) \\ = 2(k^2 + 8k + 7) = 2(k + 1)(k + 7) \\ = 2(k + 1)(k + 1 + 6), \text{ so } P_{k+1} \text{ is true. Therefore, } P_n \\ \text{is true for all } n \geq 1. \end{aligned}$$

5.  $P_n$ :  $5n - 2$ .  $P_1$  is true:  $a_1 = 5 \cdot 1 - 2 = 3$ .

Now assume  $P_k$  is true:  $a_k = 5k - 2$ .

To get  $a_{k+1}$ , add 5 to  $a_k$ ; that is,

$$a_{k+1} = (5k - 2) + 5 = 5(k + 1) - 2 \text{ This shows that } P_{k+1} \text{ is true. Therefore, } P_n \text{ is true for all } n \geq 1.$$

6.  $P_n$ :  $a_n = 2n + 5$ .  $P_1$  is true:  $a_1 = 2 \cdot 1 + 5 = 7$ .

Now assume  $P_k$  is true:  $a_k = 2k + 5$ .

To get  $a_{k+1}$ , add 2 to  $a_k$ ; that is,

$$a_{k+1} = (2k + 5) + 2 = 2(k + 1) + 5 \text{ This shows that } P_{k+1} \text{ is true. Therefore, } P_n \text{ is true for all } n \geq 1.$$

7.  $P_n$ :  $a_n = 2 \cdot 3^{n-1}$ .

$P_1$  is true:  $a_1 = 2 \cdot 3^{1-1} = 2 \cdot 3^0 = 2$ .

Now assume  $P_k$  is true:  $a_k = 2 \cdot 3^{k-1}$ .

To get  $a_{k+1}$ , multiply  $a_k$  by 3; that is,

$$a_{k+1} = 3 \cdot 2 \cdot 3^{k-1} = 2 \cdot 3^k = 2 \cdot 3^{(k+1)-1}. \text{ This shows that } P_{k+1} \text{ is true. Therefore, } P_n \text{ is true for all } n \geq 1.$$

8.  $P_n$ :  $a_n = 3 \cdot 5^{n-1}$ .

$P_1$  is true:  $a_1 = 3 \cdot 5^{1-1} = 3 \cdot 5^0 = 3$ .

Now assume  $P_k$  is true:  $a_k = 3 \cdot 5^{k-1}$ .

To get  $a_{k+1}$ , multiply  $a_k$  by 5; that is,

$$a_{k+1} = 5 \cdot 3 \cdot 5^{k-1} = 3 \cdot 5^k = 3 \cdot 5^{(k+1)-1}. \text{ This shows that } P_{k+1} \text{ is true. Therefore, } P_n \text{ is true for all } n \geq 1.$$

9.  $P_1$ :  $1 = \frac{1(1+1)}{2}$ .

$$P_k: 1 + 2 + \dots + k = \frac{k(k+1)}{2}$$

$$P_{k+1}: 1 + 2 + \dots + k + (k+1) = \frac{(k+1)(k+2)}{2}$$

10.  $P_1$ :  $(2(1) - 1)^2 = \frac{1(2-1)(2+1)}{3}$ .

$$P_k: 1^2 + 3^2 + \dots + (2k-1)^2 = \frac{k(2k-1)(2k+1)}{3}$$

$$P_{k+1}: 1^2 + 3^2 + \dots + (2k-1)^2 + (2k+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3}$$

11.  $P_1$ :  $\frac{1}{1 \cdot 2} = \frac{1}{1+1}$ .

$$P_k: \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

$$P_{k+1}: \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

12.  $P_1$ :  $1^4 = \frac{1(1+1)(2+1)(3+3-1)}{30}$ .

$$P_k: 1^4 + 2^4 + \dots + k^4 = \frac{k(k+1)(2k+1)(3k^2+3k-1)}{30}$$

$$P_{k+1}: 1^4 + 2^4 + \dots + k^4 + (k+1)^4 = \frac{(k+1)(k+2)(2k+3)(3k^2+9k+5)}{30}$$

13.  $P_n$ :  $1 + 5 + 9 + \dots + (4n - 3) = n(2n - 1)$ .

$P_1$  is true:  $4(1) - 3 = 1 \cdot (2 \cdot 1 - 1)$ .

Now assume  $P_k$  is true:

$$1 + 5 + 9 + \dots + (4k - 3) = k(2k - 1).$$

Add  $4(k + 1) - 3 = 4k + 1$  to both sides:

$$\begin{aligned} 1 + 5 + 9 + \dots + (4k - 3) + [4(k + 1) - 3] \\ = k(2k - 1) + 4k + 1 = 2k^2 + 3k + 1 \\ = (k + 1)(2k + 1) = (k + 1)[2(k + 1) - 1], \end{aligned}$$

so  $P_{k+1}$  is true.

Therefore,  $P_n$  is true for all  $n \geq 1$ .

14.  $P_n$ :  $1 + 2 + 2^2 + \dots + 2^{n-1} = 2^n - 1$ .

$P_1$  is true:  $2^{1-1} = 2^1 - 1$ .

Now assume  $P_k$  is true:

$$1 + 2 + 2^2 + \dots + 2^{k-1} = 2^k - 1.$$

Add  $2^k$  to both sides,

$$1 + 2 + 2^2 + \dots + 2^{k-1} + 2^k = 2^k - 1 + 2^k = 2 \cdot 2^k - 1 = 2^{k+1} - 1, \text{ so } P_{k+1} \text{ is true.}$$

Therefore,  $P_n$  is true for all  $n \geq 1$ .

15.  $P_n$ :  $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$ .

$$P_1 \text{ is true: } \frac{1}{1 \cdot 2} = \frac{1}{1+1}.$$

Now assume  $P_k$  is true:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{k(k+1)}$$

Add  $\frac{1}{(k+1)(k+2)}$  to both sides:

$$\begin{aligned} \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} \\ = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k(k+2) + 1}{(k+1)(k+2)} \\ = \frac{(k+1)(k+1)}{(k+1)(k+2)} = \frac{k+1}{k+2} = \frac{k+1}{(k+1)+1}, \end{aligned}$$

so  $P_{k+1}$  is true.

Therefore,  $P_n$  is true for all  $n \geq 1$ .

16.  $P_n: \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \dots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$ .

$P_1$  is true: it says that  $\frac{1}{1 \cdot 3} = \frac{1}{2 \cdot 1 + 1}$ .

Now assume  $P_k$  is true:

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \dots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}$$

Add  $\frac{1}{[2(k+1)-1][2(k+1)+1]}$  to both sides, and we have

$$\begin{aligned} &\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \dots + \frac{1}{(2k-1)(2k+1)} \\ &\quad + \frac{1}{[2(k+1)-1][2(k+1)+1]} \\ &= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} = \frac{k(2k+3)+1}{(2k+1)(2k+3)} \\ &= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} = \frac{k+1}{2(k+1)+1}, \end{aligned}$$

so  $P_{k+1}$  is true.

Therefore,  $P_n$  is true for all  $n \geq 1$ .

17.  $P_n: 2^n \geq 2n$ .

$P_1$  is true:  $2^1 \geq 2 \cdot 1$  (in fact, they are equal). Now assume

$P_k$  is true:  $2^k \geq 2k$ .

Then  $2^{k+1} = 2 \cdot 2^k \geq 2 \cdot 2k$

$= 2 \cdot (k+k) \geq 2(k+1)$ , so  $P_{k+1}$  is true. Therefore,

$P_n$  is true for all  $n \geq 1$ .

18.  $P_n: 3^n \geq 3n$ .

$P_1$  is true:  $3^1 \geq 3 \cdot 1$  (in fact, they are equal). Now assume

$P_k$  is true:  $3^k \geq 3k$ .

Then  $3^{k+1} = 3 \cdot 3^k \geq 3 \cdot 3k$

$= 3 \cdot (k+2k) \geq 3(k+1)$ , so  $P_{k+1}$  is true.

Therefore,  $P_n$  is true for all  $n \geq 1$ .

19.  $P_n: 3$  is a factor of  $n^3 + 2n$ .

$P_1$  is true: 3 is a factor of  $1^3 + 2 \cdot 1 = 3$ .

Now assume  $P_k$  is true: 3 is a factor of  $k^3 + 2k$ .

Then  $(k+1)^3 + 2(k+1)$

$$= (k^3 + 3k^2 + 3k + 1) + (2k + 2)$$

$$= (k^3 + 2k) + 3(k^2 + k + 1).$$

Since 3 is a factor of both terms, it is a factor of the sum,

so  $P_{k+1}$  is true. Therefore,  $P_n$  is true for all  $n \geq 1$ .

20.  $P_n: 6$  is a factor of  $7^n - 1$ .

$P_1$  is true: 6 is a factor of  $7^1 - 1 = 6$ .

Now assume  $P_k$  is true, so that 6 is a factor of

$$7^k - 1 = 6.$$

Then  $7^{k+1} - 1 = 7 \cdot 7^k - 1 = 7(7^k - 1) + 6$ . Since 6 is

a factor of both terms of this sum, it is a factor of the sum,

so  $P_{k+1}$  is true.

Therefore,  $P_n$  is true for all positive integers  $n$ .

21.  $P_n$ : The sum of the first  $n$  terms of a geometric sequence

with first term  $a_1$  and common ratio  $r \neq 1$  is  $\frac{a_1(1-r^n)}{1-r}$ .

$P_1$  is true:  $a_1 = \frac{a_1(1-r^1)}{1-r}$ .

Now assume  $P_k$  is true so that

$$a_1 + a_1r + \dots + a_1r^{k-1} = \frac{a_1(1-r^k)}{(1-r)}.$$

Add  $a_1r^k$  to both sides:  $a_1 + a_1r + \dots = a_1r^{k-1}$

$$+ a_1r^k = \frac{a_1(1-r^k)}{(1-r)} + a_1r^k$$

$$= \frac{a_1(1-r^k) + ar^k(1-r)}{1-r}$$

$$= \frac{a_1 - a_1r^k - a_1r^{k+1}}{1-r} = \frac{a_1 - a_1r^{k+1}}{1-r},$$

so  $P_{k+1}$  is true. Therefore,  $P_n$  is true for all positive integers  $n$ .

22.  $P_n: S_n = \frac{n}{2}[2a_1 + (n-1)d]$ .

First note that  $a_n = a_1 + (n-1)d$ .  $P_1$  is true:

$$S_1 = \frac{1}{2}[2a_1 + (1-1)d] = \frac{1}{2}(2a_1) = a_1.$$

Now assume  $P_k$  is true:  $S_k = \frac{k}{2}[2a_1 + (k-1)d]$ .

Add  $a_{k+1} = a_1 + kd$  to both sides, and observe that

$$S_k + a_{k+1} = S_{k+1}.$$

Then we have

$$S_{k+1} = \frac{k}{2}[2a_1 + (k-1)d] + a_1 + kd$$

$$= ka_1 + \frac{1}{2}k(k-1)d + a_1 + kd$$

$$= (k+1)a_1 + \frac{1}{2}k(k+1)d$$

$$= \frac{k+1}{2}[2a_1 + (k+1-1)d].$$

Therefore,  $P_{k+1}$  is true, so  $P_n$  is true for all  $n \geq 1$ .

