Chapter 5
Analytic Trigonometry

■ Section 5.1 Fundamental Identities

Exploration 1

1. $\cos\theta = 1/\sec\theta$, $\sec\theta = 1/\cos\theta$, and $\tan\theta = \sin\theta/\cos\theta$
2. $\sin\theta = 1/\csc\theta$ and $\tan\theta = 1/\cot\theta$
3. $\csc\theta = 1/\sin\theta$, $\cot\theta = 1/\tan\theta$, and $\cot\theta = \cos\theta/\sin\theta$

Quick Review 5.1

For #1–4, use a calculator.

1. $1.1760 \text{ rad} = 67.380°$
2. $0.9273 \text{ rad} = 53.130°$
3. $2.4981 \text{ rad} = 143.130°$
4. $-0.3948 \text{ rad} = -22.620°$
5. $a^2 - 2ab + b^2 = (a-b)^2$
6. $4u^2 + 4u + 1 = (2u+1)^2$
7. $2x^2 - 3xy - 2y^2 = (2x+y)(x-2y)$
8. $2v^2 - 5v - 3 = (2v+1)(v-3)$
9. $\frac{1}{x}\cdot\frac{y}{y} - \frac{2}{y}\cdot\frac{x}{x} = \frac{y-2x}{xy}$
10. $\frac{a}{x}\cdot\frac{y}{y} + \frac{b}{y}\cdot\frac{x}{x} = \frac{ay+bx}{xy}$
11. $\frac{x+y}{\frac{1}{x}+\frac{1}{y}} = (x+y)\cdot\left(\frac{xy}{x+y}\right) = xy$
12. $\frac{x}{x-y}\cdot\frac{x+y}{x+y} - \frac{y}{x+y}\cdot\frac{x-y}{x-y} = \frac{x^2+y^2}{x^2-y^2}$

Section 5.1 Exercises

1. $\sec^2\theta = 1 + \tan^2\theta = 1 + (3/4)^2 = 25/16$, so $\sec\theta = \pm 5/4$. Then $\cos\theta = 1/\sec\theta = \pm 4/5$. But $\sin\theta, \tan\theta > 0$ implies $\cos\theta > 0$. So $\cos\theta = 4/5$. Finally,
$$\tan\theta = \frac{3}{4}$$
$$\frac{\sin\theta}{\cos\theta} = \frac{3}{4}$$
$$\sin\theta = \frac{3}{4}\cos\theta = \frac{3}{4}\left(\frac{4}{5}\right) = \frac{3}{5}.$$
2. $\sec^2\theta = 1 + \tan^2\theta = 1 + 3^2 = 10$, so $\sec\theta = \pm\sqrt{10}$. But $\cos\theta > 0$ implies $\sec\theta > 0$, so $\sec\theta = \sqrt{10}$. Finally,
$$\tan\theta = 3$$
$$\frac{\sec\theta}{\csc\theta} = 3$$
$$\csc\theta = \frac{1}{3}\sec\theta = \frac{1}{3}\sqrt{10} = \frac{\sqrt{10}}{3}.$$
3. $\tan^2\theta = \sec^2\theta - 1 = 4^2 - 1 = 15$, so $\tan\theta = \pm\sqrt{15}$. But $\sec\theta > 0, \sin\theta < 0$ implies $\tan\theta < 0$, so $\tan\theta = -\sqrt{15}$. And $\cot\theta = 1/\tan\theta = -1/\sqrt{15} = -\sqrt{15}/15$.
4. $\sin^2\theta = 1 - \cos^2\theta = 1 - (0.8)^2 = 0.36$, so $\sin\theta = \pm 0.6$. But $\cos\theta > 0$, $\tan\theta < 0$ implies $\sin\theta < 0$, so $\sin\theta = -0.6$. Finally, $\tan\theta = \sin\theta/\cos\theta = -0.6/0.8 = -0.75$.
5. $\cos(\pi/2 - \theta) = \sin\theta = 0.45$
6. $\cot\theta = \tan(\pi/2 - \theta) = -5.32$
7. $\cos(-\theta) = \cos\theta = \sin(\pi/2 - \theta) = -\sin(\theta - \pi/2) = -0.73$
8. $\cot(-\theta) = -\cot\theta = -\tan(\pi/2 - \theta) = \tan(\theta - \pi/2) = 7.89$
9. $\tan x\cos x = \frac{\sin x}{\cos x}\cdot\cos x = \sin x$
10. $\cot x\tan x = \frac{\cos x}{\sin x}\cdot\frac{\sin x}{\cos x} = 1$
11. $\sec y\sin\left(\frac{\pi}{2} - y\right) = \frac{1}{\cos y}\cdot\cos y = 1$
12. $\cot u\sin u = \frac{\cos u}{\sin u}\cdot\sin u = \cos u$
13. $\frac{1+\tan^2 x}{\csc^2 x} = \frac{\sec^2 x}{\csc^2 x} = \frac{1/\cos^2 x}{1/\sin^2 x} = \frac{\sin^2 x}{\cos^2 x} = \tan^2 x$
14. $\frac{1-\cos^2\theta}{\sin\theta} = \frac{\sin^2\theta}{\sin\theta} = \sin\theta$
15. $\cos x - \cos^3 x = \cos x(1 - \cos^2 x) = \cos x\sin^2 x$
16. $\frac{\sin^2 u + \tan^2 u + \cos^2 u}{\sec u} = \frac{1+\tan^2 u}{\sec u} = \frac{\sec^2 u}{\sec u} = \sec u$
17. $\sin x\csc(-x) = \sin x\cdot\frac{1}{\sin(-x)} = -1$
18. $\sec(-x)\cos(-x) = \frac{1}{\cos(-x)}\cdot\cos(-x) = 1$
19. $\cot(-x)\cot\left(\frac{\pi}{2} - x\right) = \frac{\cos(-x)}{\sin(-x)}\cdot\frac{\cos\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)}$
$= \frac{\cos(-x)}{\sin(-x)}\cdot\frac{\sin(x)}{\cos(x)} = -1$
20. $\cot(-x)\tan(-x) = \frac{\cos(-x)}{\sin(-x)}\cdot\frac{\sin(-x)}{\cos(-x)} = 1$
21. $\sin^2(-x) + \cos^2(-x) = 1$
22. $\sec^2(-x) - \tan^2 x = \sec^2 x - \tan^2 x = 1$
23. $\frac{\tan\left(\frac{\pi}{2}-x\right)\csc x}{\csc^2 x} = \frac{\cot x}{\csc x} = \frac{\cos x}{\sin x}\cdot\frac{\sin x}{1} = \cos x$