23.  $P_n: \sum_{k=1}^n k = \frac{n(n+1)}{2}$ .

$P_1$  is true:  $\sum_{k=1}^1 k = 1 = \frac{1 \cdot 2}{2}$ .

Now assume  $P_k$  is true:  $\sum_{i=1}^k i = \frac{k(k+1)}{2}$ .

Add  $(k+1)$  to both sides, and we have

$$\sum_{i=1}^{k+1} i = \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

$$= \frac{(k+1)(k+1+1)}{2}, \text{ so } P_{k+1} \text{ is true.}$$

Therefore,  $P_n$  is true for all  $n \geq 1$ .

24.  $P_n: \sum_{k=1}^n k^3 = \frac{n^2(n+1)^2}{4}$ .  $P_1$  is true:  $1^3 = \frac{1^2 \cdot 2^2}{4}$ .

Now assume  $P_k$  is true so that

$$1^3 + 2^3 + \dots + k^3 = \frac{k^2(k+1)^2}{4}. \text{ Add } (k+1)^3 \text{ to both}$$

sides and we have

$$1^3 + 2^3 + \dots + k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$

$$= \frac{(k+1)^2(k^2 + 4k + 4)}{4} = \frac{(k+1)^2((k+1)+1)^2}{4}$$

so  $P_{k+1}$  is true. Therefore,  $P_n$  is true for all positive integers.



25. Use the formula in 23:  $\sum_{k=1}^{500} k = \frac{(500)(501)}{2} = 125,250$
26. Use the formula in Example 2:  $\sum_{k=1}^{250} k^2 = \frac{(250)(251)(501)}{6} = 5,239,625$
27. Use the formula in 23:  $\sum_{k=4}^n k = \sum_{k=1}^n k - \sum_{k=1}^3 k = \frac{n(n+1)}{2} - \frac{3 \cdot 4}{2} = \frac{n^2 + n - 12}{2} = \frac{(n-3)(n+4)}{2}$
28. Use the formula in 24:  $\sum_{k=1}^{75} k^3 = \frac{(75^2)(76^2)}{4} = 8,122,500$
29. Use the formula in 14:  $\sum_{k=1}^{35} 2^{k-1} = 2^{35} - 1 \approx 3.44 \times 10^{10}$
30. Use the formula in 24:  $\sum_{k=1}^{15} k^3 = \frac{(15^2)(16^2)}{4} = 14,400$
31.  $\sum_{k=1}^n (k^2 - 3k + 4) = \sum_{k=1}^n k^2 - \sum_{k=1}^n 3k + \sum_{k=1}^n 4 = \frac{n(n+1)(2n+1)}{6} - 3 \left[ \frac{n(n+1)}{2} \right] + 4n = \frac{n(n^2 - 3n + 8)}{3}$
32.  $\sum_{k=1}^n (2k^2 + 5k - 2) = \sum_{k=1}^n 2k^2 + \sum_{k=1}^n 5k - \sum_{k=1}^n 2 = 2 \left[ \frac{n(n+1)(2n+1)}{6} \right] + 5 \left[ \frac{n(n+1)}{2} \right] - 2n = \frac{n(4n^2 + 21n + 5)}{6} = \frac{n(n+5)(4n+1)}{6}$
33.  $\sum_{k=1}^n (k^3 - 1) = \sum_{k=1}^n k^3 - \sum_{k=1}^n 1 = \frac{n^2(n+1)^2}{4} - n = \frac{n(n^3 + 2n^2 + n - 4)}{4} = \frac{n(n-1)(n^2 + 3n + 4)}{4}$
34.  $\sum_{k=1}^n (k^3 + 4k - 5) = \sum_{k=1}^n k^3 + \sum_{k=1}^n 4k - \sum_{k=1}^n 5 = \frac{n^2(n+1)^2}{4} + 4 \left[ \frac{n(n+1)}{2} \right] - 5n = \frac{n(n^3 + 2n^2 + 9n - 12)}{4} = \frac{n(n-1)(n^2 + 3n + 12)}{4}$
35. The inductive step does not work for 2 people. Sending them alternately out of the room leaves 1 person (and one blood type) each time, but we cannot conclude that their blood types will match *each other*.
36. The number  $k$  is a fixed number for which the statement  $P_k$  is known to be true. Once the anchor is established, we can assume that such a number  $k$  exists. We can not assume that  $P_n$  is true, because  $n$  is not fixed.
37. False. Mathematical induction is used to show that a statement  $P_n$  is true for all positive integers.
38. True.  $(1+1)^2 = 4 = 4(1)$ .  $P_n$  is false, however, for all other values of  $n$ .
39. The inductive step assumes that the statement is true for some positive integer  $k$ . The answer is E.
40. The anchor step, proving  $P_1$ , comes first. The answer is A.
41. Mathematical induction could be used, but the formula for a finite arithmetic sequence with  $a_1 = 1$ ,  $d = 2$  would also work. The answer is B.
42. The first two partial sums are 1 and 9. That eliminates all answers except C. Mathematical induction can be used to show directly that C is the correct answer.
43.  $P_n$ : 2 is a factor of  $(n+1)(n+2)$ .  $P_1$  is true because 2 is a factor of  $(2)(3)$ . Now assume  $P_k$  is true so that 2 is a factor of  $(k+1)(k+2)$ . Then  $[(k+1)+1][(k+2)+2] = (k+2)(k+3) = k^2 + 5k + 6 = k^2 + 3k + 2 + 2k + 4 = (k+1)(k+2) + 2(k+2)$ . Since 2 is a factor of both terms of this sum, it is a factor of the sum, and so  $P_{k+1}$  is true. Therefore,  $P_n$  is true for all positive integers  $n$ .
44.  $P_n$ : 6 is a factor of  $n(n+1)(n+2)$ .  $P_1$  is true because 6 is a factor of  $(1)(2)(3)$ . Now assume  $P_k$  is true so that 6 is a factor of  $k(k+1)(k+2)$ . Then  $(k+1)[(k+1)+1][(k+1)+2] = k(k+1)(k+2) + 3(k+1)(k+2)$ . Since 2 is a factor of  $(k+1)(k+2)$ , 6 is a factor of both terms of the sum and thus of the sum itself, and so  $P_{k+1}$  is true.
45. Given any two consecutive integers, one of them must be even. Therefore, their product is even. Since  $n+1$  and  $n+2$  are consecutive integers, their product is even. Therefore, 2 is a factor of  $(n+1)(n+2)$ .
46. Given any three consecutive integers, one of them must be a multiple of 3, and at least one of them must be even. Therefore, their product is a multiple of 6. Since  $n$ ,  $n+1$  and  $n+2$  are three consecutive integers, 6 is a factor of  $n(n+1)(n+2)$ .
47.  $P_n$ :  $F_{n+2} - 1 = \sum_{k=1}^n F_k$ .  $P_1$  is true since  $F_{1+2} - 1 = F_3 - 1 = 2 - 1 = 1$ , which equals  $\sum_{k=1}^1 F_k = 1$ . Now assume that  $P_k$  is true:  $F_{k+2} - 1 = \sum_{i=1}^k F_i$ . Then  $F_{(k+1)+2} - 1 = F_{k+3} - 1 = F_{k+1} + F_{k+2} - 1 = (F_{k+2} - 1) + F_{k+1} = \left( \sum_{i=1}^k F_i \right) + F_{k+1} = \sum_{i=1}^{k+1} F_i$ , so  $P_{k+1}$  is true. Therefore,  $P_n$  is true for all  $n \geq 1$ .
48.  $P_n$ :  $a_n < 2$ .  $P_1$  is easy:  $a_1 = \sqrt{2} < 2$ . Now assume that  $P_k$  is true:  $a_k < 2$ . Note that  $a_{k+1} = \sqrt{2 + a_k}$ , so that  $a_{k+1}^2 < 2 + a_k < 2 + 2 = 4$ ; therefore  $a_{k+1} < 2$ , so  $P_{k+1}$  is true. Therefore,  $P_n$  is true for all  $n \geq 1$ .
49.  $P_n$ :  $a - 1$  is a factor of  $a^n - 1$ .  $P_1$  is true because  $a - 1$  is a factor of  $a - 1$ . Now assume  $P_k$  is true so that  $a - 1$  is a factor of  $a^k - 1$ . Then  $a^{k+1} - 1 = a \cdot a^k - 1 = a(a^k - 1) + (a - 1)$ . Since  $a - 1$  is a factor of both terms in the sum, it is a factor of the sum, and so  $P_{k+1}$  is true. Therefore,  $P_n$  is true for all positive integers  $n$ .

50. Let  $P(a) = a^n - 1$ . Since  $P(1) = 1^n - 1 = 0$ , the Factor Theorem for polynomials allows us to conclude that  $a - 1$  is a factor of  $P$ .
51.  $P_n: 3n - 4 \geq n$  for  $n \geq 2$ .  $P_2$  is true since  $3 \cdot 2 - 4 \leq 2$ . Now assume that  $P_k$  is true:  $3k - 4 \geq 2$ . Then  $3(k + 1) - 4 = 3k + 3 - 4 = (3k - 4) + 3 \geq k + 3 \geq k + 1$ , so  $P_{k+1}$  is true. Therefore,  $P_n$  is true for all  $n \geq 2$ .
52.  $P_n: 2^n \geq n^2$  for  $n \geq 4$ .  $P_4$  is true since  $2^4 \geq 4^2$ . Now assume that  $P_k$  is true:  $2^k \geq k^2$ . Then  $2^{k+1} = 2 \cdot 2^k \geq 2 \cdot k^2 \geq 2 \cdot k^2 \geq k^2 + 2k + 1 = (k + 1)^2$ . The inequality  $2k^2 \geq k^2 + 2k + 1$ , or equivalently,  $k^2 \geq 2k + 1$ , is true for all  $k \geq 4$  because  $k^2 = k \cdot k \geq 4k = 2k + 2k > 2k + 1$ .) Thus  $P_{k+1}$  is true, so  $P_n$  is true for all  $n \geq 4$ .
53. Use  $P_3$  as the anchor and obtain the inductive step by representing any  $n$ -gon as the union of a triangle and an  $(n - 1)$ -gon.

### Section 9.7 Statistics and Data (Graphical)

#### Exploration 1

- We observe that the numbers seem to be centered a bit below 13. We would need to take into account the different state populations (not given in the table) in order to compute the national average exactly; but, just for the record, it was about 12.5 percent.
- We observe in the stemplot that 3 states have percentages above 15.
- We observe in the stemplot that the bottom five states are all below 10%. Returning to the table, we pick these out as Alaska, Colorado, Georgia, Texas, and Utah.
- The low outlier is Alaska, where older people would be less willing or able to cope with the harsh winter conditions. The high outlier is Florida, where the mild weather and abundant retirement communities attract older residents.

#### Quick Review 9.7

- $\approx 15.48\%$
- $\approx 20.94\%$
- $\approx 14.44\%$
- $\approx 27.22\%$
- $\approx 1723$
- 9200
- 235 thousand
- 238 million
- 1 million
- 1 billion

#### Section 9.7 Exercises

1.  $\begin{array}{c|c} 0 & 5\ 8\ 9 \\ 1 & 3\ 4\ 6 \\ 2 & 3\ 6\ 8 \\ 3 & 3\ 9 \\ 4 & \\ 5 & \\ 6 & 1 \\ \hline \end{array}$   
61 is an outlier.

2. 

Maris	Aaron
$\begin{array}{c c} 9\ 8\ 5 & 0 \\ 6\ 4\ 3 & 1\ 0\ 2\ 3 \\ 8\ 6\ 3 & 2\ 0\ 4\ 6\ 7\ 9 \\ 9\ 3 & 3\ 0\ 2\ 4\ 4\ 8\ 9\ 9 \\ & 4\ 0\ 0\ 4\ 4\ 4\ 4\ 5\ 7 \\ & 5 \\ & 1\ 6 \end{array}$	

Except for Maris's one record-breaking year, his home run output falls well short of Aaron's.

3. 

Males
$\begin{array}{c c} 6 & 0\ 3 \\ 6 & 7\ 8\ 8\ 8 \\ 7 & 1\ 2\ 2\ 3\ 3\ 3 \\ 7 & \end{array}$

This stemplot shows the life expectancies of males in the nations of South America are clustered near 70, with two lower values clustered near 60.

4. 

Females
$\begin{array}{c c} 6 & 5\ 8 \\ 7 & 1\ 2 \\ 7 & 5\ 6\ 7\ 7\ 9\ 9 \\ 8 & 0\ 0 \end{array}$

This stemplot shows the life expectancies of females in the nations of South America are clustered in the high 70's and at 80, with four lower values in the high 60's and low 70's.

5. 

Males	$\begin{array}{c c} 3\ 0 \\ 8\ 8\ 8\ 7 \\ 3\ 3\ 3\ 2\ 2\ 1 \end{array}$	Females	$\begin{array}{c c} 6 \\ 6\ 5\ 8 \\ 7\ 1\ 2 \\ 7\ 5\ 6\ 7\ 7\ 9\ 9 \\ 8\ 0\ 0 \end{array}$
-------	---	---------	--

This stemplot shows that the life expectancies of the women in the nations of South America are about 5–6 years higher than that of the men in the nations of South America.

6. Female-Male Difference

- $\begin{array}{c|c} 3 & 5 \\ 4 & 7\ 8 \\ 5 & 2\ 4\ 8 \\ 6 & 3\ 5\ 7 \\ 7 & 7\ 8 \\ 8 & 1 \end{array}$

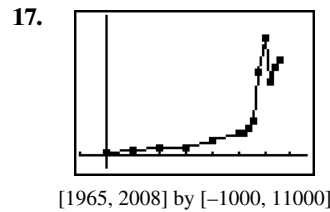
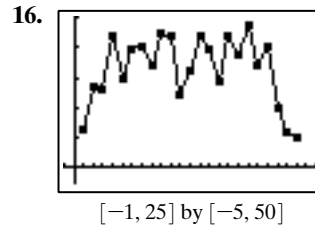
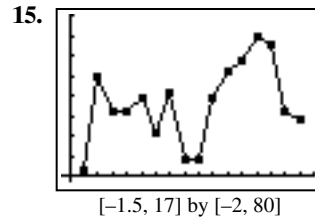
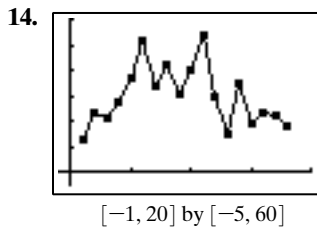
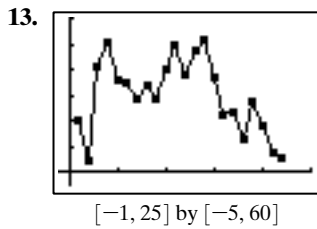
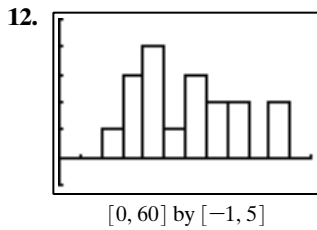
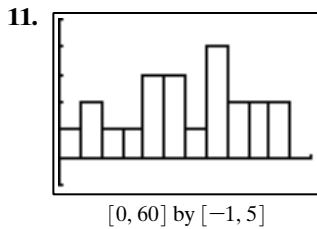
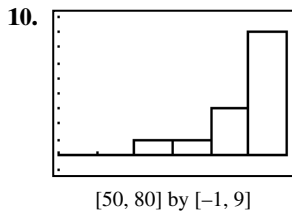
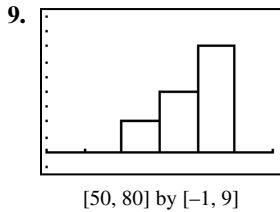
This stemplot of differences shows that the women in the nations of South America have higher life expectancies than the men, by about 5–6 years.

7. 

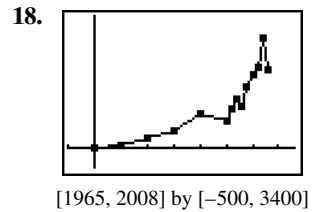
Life expectancy (years)	Frequency
60.0–64.9	2
65.0–69.9	4
70.0–74.9	6

8.

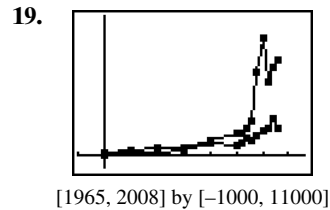
Life expectancy (years)	Frequency
60.0–64.9	1
65.0–69.9	1
70.0–74.9	2
75.0–79.9	8



The top male's earnings appear to be growing exponentially, with unusually high earnings in 1999 and 2000. Since the graph shows the earnings of only the top player (as opposed to a mean or median for all players), it can behave strangely if the top player has a very good year — as Tiger Woods did in 1999 and 2000.

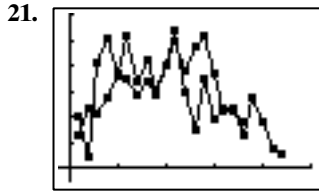


The top female's earnings appear to be growing faster than linearly, but with more fluctuation since 1990. Since the graph shows the earnings of only the top player (as opposed to a mean or median for all players), it is strongly affected by how well the top player does in a given year. The top player in 2002 and 2003, Annika Sorenstam, played in fewer tournaments in 2003 and hence won less money.



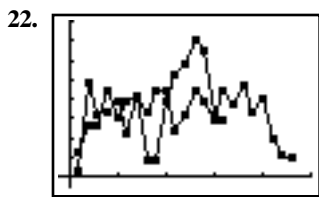
After approaching parity in 1985, the top PGA player's earnings have grown much faster than the top LPGA player's earnings, even if the unusually good years for Tiger Woods (1999 and 2000) are not considered part of the trend.

20. The 1985 male total was unusually low in the context of the rest of the time plot. If it had been in line with the earlier and later data, the difference between the PGA and LPGA growth rates would not have been that large. (The unusual PGA point in 1985, oddly enough, belonged to Curtis Strange.)



$[-1, 25]$  by  $[-5, 60]$

The two home run hitters enjoyed similar success, with Mays enjoying a bit of an edge in the earlier and later years of his career, and Mantle enjoying an edge in the middle years.



$[-1, 25]$  by  $[-5, 80]$

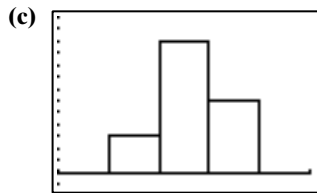
The two home run hitters were comparable for the first seven years of their careers.

23. (a)

Stem	Leaf
28	2
29	3 7
30	
31	6 7
32	7 8
33	5 5 5 8
34	2 8 8
35	3 3 4
36	3 7
37	
38	5

(b)

Interval	Frequency
25.0–29.9	3
30.0–34.9	11
35.0–39.9	6



$[20, 45]$  by  $[-1, 13]$

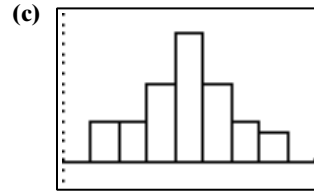
(d) Time is not a variable in the data.

24. (a)

Stem	Leaf
6	2 3 9 9
7	7 9 9 9
8	0 2 3 7 7 8 9 9
9	0 0 0 1 1 2 3 3 3 4 5 6 7
10	2 3 3 6 6 6 7 9
11	0 2 4 6
12	0 5 9

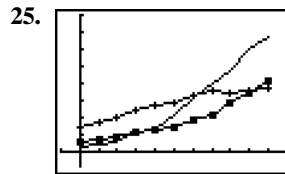
(b)

Interval	Frequency
6.0–6.9	4
7.0–7.9	4
8.0–8.9	8
9.0–9.9	13
10.0–10.9	8
11.0–11.9	4
12.0–12.9	3



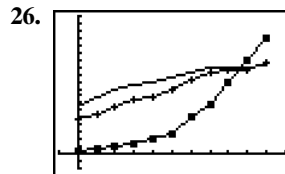
$[5, 14]$  by  $[-2, 15]$

(d) The data are not categorical.



$[1890, 2010]$  by  $[-4, 40]$

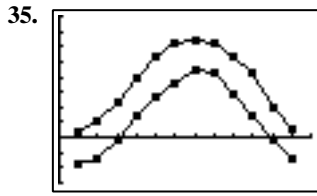
— = CA      + = NY      ■ = TX



$[1890, 2010]$  by  $[-4, 40]$

— = PA      + = IL      ■ = FL

- 27. False. The empty branches are important for visualizing the true distribution of the data values.
- 28. False. They are outliers only if they are significantly higher or lower than the other numbers in the data set.
- 29. A time plot uses a continuous line. The answer is C.
- 30. Back-to-back stemplots are designed for comparing data sets. The answer is B.
- 31. The histogram suggests data values clustered near an upper limit — such as the maximum possible score on an easy test. The answer is A.
- 32.  $45^\circ$  is  $1/8$ , or 12.5%, of  $360^\circ$ . The answer is B.
- 33. Answers will vary. Possible outliers could be the pulse rates of long-distance runners and swimmers, which are often unusually low. Students who have had to run to class from across campus might have pulse rates that are unusually high.
- 34. Answers will vary. Female heights typically have a distribution that is uniformly lower than male heights, but the difference might not be apparent from a stem plot, especially if the sample is small.



[0, 13] by [-15, 40]

36. Using the form  $f(t) = k + a \sin[b(t - h)]$ , we have  $b = \pi/6$  for both graphs since the period is 12.  $a$  is half the difference between the extremes, and  $k$  is the average of the extremes. Finally,  $h$  is the offset to the first time this average occurs.