24. $\dfrac{1+\tan x}{1+\cot x}\cdot\dfrac{\sin x\cos x}{\sin x\cos x}=\dfrac{\sin x\cos x+\sin^2 x}{\sin x\cos x+\cos^2 x}$

$=\dfrac{\sin x(\cos x+\sin x)}{\cos x(\sin x+\cos x)}=\tan x$

25. $(\sec^2 x+\csc^2 x)-(\tan^2 x+\cot^2 x)$

$=(\sec^2 x-\tan^2 x)+(\csc^2 x-\cot^2 x)=1+1=2$

26. $\dfrac{\sec^2 u-\tan^2 u}{\cos^2 v+\sin^2 v}=\dfrac{1}{1}=1$

27. $(\sin x)(\tan x+\cot x)=(\sin x)\left(\dfrac{\sin x}{\cos x}+\dfrac{\cos x}{\sin x}\right)$

$=\sin x\left(\dfrac{\sin^2 x+\cos^2 x}{(\cos x)(\sin x)}\right)=\dfrac{1}{\cos x}=\sec x$

28. $\sin\theta-\tan\theta\cos\theta+\cos\left(\dfrac{\pi}{2}-\theta\right)$

$=\sin\theta-\dfrac{\sin\theta}{\cos\theta}\cdot\cos\theta+\sin\theta=\sin\theta$

29. $(\sin x)(\cos x)(\tan x)(\sec x)(\csc x)$

$=(\sin x)(\cos x)\left(\dfrac{\sin x}{\cos x}\right)\left(\dfrac{1}{\cos x}\right)\left(\dfrac{1}{\sin x}\right)=\dfrac{\sin x}{\cos x}$

$=\tan x$

30. $\dfrac{(\sec y-\tan y)(\sec y+\tan y)}{\sec y}$

$=\dfrac{\left(\dfrac{1}{\cos y}-\dfrac{\sin y}{\cos y}\right)\left(\dfrac{1}{\cos y}+\dfrac{\sin y}{\cos y}\right)}{\left(\dfrac{1}{\cos y}\right)}$

$=\dfrac{1+\sin y-\sin y-\sin^2 y}{\cos^2 y}\cdot\dfrac{\cos y}{1}=\dfrac{1-\sin^2 y}{\cos y}$

$=\dfrac{\cos^2 y}{\cos y}=\cos y$

31. $\dfrac{\tan x}{\csc^2 x}+\dfrac{\tan x}{\sec^2 x}$

$=\left(\dfrac{\sin x}{\cos x}\right)(\sin^2 x)+\left(\dfrac{\sin x}{\cos x}\right)\cdot\cos^2 x$

$=\left(\dfrac{\sin x}{\cos x}\right)(\sin^2 x+\cos^2 x)=\dfrac{\sin x}{\cos x}=\tan x.$

32. $\dfrac{\sec^2 x\csc x}{\sec^2 x+\csc^2 x}=\dfrac{\left(\dfrac{1}{\cos^2 x}\cdot\dfrac{1}{\sin x}\right)}{\left(\dfrac{1}{\cos^2 x}\right)+\left(\dfrac{1}{\sin^2 x}\right)}$

$=\dfrac{1}{\cos^2 x\cdot\sin x}\cdot\dfrac{\cos^2 x\sin^2 x}{\sin^2 x+\cos^2 x}=\dfrac{\sin x}{1}=\sin x$

33. $\dfrac{1}{\sin^2 x}+\dfrac{\sec^2 x}{\tan^2 x}=\csc^2 x+\dfrac{1}{\cos^2 x\left(\dfrac{\sin^2 x}{\cos^2 x}\right)}$

$=\csc^2 x+\dfrac{1}{\sin^2 x}=2\csc^2 x$

34. $\dfrac{1}{1-\sin x}+\dfrac{1}{1+\sin x}$

$=\dfrac{1+\sin x}{(1-\sin x)(1+\sin x)}+\dfrac{1-\sin x}{(1-\sin x)(1+\sin x)}$

$=\dfrac{2}{1-\sin^2 x}=\dfrac{2}{\cos^2 x}=2\sec^2 x$

35. $\dfrac{\sin x}{\cot^2 x}-\dfrac{\sin x}{\cos^2 x}=(\sin x)(\tan^2 x)-(\sin x)(\sec^2 x)$

$=(\sin x)(\tan^2 x-\sec^2 x)=(\sin x)(-1)=-\sin x$

36. $\dfrac{1}{\sec x-1}-\dfrac{1}{\sec x+1}=\dfrac{\sec x+1-\sec x+1}{\sec^2 x-1}$

$=\dfrac{2}{\tan^2 x}=2\cot^2 x$

37. $\dfrac{\sec x}{\sin x}-\dfrac{\sin x}{\cos x}=\dfrac{\sec x\cos x-\sin^2 x}{\sin x\cos x}=\dfrac{1-\sin^2 x}{\sin x\cos x}$

$=\dfrac{\cos^2 x}{\sin x\cos x}=\dfrac{\cos x}{\sin x}=\cot x$

38. $\dfrac{\sin x}{1-\cos x}+\dfrac{1-\cos x}{\sin x}=\dfrac{\sin^2 x+(1-\cos x)^2}{\sin x(1-\cos x)}$

$=\dfrac{\sin^2 x+\cos x^2+1-2\cos x}{\sin x(1-\cos x)}=\dfrac{2(1-\cos x)}{\sin x(1-\cos x)}$