The maxima of both graphs occur in July ( $t = 7$ ), while the minima occur in January ( $t = 1$ ), so we should choose  $b = 3.5$  or 4 (if the latter is halfway between the maximum and the minimum, but the former gives a better match for some values of  $t$ ). Thus, for the high

temperature, use  $f(t) \approx 17 + 15 \sin\left[\frac{\pi}{6}(t - 3.5)\right]$  or

$17 + 15 \sin\left[\frac{\pi}{6}(t - 4)\right]$ , and for the low temperature, use

$$g(t) \approx 6.5 + 15.5 \sin\left[\frac{\pi}{6}(t - 3.5)\right] \text{ or}$$

$$6.5 + 15.5 \sin\left[\frac{\pi}{6}(t - 4)\right].$$

## ■ Section 9.8 Statistics and Data (Algebraic)

### Exploration 1

- In Figure 9.19 (a) the extreme values will cause the range to be big, but the compact distribution of the rest of the data indicate a small interquartile range. The data in Figure 9.19 (b) exhibit high variability.
- Figure 9.20 (b) has a longer “tail” to the right (skewed right), so the values in the tail pull the mean to the right of the (resistant) median. Figure 9.20 (c) is skewed left, so the values in the tail pull the mean to the left of the median. Figure 9.20 (a) is symmetric about a vertical line, so the median and the mean are close together.

### Quick Review 9.8

1.  $\sum_{i=1}^7 x_i = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$

2.  $\sum_{i=1}^5 (x_i - \bar{x}) = (x_1 - \bar{x}) + (x_2 - \bar{x}) + (x_3 - \bar{x}) + (x_4 - \bar{x}) + (x_5 - \bar{x}) = x_1 + x_2 + x_3 + x_4 + x_5 - 5\bar{x}$ .

Note that, since  $\bar{x} = \frac{1}{5} \sum_{i=1}^5 x_i$ , this simplifies to 0.

3.  $\frac{1}{7} \sum_{i=1}^7 x_i = \frac{1}{7}(x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7)$

4.  $\frac{1}{5} \sum_{i=1}^5 (x_i - \bar{x}) = \frac{1}{5}(x_1 + x_2 + x_3 + x_4 + x_5 - 5\bar{x}) = \frac{1}{5}(x_1 + x_2 + x_3 + x_4 + x_5) - \bar{x}$ . Note that, since  $\bar{x} = \frac{1}{5} \sum_{i=1}^5 x_i$ , this simplifies to 0.

5. The expression at the end of the first line is a simple expansion of the sum (and is a reasonable answer to the given question). By expanding further, we can also arrive at the final expression below, which is somewhat simpler.

$$\begin{aligned} \frac{1}{5} \sum_{i=1}^5 (x_i - \bar{x})^2 &= \frac{1}{5} [(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_5 - \bar{x})^2] \\ &= \frac{1}{5} [(x_1^2 - 2x_1\bar{x} + \bar{x}^2) + (x_2^2 - 2x_2\bar{x} + \bar{x}^2) + \cdots + (x_5^2 - 2x_5\bar{x} + \bar{x}^2)] \\ &= \frac{1}{5} [x_1^2 + x_2^2 + \cdots + x_5^2 - 2\bar{x}(x_1 + x_2 + \cdots + x_5)] + \bar{x}^2 \\ &= \frac{1}{5} (x_1^2 + x_2^2 + \cdots + x_5^2) - 2\bar{x}^2 + \bar{x}^2 = \frac{1}{5} (x_1^2 + x_2^2 + \cdots + x_5^2) - \bar{x}^2 \end{aligned}$$

6. The square root does not allow for further simplification. The final answer is the square root of the expression from #5: either

$$\sqrt{\frac{1}{5} [(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_5 - \bar{x})^2]} \text{ or } \sqrt{\frac{1}{5} (x_1^2 + x_2^2 + \cdots + x_5^2) - \bar{x}^2}.$$

7.  $\sum_{i=1}^8 x_i f_i$

8.  $\sum_{i=1}^{10} (x_i - \bar{x})^2$

9.  $\frac{1}{50} \sum_{i=1}^{50} (x_i - \bar{x})^2$

10.  $\sqrt{\frac{1}{7} \sum_{i=1}^7 (x_i - \bar{x})^2}$

## Section 9.8 Exercises

1. (a) Statistic. The number characterizes a set of known data values.  
 (b) Parameter. The number describes an entire population, on the basis of some statistical inference.  
 (c) Statistic. The number is calculated from information about all rats in a small, experimental population.
2. (a) Mode. No quantitative measure beyond what is “most common” is implied.  
 (b) Mean. A pitcher’s earned run average (ERA) is total earned runs divided by number of nine-inning blocks pitched.  
 (c) Median. The middle height is considered average.
3.  $\bar{x} = \frac{1}{5}(12 + 23 + 15 + 48 + 36) = \frac{134}{5} = 26.8$
4.  $\bar{x} = \frac{1}{6}(4 + 8 + 11 + 6 + 21 + 7) = \frac{57}{6} = 9.5$
5.  $\bar{x} = \frac{1}{6}(32.4 + 48.1 + 85.3 + 67.2 + 72.4 + 55.3) = \frac{360.7}{6} = \frac{3607}{60} \approx 60.12$
6.  $\bar{x} = \frac{1}{7}(27.4 + 3.1 + 9.7 + 32.3 + 12.8 + 39.4 + 73.7) = \frac{198.4}{7} = \frac{992}{35} \approx 28.3$
7.  $\bar{x} = \frac{1}{6}(1.5 + 0.5 + 4.8 + 7.3 + 6.3 + 3.0) = 3.9$  million, or 3,900,000
8.  $\bar{x} = \frac{1}{6}(29.8 + 12.9 + 11.4 + 18.0 + 11.9 + 17.0) \approx 16.83$  million, or about 16,830,000
9.  $\bar{x} = \frac{1}{9}(0 + 0 + 1 + 2 + 61 + 33 + 26 + 13 + 1) = \frac{137}{9} \approx 15.2$  satellites
10.  $\bar{x} = \frac{1}{7}(30,065,000 + 13,209,000 + 44,579,000 + \dots + 17,819,000) = \frac{1}{7}(148,196,000) = 21,171,000 \text{ km}^2$
11. There are 9 data values, which is an odd number, so the median is the middle data value when they are arranged in order. In order, the data values are  $\{0, 0, 1, 1, 2, 8, 21, 28, 30\}$ . The median is 2.
12. There are 7 data values, which is an odd number, so the median is the middle data value when they are arranged in order. In order, the data values are  $\{7687, 9938, 13,209, 17,819, 24,256, 30,065, 44,579\}$  in thousands of  $\text{km}^2$ . The median is 17,819,000  $\text{km}^2$ .
13. Mays:  $\frac{660}{22} = 30$  home runs/year. Mantle:  $\frac{536}{18} \approx 29.8$  home runs/year. Mays had the greater production rate.
14. State College:  $\frac{12}{5} = 2.4$  houses/day. College Station:  $\frac{15}{7} \approx 2.14$  houses/day. The State College workers were faster.
15. No computations needed — Hip-Hop House: 1147 skirts in 4 weeks. What-Next Fashions: 1516 skirts in 4 weeks. What-Next Fashions had the greater production rate.
16. India:  $\frac{311,000,000,000}{882,575,000} \approx \$352$ .  
 Mexico:  $\frac{218,000,000,000}{87,715,000} \approx \$2485$ . Mexico has the higher PCI.
17. For  $\{79.5, 82.1, 82.3, 82.9, 84.0, 84.5, 84.8, 85.4, 85.7, 85.7, 86.0, 86.4, 87.0, 87.5, 87.7, 88.0, 88.0, 88.2, 88.2, 88.5, 89.2, 89.2, 89.8, 89.8, 91.1, 91.3, 91.6, 91.8, 94.1, 94.4\}$ , the median is  $\frac{87.7 + 88.0}{2} = 87.85$ , and there is no mode.
18.  $\bar{x} = \frac{5(18,522) + 4(27,102) + 3(31,286) + 2(20,732) + 1(18,875)}{18,522 + 27,102 + 31,286 + 20,732 + 18,875} \approx 3.05$
19.  $\bar{x} = \frac{5(9879) + 4(5119) + 3(6143) + 2(2616) + 1(3027)}{9879 + 5119 + 6143 + 2616 + 3027} \approx 3.61$
20. (a) Non-weighted:  $\bar{x} = \frac{1}{12}(2 + 5 + 12 + 20 + \dots + 10 + 3) = \frac{221}{12} \approx 18.42^\circ\text{C}$   
 (b) Weighted:  $\bar{x} = \frac{(2)(31) + (5)(28) + (12)(31) + (20)(30) + \dots + (10)(30) + (3)(31)}{31 + 28 + 31 + 30 + \dots + 30 + 31} = \frac{6748}{365} \approx 18.49^\circ\text{C}$   
 (c) The weighted average is the better indicator.

21. (a) Non-weighted:  $\bar{x} = \frac{1}{12}(-9 - 7 - 1 + 7 + \dots - 1 - 7) = \frac{77}{12} \approx 6.42^\circ\text{C}$

(b) Weighted:  $\bar{x} = \frac{(-9)(31) + (-7)(28) + (-1)(31) + (7)(30) + \dots + (-1)(30) + (-7)(31)}{31 + 28 + 31 + 30 + \dots + 30 + 31} = \frac{2370}{365} \approx 6.49^\circ\text{C}$

(c) The weighted average is the better indicator.

22. The ordered data for Mark McGwire is {3, 9, 9, 22, 29, 32, 32, 33, 39, 39, 42, 49, 52, 58, 65, 70} so the median is

$$\frac{33 + 39}{2} = 36,$$

$$Q_3 = \frac{49 + 52}{2} = 50.5, Q_1 = \frac{22 + 29}{2} = 25.5.$$

Five-number summary: {3, 25.5, 36, 50.5, 70}

Range:  $70 - 3 = 67$

IQR:  $50.5 - 25.5 = 25$

No outliers

The ordered data for Barry Bonds is:

{16, 19, 24, 25, 25, 33, 33, 34, 34, 37, 37, 40, 42, 46, 49, 73} so

the median is  $\frac{34 + 34}{2} = 34,$

$$Q_3 = \frac{40 + 42}{2} = 41, \text{ and } Q_1 = \frac{25 + 25}{2} = 25$$

Five-number summary: {16, 25, 34, 41, 73}

Range:  $73 - 16 = 57$

IQR:  $41 - 25 = 16$

Outlier: 73

23. The ordered data for Willie Mays is: {4, 6, 8, 13, 18, 20, 22, 23, 28, 29, 29, 34, 35, 36, 37, 38, 40, 41, 47, 49, 51, 52} so the median is

$$\frac{29 + 34}{2} = 31.5, Q_3 = 40, \text{ and } Q_1 = 20$$

Five-number summary: {4, 20, 31.5, 40, 52}

Range:  $52 - 4 = 48$

IQR:  $40 - 20 = 20$

No outliers

The ordered data for Mickey Mantle is:

{13, 15, 18, 19, 21, 22, 23, 23, 27, 30, 31, 34, 35, 37, 40, 42, 52, 54} so the median is

$$\frac{27 + 30}{2} = 28.5, Q_3 = 37, \text{ and } Q_1 = 21$$

Five-number summary: {13, 21, 28.5, 37, 54}

Range:  $54 - 13 = 41$

IQR:  $37 - 21 = 16$

No outliers

24. While a stemplot is not needed to answer this question, the sorted stemplot below is more compact than a sorted list of the 44 numbers. The underlined numbers are the ones used for the five-number summary, which is

Min	$Q_1$	Median	$Q_3$	Max
6.2	8.5	9.25	10.6	12.9
$= \frac{8.3 + 8.7}{2} = \frac{9.2 + 9.3}{2} = \frac{10.6 + 10.6}{2}$				

The range is  $12.9 - 6.2 = 6.7$  and

IQR =  $10.6 - 8.5 = 2.1$ . There are no outliers, since none of the numbers fall below  $Q_1 - 1.5 \times \text{IQR} = 5.35$  or above  $Q_3 + 1.5 \times \text{IQR} = 13.75$ .

Stem	Leaf
6	<u>2</u> 3 9 9
7	7 9 9 9
8	0 2 <u>3</u> <u>7</u> 7 8 9 9
9	0 0 0 1 1 <u>2</u> <u>3</u> 3 3 4 5 6 7
10	2 3 3 <u>6</u> <u>6</u> 6 7 9
11	0 2 4 6
12	0 5 9

25. The sorted list is {28.2, 29.3, 29.7, 31.6, 31.7, 32.7, 32.8, 33.5, 33.5, 33.5, 33.8, 34.2, 34.8, 34.8, 35.3, 35.4, 36.7, 37.3, 38.5}. Since there are 19 numbers, the median is the 10th,  $Q_1$  is the 5th, and  $Q_3$  is the 15th; no additional computations are needed. All these numbers are underlined above, and are summarized below:

Min	$Q_1$	Median	$Q_3$	Max
28.2	31.7	33.5	35.3	38.5

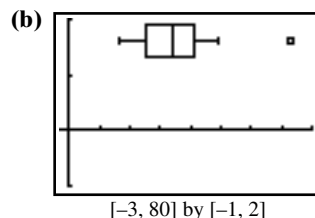
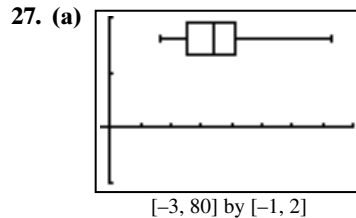
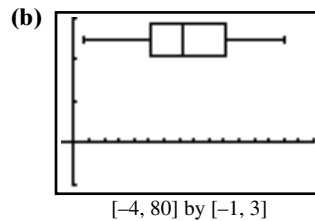
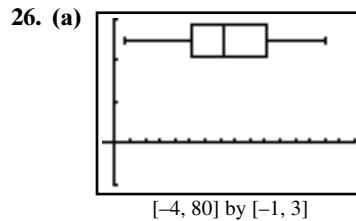
The range is  $38.5 - 28.2 = 10.3$ , and

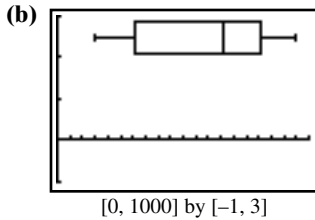
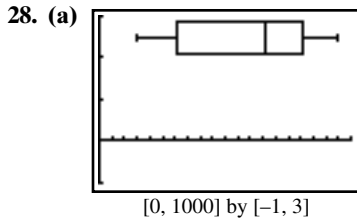
IQR =  $35.3 - 31.7 = 3.6$ . There are no outliers, since none of the numbers fall below  $Q_1 - 1.5 \times \text{IQR} = 26.3$  or above  $Q_3 + 1.5 \times \text{IQR} = 40.7$ .

Note that some computer software may return

$$Q_1 = 32.2 = \frac{31.7 + 32.7}{2} \text{ and } Q_3 = 35.05 = \frac{34.8 + 35.3}{2}$$

(The TI calculator which computes the five-number summary produces the results shown in the table.)



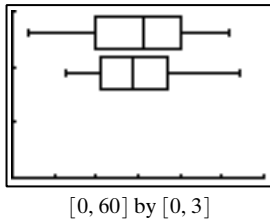


29. 12 of the 44 numbers are more than 10.5:  $\frac{12}{44} = \frac{3}{11}$

30. All but 4 of the 44 numbers are more than 7.5:  $\frac{40}{44} = \frac{10}{11}$

31. The five-number summaries are:  
Mays: 4, 20, 31.5, 40, 52 (top of graph)  
Mantle: 13, 21, 28.5, 37, 54 (bottom of graph)

- (a) Mays's data set has the greater range.
- (b) Mays's data set also has the greater *IQR*.



46.  $\bar{x} = \frac{10(3) + 9(3) + 8(5) + 7(6) + 6(4) + 5(3) + 4(1) + 3(0) + 2(0) + 1(0)}{25}$

$= 7.28$

The answer is B.

47. The total number of points from all 30 exams combined is  $30 \times 81.3 = 2439$ . Adding 9 more points and recalculating produces a new mean of  $(2439 + 9)/30 = 81.6$ . The median will be unaffected by an adjustment in the top score. The answer is B.

48. In a normal distribution, 95% of the data values lie within 2 standard deviations of the mean. The answer is C.

49. There are many possible answers; examples are given.

(a) {2, 2, 2, 3, 6, 8, 20} — mode = 2, median = 3, and  $\bar{x} = \frac{43}{7} \approx 6.14$ .

(b) {1, 2, 3, 4, 6, 48, 48} — median = 4,  $\bar{x} = 16$ , and mode = 48

(c) {−20, 1, 1, 1, 2, 3, 4, 5, 6} —  $\bar{x} = \frac{1}{3}$ , mode = 1, and median = 2.

32. Although Mays' best year was not as good as Mantle's best year, Mays was in some ways a better home-run hitter: his median season was 31.5 home runs, compared to 28.5 for Mantle. However, Mays had three extremely bad years when he hit fewer than 10 home runs; these make him look worse by comparison.

For #33–38, the best way to do the computation is with the statistics features of a calculator.

33.  $\sigma \approx 9.08, \sigma^2 = 82.5$

34.  $\sigma \approx 23.99, \sigma^2 = 575.64$

35.  $\sigma \approx 186.36; \sigma^2 \approx 34,828.12$

36.  $\sigma \approx 126.84; \sigma^2 \approx 16,088.08$

37.  $\sigma \approx 1.53, \sigma^2 \approx 2.34$

38.  $\sigma \approx 2.60, \sigma^2 \approx 6.77$

39. No. An outlier would need to be less than  $25.5 - 1.5(25) = -12$  or greater than  $50.5 + 1.5(25) = 88$ .

40. The standard deviation of a set can never be negative, since it is the (positive) square root of the variance. It is possible for the standard deviation of a set to be zero, but all the numbers in the set would have to be the same.

41. (a) 68%

(b) 2.5%

(c) A parameter, since it applies to the entire population.

42. (a) 16%

(b) 32.7

(c) No. The mean would have to be *weighted* according to the number of people in each state who took the ACT.

43. False. The median is a resistant measure. The *mean* is strongly affected by outliers.

44. True. The box extends from the first quartile,  $Q_1$ , to the third quartile,  $Q_3$ , and  $Q_3 - Q_1$  is the interquartile range.

45. The plot of an ideal normal distribution is a symmetric "bell curve." The answer is A.

50. There are many possible answers; examples are given.

(a) {2, 4, 6, 8} —  $\sigma \approx 2.24$  and  $IQR = 7 - 3 = 4$ .

(b) {1, 5, 5, 6, 6, 9} —  $IQR = 6 - 5 = 1$ , and  $\sigma \approx 2.36$ .

(c) {3, 3, 3, 9, 9, 9} —  $IQR = 9 - 3 = 6$  and range =  $9 - 3 = 6$ .

51. No:  $(x_i - \bar{x}) \leq (\max - \min)^2 = (\text{range})^2$ , so that

$$\sigma^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 \leq \frac{1}{n} \sum_{i=1}^n (\text{range})^2 = (\text{range})^2. \text{ Then}$$

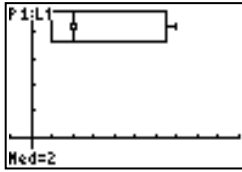
$$\sigma = \sqrt{\sigma^2} \leq \sqrt{(\text{range})^2} = \text{range}.$$

52. The mode does not have to be a numerical value because it is the data value that occurs most often; both the mean and the median are for numerical data only, since they both involve calculations of numerical data values.



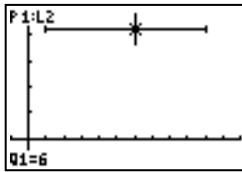
53. There are many possible answers; example data sets are given.

(a) {1, 1, 2, 6, 7} — median = 2 and  $\bar{x} = 3.4$ .



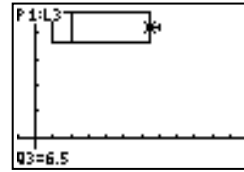
[-1, 10] by [-1, 5]

(b) {1, 6, 6, 6, 6, 10} —  $2 \times \text{IQR} = 2(6 - 6) = 0$  and range =  $10 - 1 = 9$ .



[-1, 12] by [-1, 5]

(c) {1, 1, 2, 6, 7} — range =  $7 - 1 = 6$  and  $2 \times \text{IQR} = 2(6 - 1) = 10$ .



[-1, 12] by [-1, 5]

54. One possible answer: {1, 2, 3, 4, 5, 6, 6, 6, 30}.

55. For women living in South American nations, the mean life expectancy is

$$\bar{x} = \frac{(79.7)(39.1) + (67.9)(8.7) + \cdots + (79.2)(3.4) + (77.3)(25.0)}{39.1 + 8.7 + \cdots + 3.4 + 25.0} = \frac{27,825.56}{366.4} \approx 75.9 \text{ years.}$$

56. For men living in South American nations, the mean life expectancy is

$$\bar{x} = \frac{(72.0)(39.1) + (62.5)(8.7) + \cdots + (72.7)(3.4) + (71.0)(25.0)}{39.1 + 8.7 + \cdots + 3.4 + 25.0} = \frac{25,160.64}{366.4} \approx 68.7 \text{ years.}$$

57. Since  $\sigma = 0.05$  mm, we have  $2\sigma = 0.1$  mm, so 95% of the ball bearings will be acceptable. Therefore, 5% will be rejected.

58. Use  $\mu = 12.08$  and  $\sigma = 0.04$ .

Then  $\mu - 2\sigma = 12.00$  and  $\mu + 2\sigma = 12.16$ , so 95% of the cans contain 12 to 12.16 oz of cola, 2.5% contain less than 12 oz, and 2.5% contain more than 12 oz. Therefore, 2.5% of the cans contain less than the advertised amount.