$=2\csc x$

39. $\cos^2 x+2\cos x+1=(\cos x+1)^2$

40. $1-2\sin x+\sin^2 x=(1-\sin x)^2$

41. $1-2\sin x+(1-\cos^2 x)=1-2\sin x+\sin^2 x$

$=(1-\sin x)^2$

42. $\sin x-\cos^2 x-1=\sin x+\sin^2 x-2$

$=(\sin x-1)(\sin x+2)$

43. $\cos x-2\sin^2 x+1=\cos x-2+2\cos^2 x+1$

$=2\cos^2 x+\cos x-1=(2\cos x-1)(\cos x+1)$

44. $\sin^2 x+\dfrac{2}{\csc x}+1=\sin^2 x+2\sin x+1$

$=(\sin x+1)^2$

45. $4\tan^2 x-\dfrac{4}{\cot x}+\sin x\csc x$

$=4\tan^2 x-4\tan x+\sin x\cdot\dfrac{1}{\sin x}$

$=4\tan^2 x-4\tan x+1=(2\tan x-1)^2$

46. $\sec^2 x-\sec x+\tan^2 x=\sec^2 x-\sec x+\sec^2 x-1$

$=2\sec^2 x-\sec x-1=(2\sec x+1)(\sec x-1)$

47. $\dfrac{1-\sin^2 x}{1+\sin x}=\dfrac{(1-\sin x)(1+\sin x)}{1+\sin x}=1-\sin x$

48. $\dfrac{\tan^2\alpha-1}{1+\tan\alpha}=\dfrac{(\tan\alpha-1)(\tan\alpha+1)}{1+\tan\alpha}=\tan\alpha-1$

49. $\dfrac{\sin^2 x}{1+\cos x}=\dfrac{1-\cos^2 x}{1+\cos x}=\dfrac{(1-\cos x)(1+\cos x)}{1+\cos x}$

$=1-\cos x$

50. $\dfrac{\tan^2 x}{\sec x+1}=\dfrac{\sec^2 x-1}{\sec x+1}=\dfrac{(\sec x-1)(\sec x+1)}{\sec x+1}$

$=\sec x-1$

51. $(\cos x)(2\sin x-1)=0$, so either $\cos x=0$ or $\sin x=\dfrac{1}{2}$. Then $x=\dfrac{\pi}{2}+n\pi$ or $x=\dfrac{\pi}{6}+2n\pi$ or $x=\dfrac{5\pi}{6}+2n\pi$, n an integer. On the interval:

$x=\left\{\dfrac{\pi}{6},\dfrac{\pi}{2},\dfrac{5\pi}{6},\dfrac{3\pi}{2}\right\}$

52. $(\tan x)(\sqrt{2}\cos x - 1) = 0$, so either $\tan x = 0$ or $\cos x = \frac{1}{\sqrt{2}}$. Then $x = n\pi$ or $x = \pm\frac{\pi}{4} + 2n\pi$, n an integer. On the interval: $x = \left\{0, \frac{\pi}{4}, \pi, \frac{7\pi}{4}\right\}$

53. $(\tan x)(\sin^2 x - 1) = 0$, so either $\tan x = 0$ or $\sin^2 x = 1$. Then $x = n\pi$ or $x = \frac{\pi}{2} + n\pi$, n an interger. However, $\tan x$ excludes $x = \frac{\pi}{2} + n\pi$, so we have only $x = n\pi$, n an integer. On the interval: $x = \{0, \pi\}$

54. $(\sin x)(\tan^2 x - 1) = 0$, so either $\sin x = 0$ or $\tan^2 x = 1$. Then $x = n\pi$ or $x = \frac{\pi}{4} + n\frac{\pi}{2}$, n an integer. Put another way, all multiples of $\frac{\pi}{4}$ except for $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$, etc. On the interval: $x = \left\{0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$

55. $\tan x = \pm\sqrt{3}$, so $x = \pm\frac{\pi}{3} + n\pi$, n an integer. On the interval: $x = \left\{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}$

56. $\sin x = \pm\frac{1}{\sqrt{2}}$, so $x = \frac{\pi}{4} + n\frac{\pi}{2}$, n an integer. On the interval: $x = \left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$

57. $(2\cos x - 1)^2 = 0$, so $\cos x = \frac{1}{2}$; therefore $x = \pm\frac{\pi}{3} + 2n\pi$, n an integer.

58. $(2\sin x + 1)(\sin x + 1) = 0$, so $\sin x = -\frac{1}{2}$ or $\sin x = -1$. Then $x = -\frac{\pi}{6} + 2n\pi$, $x = -\frac{5\pi}{6} + 2n\pi$ or $x = -\frac{\pi}{2} + 2n\pi$, n an integer.

59. $(\sin\theta)(\sin\theta - 2) = 0$, so $\sin\theta = 0$ or $\sin\theta = 2$. Then $\theta = n\pi$, n an integer.

60. $3\sin t = 2 - 2\sin^2 t$, or $2\sin^2 t + 3\sin t - 2 = 0$. This factors to $(2\sin t - 1)(\sin t + 2) = 0$, so $\sin t = \frac{1}{2}$ or $\sin t = -2$. Then $t = \frac{\pi}{6} + 2n\pi$ or $t = \frac{5\pi}{6} + 2n\pi$, n an integer.

61. $\cos(\sin x) = 1$ if $\sin x = n\pi$. Only $n = 0$ gives a value between -1 and $+1$, so $\sin x = 0$, or $x = n\pi$, n an integer.

62. This can be rewritten as $(2\sin x - 1)(\sin x + 2) = 0$, so $\sin x = \frac{1}{2}$ or $\sin x = -2$. Then $x = \frac{\pi}{6} + 2n\pi$ or $x = \frac{5\pi}{6} + 2n\pi$, n an integer. See also #60.

63. $\cos^{-1} 0.37 \approx 1.1918$, so the solution set is $\{\pm 1.1918 + 2n\pi \mid n = 0, \pm 1, \pm 2, \ldots\}$.