## Chapter 9 Review

- $\binom{12}{5} = \frac{12!}{5!(12-5)!} = \frac{12!}{5!7!} = 792$
- $\binom{789}{787} = \frac{789!}{787!(789-787)!} = \frac{789!}{787!2!} = 310,866$
- ${}_{18}C_{12} = \frac{18!}{12!(18-12)!} = \frac{18!}{12!6!} = 18,564$
- ${}_{35}C_{28} = \frac{35!}{28!(35-28)!} = \frac{35!}{28!7!} = 6,724,520$
- ${}_{12}P_7 = \frac{12!}{(12-7)!} = \frac{12!}{5!} = 3,991,680$
- ${}_{15}P_8 = \frac{15!}{(15-8)!} = \frac{15!}{7!} = 259,459,200$
- $26 \cdot 36^4 = 43,670,016$  code words
- $3 + (3 \cdot 4) = 15$  trips
- ${}_{26}P_2 \cdot {}_{10}P_4 + {}_{10}P_3 \cdot {}_{26}P_3 = 14,508,000$  license plates
- ${}_{45}C_3 = 14,190$  committees
- Choose 10 more cards from the other 49:  
 ${}_3C_3 \cdot {}_{49}C_{10} = {}_{49}C_{10} = 8,217,822,536$  hands

12. Choose a king, then 8 more cards from the other 44:

$${}_4C_4 \cdot {}_4C_1 \cdot {}_{44}C_8 = {}_4C_1 \cdot {}_{44}C_8 = 708,930,508 \text{ hands}$$

13.  ${}_5C_2 + {}_5C_3 + {}_5C_4 + {}_5C_5 = 2^5 - {}_5C_0 - {}_5C_1 = 26$  outcomes

14.  ${}_{21}C_2 \cdot {}_{14}C_2 = 19,110$  committees

15.  ${}_5P_1 + {}_5P_2 + {}_5P_3 + {}_5P_4 + {}_5P_5 = 325$

16.  $2^4 = 16$  (This includes the possibility that he has *no* coins in his pocket.)

17. (a) There are 7 letters, all different. The number of distinguishable permutations is  $7! = 5040$ . (GERMANY can be rearranged to spell MEG RYAN.)

(b) There are 13 letters, where E, R, and S each appear twice. The number of distinguishable permutations is

$$\frac{13!}{2!2!2!} = 778,377,600$$

(PRESBYTERIANS can be rearranged to spell BRITNEY SPEARS.)

18. (a) There are 7 letters, all different. The number of distinguishable permutations is  $7! = 5040$ .

(b) There are 11 letters, where A appears 3 times and L, S, and E each appear 2 times. The number of distinguishable permutations is

$$\frac{11!}{3!2!2!2!} = 831,600$$

$$\begin{aligned} 19. (2x + y)^5 &= (2x)^5 + 5(2x)^4y + 10(2x)^3y^2 \\ &\quad + 10(2x)^2y^3 + 5(2x)y^4 + y^5 \\ &= 32x^5 + 80x^4y + 80x^3y^2 + 40x^2y^3 \\ &\quad + 10xy^4 + y^5 \end{aligned}$$

$$\begin{aligned}
 20. (4a - 3b)^7 &= (4a)^7 + 7(4a)^6(-3b) + 21(4a)^5(-3b)^2 \\
 &\quad + 35(4a)^4(-3b)^3 + 35(4a)^3(-3b)^4 \\
 &\quad + 21(4a)^2(-3b)^5 + 7(4a)(-3b)^6 \\
 &\quad + (-3b)^7 \\
 &= 16,384a^7 - 86,016a^6b + 193,536a^5b^2 \\
 &\quad - 241,920a^4b^3 + 181,440a^3b^4 \\
 &\quad - 81,648a^2b^5 + 20,412ab^6 - 2187b^7
 \end{aligned}$$

$$\begin{aligned}
 21. (3x^2 + y^3)^5 &= (3x^2)^5 + 5(3x^2)^4(y^3) \\
 &\quad + 10(3x^2)^3(y^3)^2 + 10(3x^2)^2(y^3)^3 \\
 &\quad + 5(3x^2)(y^3)^4 + (y^3)^5 \\
 &= 243x^{10} + 405x^8y^3 + 270x^6y^6 \\
 &\quad + 90x^4y^9 + 15x^2y^{12} + y^{15}
 \end{aligned}$$

$$\begin{aligned}
 22. \left(1 + \frac{1}{x}\right)^6 &= 1 + 6(x^{-1}) + 15(x^{-1})^2 + 20(x^{-1})^3 \\
 &\quad + 15(x^{-1})^4 + 6(x^{-1})^5 + (x^{-1})^6 \\
 &= 1 + 6x^{-1} + 15x^{-2} + 20x^{-3} \\
 &\quad + 15x^{-4} + 6x^{-5} + x^{-6}
 \end{aligned}$$

$$\begin{aligned}
 23. (2a^3 - b^2)^9 &= (2a^3)^9 + 9(2a^3)^8(-b^2) + 36(2a^3)^7(-b^2)^2 \\
 &\quad + 84(2a^3)^6(-b^2)^3 + 126(2a^3)^5(-b^2)^4 \\
 &\quad + 126(2a^3)^4(-b^2)^5 + 84(2a^3)^3(-b^2)^6 \\
 &\quad + 36(2a^3)^2(-b^2)^7 + 9(2a^3)(-b^2)^8 \\
 &\quad + (-b^2)^9 \\
 &= 512a^{27} - 2304a^{24}b^2 + 4608a^{21}b^4 \\
 &\quad - 5376a^{18}b^6 + 4032a^{15}b^8 \\
 &\quad - 2016a^{12}b^{10} + 672a^9b^{12} \\
 &\quad - 144a^6b^{14} + 18a^3b^{16} - b^{18}
 \end{aligned}$$

$$\begin{aligned}
 24. (x^{-2} + y^{-1})^4 &= (x^{-2})^4 + 4(x^{-2})^3(y^{-1}) \\
 &\quad + 6(x^{-2})^2(y^{-1})^2 + 4(x^{-2})(y^{-1})^3 \\
 &\quad + (y^{-1})^4 \\
 &= x^{-8} + 4x^{-6}y^{-1} + 6x^{-4}y^{-2} \\
 &\quad + 4x^{-2}y^{-3} + y^{-4}
 \end{aligned}$$

$$25. \binom{11}{8}(1)^8(-2)^3 = -\frac{11!8}{8!3!} = -\frac{11 \cdot 10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = -1320$$

$$26. \binom{8}{2}(2)^2(1)^6 = \frac{8!4}{2!6!} = \frac{8 \cdot 7 \cdot 4}{2 \cdot 1} = 112$$

$$27. \{1, 2, 3, 4, 5, 6\}$$

$$28. \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), \dots, (6, 6)\}$$

$$29. \{13, 16, 31, 36, 61, 63\}$$

$$30. \{\text{Defective, Nondefective}\}$$

$$31. \{\text{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}\}$$

$$32. \{\text{HHT, HTH, THH, TTH, THT, HTT}\}$$

$$33. \{\text{HHH, TTT}\}$$

$$\begin{aligned}
 34. P(\text{at least one head}) &= 1 - P(\text{no heads}) = 1 - \left(\frac{1}{2}\right)^3 \\
 &= \frac{7}{8}
 \end{aligned}$$

$$35. P(\text{HHTHTT}) = \left(\frac{1}{2}\right)^6 = \frac{1}{2^6} = \frac{1}{64}$$

$$36. P(2 \text{ H and } 3 \text{ T}) = {}_5C_2 \cdot \left(\frac{1}{2}\right)^5 = \frac{10}{2^5} = \frac{5}{16}$$

$$37. P(1 \text{ H and } 3 \text{ T}) = {}_4C_1 \cdot \left(\frac{1}{2}\right)^4 = \frac{4}{2^4} = \frac{1}{4}$$

$$38. P(10 \text{ nondefectives in a row}) = (0.997)^{10} \approx 0.97$$

$$39. P(3 \text{ successes and } 1 \text{ failure}) = {}_4C_1 \cdot \left(\frac{1}{2}\right)^4 = \frac{4}{2^4} = \frac{1}{4} = 0.25$$

40. We can redefine the words “success” and “failure” to mean the opposite of what they meant before. In this sense, the experiment is symmetrical, because  $P(S) = P(F)$ .

$$41. P(\text{SF}) = (0.4)(0.6) = 0.24$$

$$42. P(\text{SFS}) = (0.4)(0.6)(0.4) = 0.096$$

$$43. P(\text{at least 1 success}) = 1 - P(\text{no successes}) = 1 - (0.6)^2 = 0.64$$

44. Successes are less likely than failures, so the two are not interchangeable.

$$45. \text{(a) } P(\text{brand } A) = 0.5$$

$$\text{(b) } P(\text{cashews from brand } A) = (0.5)(0.3) = 0.15$$

$$\text{(c) } P(\text{cashew}) = (0.5)(0.3) + (0.5)(0.4) = 0.35$$

$$\text{(d) } P(\text{brand } A/\text{cashew}) = \frac{0.15}{0.35} \approx 0.43$$

$$46. \text{(a) } P(\text{track wet and Mudder Earth wins}) = (0.80)(0.70) = 0.56$$

$$\text{(b) } P(\text{track dry and Mudder Earth wins}) = (0.20)(0.40) = 0.08$$

$$\text{(c) } 0.56 + 0.08 = 0.64$$

$$\text{(d) } P(\text{track wet/Mudder Earth wins}) = \frac{0.56}{0.64} = 0.875$$

For #47–48, substitute  $n = 1, n = 2, \dots, n = 6$ , and  $n = 40$ .

$$47. 0, 1, 2, 3, 4, 5; 39$$

$$48. -1, \frac{4}{3}, -2, \frac{16}{5}, -\frac{16}{3}, \frac{64}{7}; \approx 2.68 \times 10^{10}$$

For #49–54, use previously computed values of the sequence to find the next term in the sequence.

$$49. -1, 2, 5, 8, 11, 14; 32$$

$$50. 5, 10, 20, 40, 80, 160; 10,240$$

$$51. -5, -3.5, -2, -0.5, 1, 2.5; 11.5$$

$$52. 3, 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}; 3^{-10} = \frac{1}{59,049}$$

$$53. -3, 1, -2, -1, -3, -4; -76$$

$$54. -3, 2, -1, 1, 0, 1; 13$$

For #55–62, check for common difference or ratios between successive terms.

$$55. \text{Arithmetic with } d = -2.5;$$

$$a_n = 12 + (-2.5)(n - 1) = 14.5 - 2.5n$$

$$56. \text{Arithmetic with } d = 4;$$

$$a_n = -5 + 4(n - 1) = 4n - 9$$

$$57. \text{Geometric with } r = 1.2;$$

$$a_n = 10 \cdot (1.2)^{n-1}$$

$$58. \text{Geometric with } r = -2;$$

$$a_n = \frac{1}{8} \cdot (-2)^{n-1} = -\frac{1}{16}(-2)^n$$

$$59. \text{Arithmetic with } d = 4.5;$$

$$a_n = -11 + 4.5(n - 1) = 4.5n - 15.5$$

$$60. \text{Geometric with } r = \frac{1}{4};$$

$$b_n = 7 \cdot \left(\frac{1}{4}\right)^{n-1} = 28 \cdot \left(\frac{1}{4}\right)^n$$

61.  $a_n = a_1 r^{n-1}$ , so  $-192 = a_1 r^3$  and  $196,608 = a_1 r^8$ . Then

$$r^5 = -1024, \text{ so } r = -4, \text{ and } a_1 = \frac{-192}{(-4)^3} = 3;$$

$$a_n = 3(-4)^{n-1}$$

62.  $a_n = a_1 + (n-1)d$ , so  $14 = a_1 + 2d$ , and

$$-3.5 = a_1 + 7d. \text{ Then } 5d = -17.5, \text{ so}$$

$$d = -3.5, \text{ and } a_1 = 14 - 2(-3.5) = 21;$$

$$a_n = 21 - 3.5(n-1) = 24.5 - 3.5n.$$

For #63–66, use one of the formulas  $S_n = n\left(\frac{a_1 + a_n}{2}\right)$  or

$$S_n = \frac{n}{2}[2a_1 + (n-1)d]. \text{ In most cases, the first of these is}$$

easier (since the last term  $a_n$  is given); note that

$$n = \frac{a_n - a_1}{d} + 1.$$

63.  $8 \cdot \left(\frac{-11 + 10}{2}\right) = 4 \cdot (-1) = -4$

64.  $7 \cdot \left(\frac{13 - 11}{2}\right) = 7$

65.  $27 \cdot \left(\frac{2.5 - 75.5}{2}\right) = \frac{1}{2} \cdot 27 \cdot (-73) = -985.5$

66.  $31 \cdot \left(\frac{-5 + 55}{2}\right) = 31 \cdot 25 = 775$

For #67–70, use the formula  $S_n = \frac{a_1(1 - r^n)}{1 - r}$ . Note that

$$n = 1 + \log_{|r|} \left| \frac{a_n}{a_1} \right| = 1 + \frac{\ln |a_n/a_1|}{\ln |r|}.$$

67.  $\frac{4(1 - (-1/2)^6)}{1 - (-1/2)} = \frac{21}{8}$

68.  $\frac{-3(1 - (1/3)^5)}{1 - (1/3)} = -\frac{121}{27}$

69.  $\frac{2(1 - 3^{10})}{1 - 3} = 59,048$

70.  $\frac{1(1 - (-2)^{14})}{1 - (-2)} = -5461$

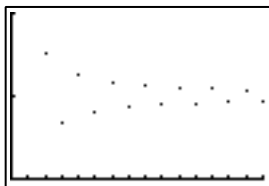
71. Geometric with  $r = \frac{1}{3}$ :

$$S_{10} = \frac{2187(1 - (1/3)^{10})}{1 - (1/3)} = \frac{29,524}{9} = 3280.\bar{4}$$

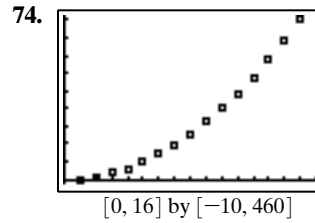
72. Arithmetic with  $d = -3$ :

$$S_{10} = \frac{10}{2}[2(94) + 9(-3)] = 5 \cdot 161 = 805$$

73.



$[0, 15]$  by  $[0, 2]$



75. With  $a_1 = \$150$ ,  $r = 1 + 0.08/12$ , and  $n = 120$ , the sum becomes

$$\frac{\$150 [1 - (1 + 0.08/12)^{120}]}{1 - (1 + 0.08/12)} = \$27,441.91$$

76. The payment amount  $P$  must be such that

$$P \left(1 + \frac{0.08}{12}\right)^0 + P \left(1 + \frac{0.08}{12}\right)^1 + \dots + P \left(1 + \frac{0.08}{12}\right)^{119} \geq \$30,000$$

Using the formula for the sum of a finite geometric series,

$$\frac{P [1 - (1 + 0.08/12)^{120}]}{1 - (1 + 0.08/12)} \geq \$30,000$$

$$\text{or } P \geq \$30,000 \frac{-0.08/12}{1 - (1 + 0.08/12)^{120}}$$

$$\approx \$163.983$$

$$\approx \$163.99 \text{ rounded up}$$

77. Converges: geometric with  $a_1 = \frac{3}{2}$  and  $r = \frac{3}{4}$ , so

$$S = \frac{3/2}{1 - (3/4)} = \frac{3/2}{1/4} = 6$$

78. Converges: geometric with  $a_1 = -\frac{2}{3}$  and  $r = -\frac{1}{3}$ , so

$$S = \frac{-2/3}{1 - (-1/3)} = \frac{-2/3}{4/3} = -\frac{1}{2}$$

79. Diverges: geometric with  $r = -\frac{4}{3}$

80. Diverges: geometric with  $r = \frac{6}{5}$

81. Converges: geometric with  $a_1 = 1.5$  and  $r = 0.5$ , so

$$S = \frac{1.5}{1 - 0.5} = \frac{1.5}{0.5} = 3$$

82. Diverges; geometric with  $r = 1.2$

83.  $\sum_{k=1}^{21} [-8 + 5(k-1)] = \sum_{k=1}^{21} (5k - 13)$

84.  $\sum_{k=1}^{10} 4(-2)^{k-1} = \sum_{k=1}^{10} (-2)^{k+1}$

85.  $\sum_{k=0}^{\infty} (2k+1)^2$  or  $\sum_{k=1}^{\infty} (2k-1)^2$

86.  $\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$  or  $\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{k-1}$

87.  $\sum_{k=1}^n (3k+1) = 3 \sum_{k=1}^n k + \sum_{k=1}^n 1$

$$= 3 \cdot \frac{n(n+1)}{2} + n = \frac{3n^2 + 5n}{2} = \frac{n(3n+5)}{2}$$

88.  $\sum_{k=1}^n 3k^2 = 3 \sum_{k=1}^n k^2 = 3 \cdot \frac{n(n+1)(2n+1)}{6}$

$$= \frac{n(n+1)(2n+1)}{2}$$

89.  $\sum_{k=1}^{25} (k^2 - 3k + 4) = \frac{25 \cdot 26 \cdot 51}{6} - 3 \cdot \frac{25 \cdot 26}{2} + 4 \cdot 25 = 4650$

90.  $\sum_{k=1}^{175} (3k^2 - 5k + 1) = 3 \cdot \frac{175 \cdot 176 \cdot 351}{6} - 5 \cdot \frac{175 \cdot 176}{2} + 175 = 5,328,575$

91.  $P_n: 1 + 3 + 6 + \dots + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}$ .

$P_1$  is true:  $\frac{1(1+1)}{2} = \frac{1(1+1)(1+2)}{6}$ .

Now assume  $P_k$  is true:  $1 + 3 + 6 + \dots + \frac{k(k+1)}{2} = \frac{k(k+1)(k+2)}{6}$ . Add  $\frac{(k+1)(k+2)}{2}$  to both sides:

$$1 + 3 + 6 + \dots + \frac{k(k+1)}{2} + \frac{(k+1)(k+2)}{2} = \frac{k(k+1)(k+2)}{6} + \frac{(k+1)(k+2)}{2}$$

$$= (k+1)(k+2) \left( \frac{k}{6} + \frac{1}{2} \right) = (k+1)(k+2) \left( \frac{k+3}{6} \right) = \frac{(k+1)((k+1)+1)((k+1)+2)}{6}$$

so  $P_{k+1}$  is true. Therefore,  $P_n$  is true for all  $n \geq 1$ .

92.  $P_n: 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + n(n+1) = \frac{n(n+1)(n+2)}{3}$ .  $P_1$  is true:

$$1(1+1) = \frac{1(1+1)(1+2)}{3}$$

Now assume  $P_k$  is true:  $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + k(k+1) = \frac{k(k+1)(k+2)}{3}$ .

Add  $(k+1)(k+2)$  to both sides:

$$1 \cdot 2 + 2 \cdot 3 + \dots + k(k+1) + (k+1)(k+2) = \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) = (k+1)(k+2) \left( \frac{k}{3} + 1 \right) = (k+1)(k+2) \left( \frac{k+3}{3} \right) = \frac{(k+1)((k+1)+1)((k+1)+2)}{3}$$

so  $P_k$  is true. Therefore,  $P_n$  is true for all  $n \geq 1$ .

93.  $P_n: 2^{n-1} \leq n!$ .  $P_1$  is true: it says that  $2^{1-1} \leq 1!$  (they are equal). Now assume  $P_k$  is true:  $2^{k-1} \leq k!$ . Then  $2^{k+1-1} = 2 \cdot 2^{k-1} \leq 2 \cdot k! \leq (k+1)k! = (k+1)!$ , so  $P_{k+1}$  is true. Therefore,  $P_n$  is true for all  $n \geq 1$ .

94.  $P_n: n^3 + 2n$  is divisible by 3.  $P_1$  is true because  $1^3 + 2 \cdot 1 = 3$  is divisible by 3. Now assume  $P_k$  is true:  $k^3 + 2k$  is divisible by 3. Then note that  $(k+1)^3 + 2(k+1) = (k^3 + 3k^2 + 3k + 1) + (2k + 2) = (k^3 + 2k) + 3(k^2 + k + 1)$ .

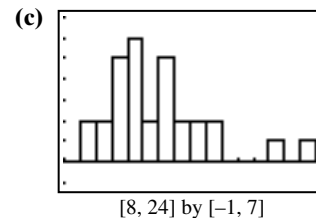
Since both terms are divisible by 3, so is the sum, so  $P_{k+1}$  is true. Therefore,  $P_n$  is true for all  $n \geq 1$ .

95. (a)

9	1 2
10	6 7
11	4 5 5 7 7
12	0 2 4 6 7 7
13	5 6
14	1 6 7 7 8
15	4 8
16	1 4
17	0 6
18	
19	
20	
21	9
22	
23	4

(b)

Price	Frequency
90,000–99,999	2
100,000–109,999	2
110,000–119,999	5
120,000–129,999	6
130,000–139,999	2
140,000–149,999	5
150,000–159,999	2
160,000–169,999	2
170,000–179,999	2
210,000–219,999	1
230,000–239,999	1

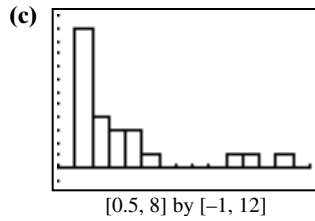


96. (a)

1	1 1 1 1 1 2 2 3 3 4 4
1	5 9 9 9
2	0 0 0
2	6 8 9
3	3
4	
4	
5	
5	8
6	1
6	
7	1

(b)

Number of Visitors	Frequency
10,000,000–14,999,999	11
15,000,000–19,999,999	4
20,000,000–24,999,999	3
25,000,000–29,999,999	3
30,000,000–34,999,999	1
35,000,000–39,999,999	0
40,000,000–44,999,999	0
45,000,000–49,999,999	0
55,000,000–59,999,999	1
60,000,000–64,999,999	1
65,000,000–69,999,999	0
70,000,000–74,999,999	1

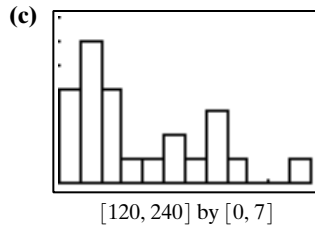


97. (a)

12	0 0 4 4
13	1 1 2 6 7 9
14	0 3 4 8
15	6
16	3
17	7 9
18	0
19	0 1 7
20	2
21	
22	0
23	0

(b)

Length (in seconds)	Frequency
120—129	4
130—139	6
140—149	4
150—159	1
160—169	1
170—179	2
180—189	1
190—199	3
200—209	1
210—219	0
220—229	0
230—239	1

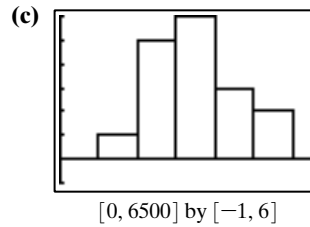


98. (a) For the stemplot (but not for the other frequency table or histogram), round to nearest hundred yards first:

Stem	Leaf
1	1
2	3 4 5 7 8
3	1 3 5 5 6
4	0 2 3 7 7
5	0 6

(b)

Yardage	Frequency
1000—1999	1
2000—2999	5
3000—3999	6
4000—4999	3
5000—5999	2



99. The ordered data is {9.1, 9.2, 10.6, 10.7, 11.4, 11.5, 11.5, 11.7, 11.7, 12, 12.2, 12.4, 12.6, 12.7, 12.7, 13.5, 13.6, 14.1, 14.6, 14.7, 14.7, 14.8, 15.4, 15.8, 16.1, 16.4, 17.0, 17.6, 21.9, 23.4} so the

median is  $\frac{12.7 + 13.5}{2} = 13.1$ ,  $Q_3 = 15.4$ , and  $Q_1 = 11.7$ .