64. $\cos^{-1} 0.75 \approx 0.7227$, so the solution set is $\{\pm 0.7227 + 2n\pi \mid n = 0, \pm 1, \pm 2, \ldots\}$.

65. $\sin^{-1} 0.30 \approx 0.3047$ and $\pi - 0.3047 \approx 2.8369$, so the solution set is $\{0.3047 + 2n\pi$ or $2.8369 + 2n\pi \mid n = 0, \pm 1, \pm 2, \ldots\}$.

66. $\tan^{-1} 5 \approx 1.3734$, so the solution set is $\{1.3734 + n\pi \mid n = 0, \pm 1, \pm 2, \ldots\}$.

67. $\sqrt{0.4} \approx 0.63246$, and $\cos^{-1} 0.63246 \approx 0.8861$, so the solution set is $\{\pm 0.8861 + n\pi \mid n = 0, \pm 1, \pm 2, \ldots\}$.

68. $\sqrt{0.4} \approx 0.63246$ and $\sin^{-1} 0.63246 \approx 0.6847$, so the solution set is $\{\pm 0.6847 + n\pi \mid n = 0, \pm 1, \pm 2, \ldots\}$.

69. $\sqrt{1 - \cos^2\theta} = |\sin\theta|$

70. $\sqrt{\tan^2\theta + 1} = |\sec\theta|$

71. $\sqrt{9\sec^2\theta - 9} = 3|\tan\theta|$

72. $\sqrt{36 - 36\sin^2\theta} = 6|\cos\theta|$

73. $\sqrt{81\tan^2\theta + 81} = 9|\sec\theta|$

74. $\sqrt{100\sec^2\theta - 100} = 10|\tan\theta|$

75. True. Since cosine is an even function, so is secant, and thus $\sec(x - \pi/2) = \sec(\pi/2 - x)$, which equals $\csc x$ by one of the cofunction identities.

76. False. The domain of validity does not include values of θ for which $\cos\theta = 0$ and $\tan\theta = \sin\theta/\cos\theta$ is undefined, namely all odd integer multiples of $\pi/2$.

77. $\tan x \sec x = \tan x/\cos x = \sin x/\cos^2 x \neq \sin x$. The answer is D.

78. sine, tangent, cosecant, and cotangent are odd, while cosine and secant are even. The answer is A.

79. $(\sec\theta + 1)(\sec\theta - 1) = \sec^2\theta - 1 = \tan^2\theta$. The answer is C.

80. By the quadratic formula, $3\cos^2 x + \cos x - 2 = 0$ implies

$$\cos x = \frac{-1 \pm \sqrt{1 - 4(3)(-2)}}{2(3)} = -1 \text{ or } \frac{2}{3}$$

There are three solutions on the interval $(0, 2\pi)$. The answer is D.

81. $\sin x$, $\cos x = \pm\sqrt{1 - \sin^2 x}$, $\tan x = \pm\frac{\sin x}{\sqrt{1 - \sin^2 x}}$,

$\csc x = \frac{1}{\sin x}$, $\sec x = \pm\frac{1}{\sqrt{1 - \sin^2 x}}$

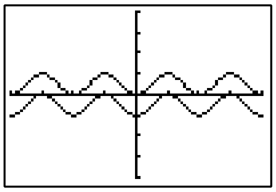
$\cot x = \pm\frac{\sqrt{1 - \sin^2 x}}{\sin x}$

82. $\sin x = \pm\sqrt{1 - \cos^2 x}$, $\cos x$, $\tan x = \pm\frac{\sqrt{1 - \cos^2 x}}{\cos x}$,

$\csc x = \pm\frac{1}{\sqrt{1 - \cos^2 x}}$, $\sec x = \frac{1}{\cos x}$,

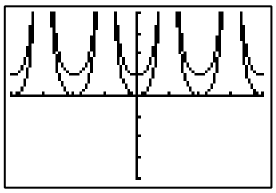
$\cot x = \pm\frac{\cos x}{\sqrt{1 - \cos^2 x}}$

83. The two functions are parallel to each other, separated by 1 unit for every x. At any x, the distance between the two graphs is $\sin^2 x - (-\cos^2 x) = \sin^2 x + \cos^2 x = 1$.



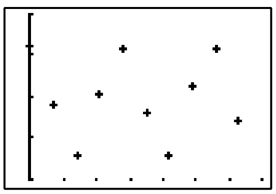
$[-2\pi, 2\pi]$ by $[-4, 4]$

84. The two functions are parallel to each other, separated by 1 unit for every x. At any x, the distance between the two graphs is $\sec^2 x - \tan^2 x = 1$.



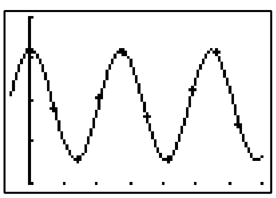
$[-2\pi, 2\pi]$ by $[-4, 4]$

85. **(a)**



$[-6, 70]$ by $[220\,000, 260\,000]$

(b) The equation is
$y = 13{,}111 \sin(0.22997x + 1.571) + 238{,}855$.



$[-6, 70]$ by $[220\,000, 260\,000]$

(c) $(2\pi)/0.22998 \approx 27.32$ days. This is the number of days that it takes the Moon to make one complete orbit of the Earth (known as the Moon's sidereal period).

(d) 225,744 miles

(e) $y = 13{,}111 \cos(-0.22997x) + 238{,}855$, or
$y = 13{,}111 \cos(0.22997x) + 238{,}855$.

86. Answers will vary.

87. Factor the left-hand side:

$$\begin{aligned}\sin^4\theta - \cos^4\theta &= (\sin^2\theta - \cos^2\theta)(\sin^2\theta + \cos^2\theta)\\ &= (\sin^2\theta - \cos^2\theta)\cdot 1\\ &= \sin^2\theta - \cos^2\theta\end{aligned}$$

88. Any k satisfying $k \geq 2$ or $k \leq -2$.

89. Use the hint:

$$\begin{aligned}\sin(\pi - x) &= \sin(\pi/2 - (x - \pi/2))\\ &= \cos(x - \pi/2) && \text{Cofunction identity}\\ &= \cos(\pi/2 - x) && \text{Since cos is even}\\ &= \sin x && \text{Cofunction identity}\end{aligned}$$

90. Use the hint:

$$\begin{aligned}\cos(\pi - x) &= \cos(\pi/2 - (x - \pi/2))\\ &= \sin(x - \pi/2) && \text{Cofunction identity}\\ &= -\sin(\pi/2 - x) && \text{Since sin is odd}\\ &= -\cos x && \text{Cofunction identity}\end{aligned}$$

91. Since A, B, and C are angles of a triangle, $A + B = \pi - C$. So: $\sin(A + B) = \sin(\pi - C)$
$= \sin C$

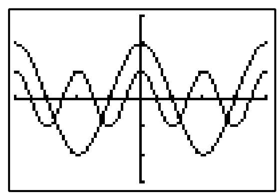
92. Using the identities from Exercises 69 and 70, we have:

$$\begin{aligned}\tan(\pi - x) &= \frac{\sin(\pi - x)}{\cos(\pi - x)}\\ &= \frac{\sin x}{-\cos x}\\ &= -\tan x\end{aligned}$$

■ Section 5.2 Proving Trigonometric Identities

Exploration 1

1. The graphs lead us to conclude that this is not an identity.

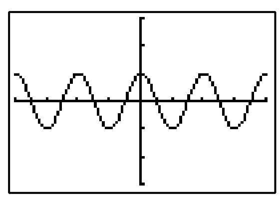


$[-2\pi, 2\pi]$ by $[-4, 4]$

2. For example, $\cos(2\cdot 0) = 1$, whereas $2\cos(0) = 2$.

3. Yes.

4. The graphs lead us to conclude that this is an identity.



$[-2\pi, 2\pi]$ by $[-3, 3]$

5. No. The graph window can not show the full graphs, so they could differ outside the viewing window. Also, the function values could be so close that the graphs *appear* to coincide.

Quick Review 5.2

1. $\csc x + \sec x = \dfrac{1}{\sin x} + \dfrac{1}{\cos x} = \dfrac{\sin x + \cos x}{\sin x \cos x}$

2. $\tan x + \cot x = \dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x} = \dfrac{\sin^2 x + \cos^2 x}{\sin x\ \cos x}$
$= \dfrac{1}{\sin x\ \cos x}$

3. $\cos x \cdot \dfrac{1}{\sin x} + \sin x \cdot \dfrac{1}{\cos x} = \dfrac{\cos^2 x + \sin^2 x}{\sin x\ \cos x}$
$= \dfrac{1}{\sin x\ \cos x}$

4. $\sin\theta\cdot\dfrac{\cos\theta}{\sin\theta}-\cos\theta\cdot\dfrac{\sin\theta}{\cos\theta}=\cos\theta-\sin\theta$

5. $\dfrac{\sin x}{1/\sin x}+\dfrac{\cos x}{1/\cos x}=\sin^2 x+\cos^2 x=1$

6. $\dfrac{1/\cos\alpha}{\cos\alpha}-\dfrac{\sin\alpha}{\cos^2\alpha/\sin\alpha}=\dfrac{1}{\cos^2\alpha}-\dfrac{\sin^2\alpha}{\cos^2\alpha}$
$=\dfrac{1-\sin^2\alpha}{\cos^2\alpha}=1$

7. No. (Any negative x.)

8. Yes.

9. No. (Any x for which $\sin x<0$, e.g. $x=-\pi/2$.)

10. No. (Any x for which $\tan x<0$, e.g. $x=-\pi/4$.)

11. Yes.

12. Yes.

Section 5.2 Exercises

1. One possible proof:
$$\frac{x^3-x^2}{x}-(x-1)(x+1)=\frac{x(x^2-x)}{x}-(x^2-1)$$
$$=x^2-x-(x^2-1)$$
$$=-x+1$$
$$=1-x$$

2. One possible proof:
$$\frac{1}{x}-\frac{1}{2}=\frac{1}{x}\left(\frac{2}{2}\right)-\frac{1}{2}\left(\frac{x}{x}\right)$$
$$=\frac{2}{2x}-\frac{x}{2x}$$
$$=\frac{2-x}{2x}$$

3. One possible proof:
$$\frac{x^2-4}{x-2}-\frac{x^2-9}{x+3}$$
$$=\frac{(x+2)(x-2)}{x-2}-\frac{(x+3)(x-3)}{x+3}$$
$$=x+2-(x-3)$$
$$=5$$

4. One possible proof:
$$(x-1)(x+2)-(x+1)(x-2)$$
$$=x^2+x-2-(x^2-x-2)$$
$$=x^2+x-2-x^2+x+2$$
$$=2x$$

5. $\dfrac{\sin^2 x+\cos^2 x}{\csc x}=\dfrac{1}{\csc x}=\sin x$. Yes.

6. $\dfrac{\tan x}{\sec x}=\dfrac{\sin x}{\cos x}\cdot\dfrac{\cos x}{1}=\sin x$. Yes.

7. $\cos x\cdot\cot x=\dfrac{\cos x}{1}\cdot\dfrac{\cos x}{\sin x}=\dfrac{\cos^2 x}{\sin x}$. No.

8. $\cos\left(x-\dfrac{\pi}{2}\right)=\cos\left(\dfrac{\pi}{2}-x\right)=\sin x$. Yes.

9. $(\sin^3 x)(1+\cot^2 x)=(\sin^3 x)(\csc^2 x)=\dfrac{\sin^3 x}{\sin^2 x}=\sin x$. Yes.