Five-number summary: {9.1, 11.7, 13.1, 15.4, 23.4}  
 Range:  $23.4 - 9.1 = 14.3$  (\$143,000)  
 IQR:  $15.4 - 11.7 = 3.7$  (\$37,000)  
 $\sigma \approx 3.19$ ,  $\sigma^2 \approx 10.14$   
 Outliers: 21.9 and 23.4 are greater than  $15.4 + 1.5(3.7) = 20.95$ .

100. The ordered data is {1.1, 1.1, 1.1, 1.1, 1.1, 1.2, 1.2, 1.3, 1.3, 1.4, 1.4, 1.5, 1.9, 1.9, 1.9, 2.0, 2.0, 2.0, 2.6, 2.8, 2.9, 3.3, 5.8, 6.1, 7.1} so the median is 1.9,

$Q_3 = \frac{2.6 + 2.8}{2} = 2.7$  and  $Q_1 = 1.2$ .

Five-number summary: {1.1, 1.2, 1.9, 2.7, 7.1}  
 Range:  $7.1 - 1.1 = 6.0$  (6 million)  
 IQR:  $2.7 - 1.2 = 1.5$  (1.5 million)  
 $\sigma \approx 1.63$ ,  $\sigma^2 \approx 2.64$   
 Outliers: 5.8, 6.1, and 7.1 are greater than  $2.7 + 1.5(1.5) = 4.95$ .

101. The ordered data is {120, 120, 124, 124, 131, 131, 132, 136, 137, 139, 140, 143, 144, 148, 156, 163, 177, 179, 180, 190, 191, 197, 202, 230} so the median is

$\frac{143 + 144}{2} = 143.5$ ,  $Q_3 = \frac{179 + 180}{2} = 179.5$ , and

$Q_1 = \frac{131 + 132}{2} = 131.5$ .

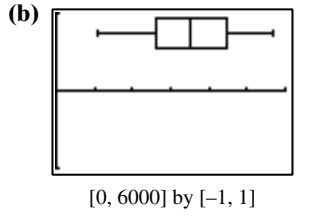
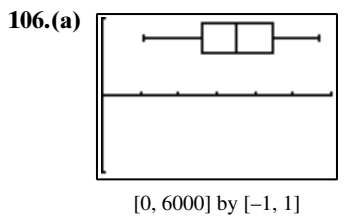
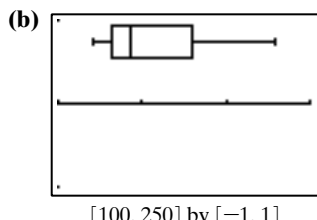
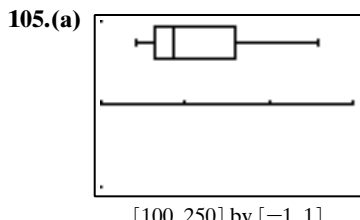
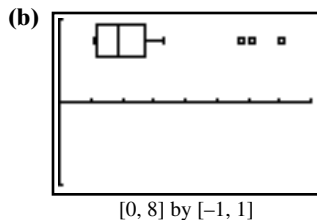
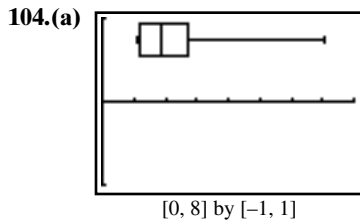
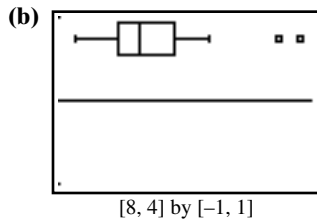
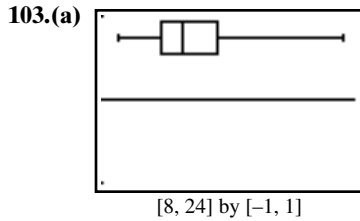
Five-number summary: {120, 131.5, 143.5, 179.5, 230}  
 Range:  $230 - 120 = 110$   
 IQR:  $179.5 - 131.5 = 48$   
 $\sigma = 29.9$ ,  $\sigma^2 = 891.4$   
 No Outliers.

102. The ordered data is: {1112, 2327, 2382, 2521, 2709, 2806, 3127, 3338, 3485, 3489, 3631, 3959, 4228, 4264, 4689, 4690, 5000, 5648},

so the median is  $\frac{3485 + 3489}{2} = 3487$ ,

$Q_3 = 4264$  and  $Q_1 = 2709$ .

Five-number summary: {1112, 2709, 3487, 4264, 5648}  
 Range:  $5648 - 1112 = 4536$   
 IQR:  $4264 - 2709 = 1555$   
 $\sigma \approx 1095$ ,  $\sigma^2 \approx 1,199,223$   
 No Outliers.

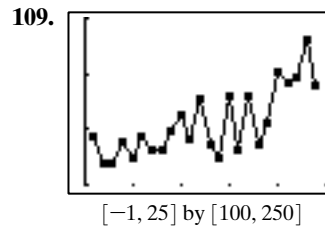
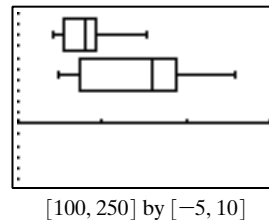


**107.**

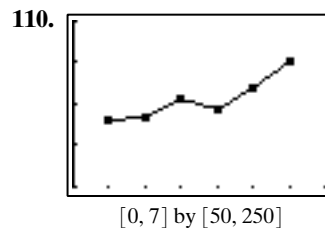
4	0	0	12	4
9	2	1	13	1 6 7
8	4	3	14	
			15	6
		3	16	
		7	17	9
			18	0
			19	0 1 7
			20	2
			21	
			22	
			23	0

The songs released in the earlier years tended to be shorter.

**108.** Earlier years are in the upper box plot. The range and interquartile range are both greater in the lower graph, which shows the times for later years.



Again, the data demonstrates that songs appearing later tended to be longer in length.



The average times are {130.5, 132.75, 157, 142.5, 168.75, 202}. The trend is clearly increasing overall, with less fluctuation than the time plot for Exercise 105.

**111.** 1 9 36 84 126 84 36 9 1

**112.** 
$${}_n P_k \times {}_{n-k} P_j = \frac{n!}{(n-k)!} \frac{(n-k)!}{[(n-k)-j]!}$$

$$= \frac{n!}{(n-k-j)!}$$

$$= \frac{n!}{[n-(k+j)]!} = {}_n P_{k+j}$$

**113. (a)**  $P(\text{no defective bats}) = (0.98)^4 \approx 0.922$

**(b)**  $P(\text{one defective bat}) = {}_4 C_1 \cdot (0.98)^3(0.02)$   
 $= 4(0.98)^3(0.02) \approx 0.075$

**114. (a)**  $P(\text{no defective light bulbs}) = (0.9996)^{10} \approx 0.996$

**(b)**  $P(\text{two defective light bulbs}) = {}_{10} C_2 \cdot (0.9996)^8(0.0004)^2 \approx 7.18 \times 10^{-6}$

**Chapter 9 Project**

Answers are based on the sample data shown in the table.

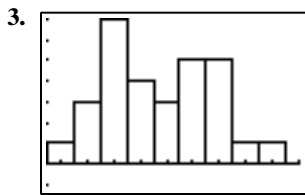
**1. Stem Leaf**

5		
5		9
6		1 1 2 3 3 3 4 4 4 4
6		5 6 6 6 7 8 8 9 9
7		0 0 1 1 1 2 2 3
7		5

The average is about 66 or 67 inches.

- 2.** A large number of students are between 63 and 64 inches and also between 69 and 72 inches.

Height	Frequency
59–60	1
61–62	3
63–64	7
65–66	4
67–68	3
69–70	5
71–72	5
73–74	1
75–76	1



[59, 78] by [-1, 7]

Again, the average appears to be about 66 or 67 inches. Since the data are not broken out by gender, one can only speculate about average heights for males and females separately. Possibly the two peaks within the distribution represent an average height of 63–64 inches for females and 70–71 inches for males.

- 4.** Mean = 66.9 in.; median = 66.5 in.; mode = 64 in.  
The mean and median both appear to be good measures of the average, but the mode is too low. Still, the mode might well be similar in other classes.
- 5.** The data set is well distributed and probably does not have outliers.

- 6.** The stem and leaf plot puts the data in order.

Minimum value: 59

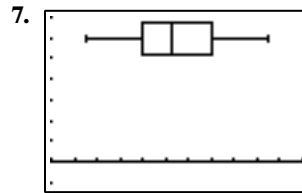
Maximum value: 75

$$\text{Median: } \frac{66 + 67}{2} = 66.5$$

$Q_1$ : 64

$Q_3$ : 70

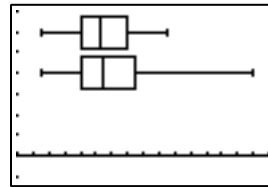
The five-number summary is {59, 64, 66.5, 70, 75}.



[56, 78] by [-1, 7]

The boxplot visually represents the five-number summary. The whisker-to-whisker size of the boxplot represents the range of the data, while the width of the box represents the interquartile range.

- 8.** Mean = 67.5; median = 67; The new five-number summary is {59, 64, 67, 71, 86}.



[56, 88] by [-1, 7]

The minimum and first quartile are unaffected, but the median, third quartile, and maximum are shifted upon to varying degrees.

- 9.** The new student's height, 86 inches, lies 15 inches away from  $Q_3$ , and that is more than  $1.5(Q_3 - Q_1) = 10.5$ . The height of the new student should probably be tossed out during prediction calculations.
- 10.** Mean = 68.9; median = 68; The new five-number summary is {59, 64, 68, 71, 86}. All three of the new students are outliers by the  $1.5 \times IQR$  test. Their heights should be left out during prediction-making.