10. No. Confirm graphically.

11. $(\cos x)(\tan x+\sin x\cot x)$
$=\cos x\cdot\dfrac{\sin x}{\cos x}+\cos x\sin x\cdot\dfrac{\cos x}{\sin x}=\sin x+\cos^2 x$

12. $(\sin x)(\cot x+\cos x\tan x)$
$=\sin x\cdot\dfrac{\cos x}{\sin x}+\sin x\cos x\cdot\dfrac{\sin x}{\cos x}=\cos x+\sin^2 x$

13. $(1-\tan x)^2=1-2\tan x+\tan^2 x$
$=(1+\tan^2 x)-2\tan x=\sec^2 x-2\tan x$

14. $(\cos x-\sin x)^2=\cos^2 x-2\sin x\cos x+\sin^2 x$
$=(\cos^2 x+\sin^2 x)-2\sin x\cos x=1-2\sin x\cos x$

15. One possible proof:
$$\frac{(1-\cos u)(1+\cos u)}{\cos^2 u}=\frac{1-\cos^2 u}{\cos^2 u}$$
$$=\frac{\sin^2 u}{\cos^2 u}$$
$$=\tan^2 u$$

16. $\tan x+\sec x=\dfrac{\sin x}{\cos x}+\dfrac{1}{\cos x}=\dfrac{\sin x+1}{\cos x}$
$=\dfrac{\cos x(\sin x+1)}{\cos^2 x}=\dfrac{\cos x(\sin x+1)}{1-\sin^2 x}=\dfrac{\cos x}{1-\sin x}$

17. $\dfrac{\cos^2 x-1}{\cos x}=\dfrac{-\sin^2 x}{\cos x}=-\dfrac{\sin x}{\cos x}\cdot\sin x=-\tan x\sin x$

18. $\dfrac{\sec^2\theta-1}{\sin\theta}=\dfrac{\tan^2\theta}{\sin\theta}=\dfrac{1}{\sin\theta}\cdot\left(\dfrac{\sin\theta}{\cos\theta}\right)^2=\dfrac{\sin\theta}{\cos^2\theta}$
$=\dfrac{\sin\theta}{1-\sin^2\theta}$

19. Multiply out the expression on the left side.

20. $\dfrac{1}{1-\cos x}+\dfrac{1}{1+\cos x}=\dfrac{(1+\cos x)+(1-\cos x)}{(1-\cos x)(1+\cos x)}$
$=\dfrac{2}{1-\cos^2 x}=\dfrac{2}{\sin^2 x}=2\csc^2 x$

21. $(\cos t-\sin t)^2+(\cos t+\sin t)^2$
$=\cos^2 t-2\cos t\sin t+\sin^2 t+\cos^2 t$
$+2\cos t\sin t+\sin^2 t=2\cos^2 t+2\sin^2 t=2$

22. $\sin^2\alpha-\cos^2\alpha=(1-\cos^2\alpha)-\cos^2\alpha=1-2\cos^2\alpha$

23. $\dfrac{1+\tan^2 x}{\sin^2 x+\cos^2 x}=\dfrac{\sec^2 x}{1}=\sec^2 x$

24. $\dfrac{1}{\tan\beta}+\tan\beta=\dfrac{\cos\beta}{\sin\beta}+\dfrac{\sin\beta}{\cos\beta}=\dfrac{\cos^2\beta+\sin^2\beta}{\cos\beta\sin\beta}$
$=\dfrac{1}{\cos\beta\sin\beta}=\sec\beta\csc\beta$

25. $\dfrac{\cos\beta}{1+\sin\beta}=\dfrac{\cos^2\beta}{\cos\beta(1+\sin\beta)}=\dfrac{1-\sin^2\beta}{\cos\beta(1+\sin\beta)}$
$=\dfrac{(1-\sin\beta)(1+\sin\beta)}{\cos\beta(1+\sin\beta)}=\dfrac{1-\sin\beta}{\cos\beta}$

26. One possible proof:
$$\frac{\sec x+1}{\tan x}=\frac{(\sec x+1)(\sec x-1)}{\tan x(\sec x-1)}$$
$$=\frac{\sec^2 x-1}{\tan x(\sec x-1)}$$
$$=\frac{\tan^2 x}{\tan x(\sec x-1)}=\frac{\dfrac{\sin x}{\cos x}}{\dfrac{1}{\cos x}-1}\cdot\frac{\cos x}{\cos x}$$
$$=\frac{\sin x}{1-\cos x}$$

27. $\frac{\tan^2 x}{\sec x + 1} = \frac{\sec^2 x - 1}{\sec x + 1} = \sec x - 1 = \frac{1}{\cos x} - 1$
$= \frac{1 - \cos x}{\cos x}$

28. $\frac{\cot v - 1}{\cot v + 1} = \frac{\cot v - 1}{\cot v + 1} \cdot \frac{\tan v}{\tan v} = \frac{\cot v \tan v - \tan v}{\cot v \tan v + \tan v}$
$= \frac{1 - \tan v}{1 + \tan v}$ (Note: $\cot v \tan v = \frac{\cos v}{\sin v} \cdot \frac{\sin v}{\cos v} = 1$)

29. $\cot^2 x - \cos^2 x = \left(\frac{\cos x}{\sin x}\right)^2 - \cos^2 x$
$= \frac{\cos^2 x(1 - \sin^2 x)}{\sin^2 x} = \cos^2 x \cdot \frac{\cos^2 x}{\sin^2 x}$
$= \cos^2 x \cot^2 x$

30. $\tan^2 \theta - \sin^2 \theta = \left(\frac{\sin \theta}{\cos \theta}\right)^2 - \sin^2 \theta$
$= \frac{\sin^2 \theta(1 - \cos^2 \theta)}{\cos^2 \theta} = \sin^2 \theta \cdot \frac{\sin^2 \theta}{\cos^2 \theta}$
$= \sin^2 \theta \tan^2 \theta$

31. $\cos^4 x - \sin^4 x = (\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x)$
$= 1(\cos^2 x - \sin^2 x) = \cos^2 x - \sin^2 x$

32. $\tan^4 t + \tan^2 t = \tan^2 t(\tan^2 t + 1) = (\sec^2 t - 1)(\sec^2 t)$
$= \sec^4 t - \sec^2 t$

33. $(x \sin \alpha + y \cos \alpha)^2 + (x \cos \alpha - y \sin \alpha)^2$
$= (x^2 \sin^2 \alpha + 2xy \sin \alpha \cos \alpha + y^2 \cos^2 \alpha)$
$+ (x^2 \cos^2 \alpha - 2xy \cos \alpha \sin \alpha + y^2 \sin^2 \alpha)$
$= x^2 \sin^2 \alpha + y^2 \cos^2 \alpha + x^2 \cos^2 \alpha + y^2 \sin^2 \alpha$
$= (x^2 + y^2)(\sin^2 \alpha + \cos^2 \alpha) = x^2 + y^2$

34. $\frac{1 - \cos \theta}{\sin \theta} = \frac{1 - \cos^2 \theta}{\sin \theta(1 + \cos \theta)} = \frac{\sin^2 \theta}{\sin \theta(1 + \cos \theta)}$
$= \frac{\sin \theta}{1 + \cos \theta}$

35. $\frac{\tan x}{\sec x - 1} = \frac{\tan x(\sec x + 1)}{\sec^2 x - 1} = \frac{\tan x(\sec x + 1)}{\tan^2 x}$
$= \frac{\sec x + 1}{\tan x}$. See also #26.

36. $\frac{\sin t}{1 + \cos t} + \frac{1 + \cos t}{\sin t} = \frac{\sin^2 t + (1 + \cos t)^2}{(\sin t)(1 + \cos t)}$
$= \frac{\sin^2 t + 1 + 2 \cos t + \cos^2 t}{(\sin t)(1 + \cos t)} = \frac{2 + 2 \cos t}{(\sin t)(1 + \cos t)}$
$= \frac{2}{\sin t} = 2 \csc t$

37. $\frac{\sin x - \cos x}{\sin x + \cos x} = \frac{(\sin x - \cos x)(\sin x + \cos x)}{(\sin x + \cos x)^2}$
$= \frac{\sin^2 x - \cos^2 x}{\sin^2 x + 2 \sin x \cos x + \cos^2 x} = \frac{\sin^2 x - (1 - \sin^2 x)}{1 + 2 \sin x \cos x}$
$= \frac{2 \sin^2 x - 1}{1 + 2 \sin x \cos x}$

38. $\frac{1 + \cos x}{1 - \cos x} = \frac{1 + \cos x}{1 - \cos x} \cdot \frac{\sec x}{\sec x} = \frac{\sec x + \cos x \sec x}{\sec x - \cos x \sec x}$
$= \frac{\sec x + 1}{\sec x - 1}$ (Note: $\cos x \sec x = \cos x \cdot \frac{1}{\cos x} = 1$.)

39. $\frac{\sin t}{1 - \cos t} + \frac{1 + \cos t}{\sin t} = \frac{\sin^2 t + (1 + \cos t)(1 - \cos t)}{(\sin t)(1 - \cos t)}$
$= \frac{\sin^2 t + 1 - \cos^2 t}{(\sin t)(1 - \cos t)} = \frac{1 - \cos^2 t + 1 - \cos^2 t}{(\sin t)(1 - \cos t)}$
$= \frac{2(1 - \cos^2 t)}{(\sin t)(1 - \cos t)} = \frac{2(1 + \cos t)}{\sin t}$

40. $\frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$
$= \left(\frac{\frac{1}{\cos A \cos B}}{\frac{1}{\cos A \cos B}}\right) \cdot \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$
$= \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - \frac{\sin A}{\cos A} \frac{\sin B}{\cos B}} = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

41. $\sin^2 x \cos^3 x = \sin^2 x \cos^2 x \cos x$
$= \sin^2 x(1 - \sin^2 x)\cos x = (\sin^2 x - \sin^4 x)\cos x$

42. $\sin^5 x \cos^2 x = \sin^4 x \cos^2 x \sin x$
$= (\sin^2 x)^2 \cos^2 x \sin x = (1 - \cos^2 x)^2 \cos^2 x \sin x$
$= (1 - 2 \cos^2 x + \cos^4 x)\cos^2 x \sin x$
$= (\cos^2 x - 2 \cos^4 x + \cos^6 x)\sin x$

43. $\cos^5 x = \cos^4 x \cos x = (\cos^2 x)^2 \cos x$
$= (1 - \sin^2 x)^2 \cos x = (1 - 2 \sin^2 x + \sin^4 x) \cos x$

44. $\sin^3 x \cos^3 x = \sin^3 x \cos^2 x \cos x$
$= \sin^3 x (1 - \sin^2 x)\cos x = (\sin^3 x - \sin^5 x)\cos x$

45. $\frac{\tan x}{1 - \cot x} + \frac{\cot x}{1 - \tan x}$
$= \frac{\tan x}{1 - \cot x} \cdot \frac{\sin x}{\sin x} + \frac{\cot x}{1 - \tan x} \cdot \frac{\cos x}{\cos x}$
$= \left(\frac{\sin^2 x/\cos x}{\sin x - \cos x} + \frac{\cos^2 x/\sin x}{\cos x - \sin x}\right)\frac{\sin x \cos x}{\sin x \cos x}$
$= \frac{\sin^3 x - \cos^3 x}{\sin x \cos x(\sin x - \cos x)}$
$= \frac{\sin^2 x + \sin x \cos x + \cos^2 x}{\sin x \cos x}$
$= \frac{1 + \sin x \cos x}{\sin x \cos x} = \frac{1}{\sin x \cos x} + 1 = \csc x \sec x + 1.$
This involves rewriting $a^3 - b^3$ as $(a - b)(a^2 + ab + b^2)$, where $a = \sin x$ and $b = \cos x$.

46. $\frac{\cos x}{1 + \sin x} + \frac{\cos x}{1 - \sin x}$
$= \frac{(\cos x)[(1 - \sin x) + (1 + \sin x)]}{(1 + \sin x)(1 - \sin x)} = \frac{2 \cos x}{1 - \sin^2 x}$
$= \frac{2 \cos x}{\cos^2 x} = 2 \sec x$

47. $\frac{2\tan x}{1-\tan^2 x}+\frac{1}{2\cos^2 x-1}$
$=\frac{2\tan x}{1-\tan^2 x}\cdot\frac{\cos^2 x}{\cos^2 x}+\frac{1}{\cos^2 x-\sin^2 x}$
$=\frac{2\sin x\cos x}{\cos^2 x-\sin^2 x}+\frac{\cos^2 x+\sin^2 x}{\cos^2 x-\sin^2 x}$
$=\frac{2\sin x\cos x+\cos^2 x+\sin^2 x}{(\cos x-\sin x)(\cos x+\sin x)}$
$=\frac{(\cos x+\sin x)^2}{(\cos x-\sin x)(\cos x+\sin x)}=\frac{\cos x+\sin x}{\cos x-\sin x}$

48. $\frac{1-3\cos x-4\cos^2 x}{\sin^2 x}=\frac{(1+\cos x)(1-4\cos x)}{1-\cos^2 x}$
$=\frac{(1+\cos x)(1-4\cos x)}{(1+\cos x)(1-\cos x)}=\frac{1-4\cos x}{1-\cos x}$

49. $\cos^3 x=(\cos^2 x)(\cos x)=(1-\sin^2 x)(\cos x)$

50. $\sec^4 x=(\sec^2 x)(\sec^2 x)=(1+\tan^2 x)(\sec^2 x)$

51. $\sin^5 x=(\sin^4 x)(\sin x)=(\sin^2 x)^2(\sin x)$
$=(1-\cos^2 x)^2(\sin x)$
$=(1-2\cos^2 x+\cos^4 x)(\sin x)$

52. (b) — divide through by $\cos x$: $\frac{1+\sin x}{\cos x}$
$=\frac{1}{\cos x}+\frac{\sin x}{\cos x}=\sec x+\tan x.$

53. (d) — multiply out: $(1+\sec x)(1-\cos x)$
$=1-\cos x+\sec x-\sec x\cos x$
$=1-\cos x+\frac{1}{\cos x}-\frac{1}{\cos x}\cdot\cos x$
$=1-\cos x+\frac{1}{\cos x}-1=\frac{1-\cos^2 x}{\cos x}=\frac{\sin^2 x}{\cos x}$
$=\frac{\sin x}{\cos x}\cdot\sin x=\tan x\sin x.$

54. (a) — put over a common denominator:
$\sec^2 x+\csc^2 x=\left(\frac{1}{\cos x}\right)^2+\left(\frac{1}{\sin x}\right)^2$
$=\frac{\sin^2 x+\cos^2 x}{\cos^2 x\sin^2 x}=\frac{1}{\cos^2 x\sin^2 x}=\left(\frac{1}{\cos x}\cdot\frac{1}{\sin x}\right)^2$
$=\sec^2 x\csc^2 x.$

55. (c) — put over a common denominator:
$\frac{1}{1+\sin x}+\frac{1}{1-\sin x}=\frac{1-\sin x+1+\sin x}{1-\sin^2 x}$
$=\frac{2}{\cos^2 x}=2\sec^2 x.$

56. (e) — multiply and divide by $\sin x\cos x$: $\frac{1}{\tan x+\cot x}$
$=\frac{\sin x\cos x}{\left(\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}\right)(\sin x\cos x)}=\frac{\sin x\cos x}{\sin^2 x+\cos^2 x}$
$=\frac{\sin x\cos x}{1}=\sin x\cos x.$

57. (b) — multiply and divide by $\sec x+\tan x$:
$\frac{1}{\sec x-\tan x}\cdot\frac{\sec x+\tan x}{\sec x+\tan x}=\frac{\sec x+\tan x}{\sec^2 x-\tan^2 x}$
$=\frac{\sec x+\tan x}{1}.$

58. False. There are numbers in the domain of both sides of the equation for which equality does not hold, namely all negative real numbers. For example, $\sqrt{(-3)^2}=3$, not -3.

59. True. If x is in the domain of both sides of the equation, then $x\geq 0$. The equation $(\sqrt{x})^2=x$ holds for all $x\geq 0$, so it is an identity.

60. By the definition of identity, all three must be true. The answer is E.

61. A proof is
$\frac{\sin x}{1-\cos x}=\frac{\sin x}{1-\cos x}\cdot\frac{1+\cos x}{1+\cos x}$
$=\frac{\sin x(1+\cos x)}{1-\cos^2 x}$
$=\frac{\sin x(1+\cos x)}{\sin^2 x}$
$=\frac{1+\cos x}{\sin x}$
The answer is E.

62. One possible proof:
$\tan\theta+\sec\theta=\frac{\sin\theta}{\cos\theta}+\frac{1}{\cos\theta}$
$=\frac{\sin\theta+1}{\cos\theta}$
$=\frac{\sin\theta+1}{\cos\theta}\cdot\frac{\sin\theta-1}{\sin\theta-1}$
$=\frac{\sin^2\theta-1}{\cos\theta(\sin\theta-1)}$
$=\frac{-\cos^2\theta}{\cos\theta(\sin\theta-1)}$
$=\frac{-\cos\theta}{\sin\theta-1}$
$=\frac{\cos\theta}{1-\sin\theta}$
The answer is C.

63. k must equal 1, so $f(x)\neq 0$. The answer is B.

64. $\cos x$; $\sin x\cot x=\sin x\cdot\frac{\cos x}{\sin x}=\cos x$

65. $\sin x$; $\cos x\tan x=\cos x\cdot\frac{\sin x}{\cos x}=\sin x$

66. 1; $\frac{\sin x}{\csc x}+\frac{\cos x}{\sec x}=\frac{\sin x}{1/\sin x}+\frac{\cos x}{1/\cos x}$
$=\sin^2 x+\cos^2 x=1$

67. 1; $\frac{\csc x}{\sin x}-\frac{\cot x\csc x}{\sec x}=\frac{1/\sin x}{\sin x}-\frac{\cos x/\sin^2 x}{1/\cos x}$
$=\frac{1}{\sin^2 x}-\frac{\cos^2 x}{\sin^2 x}=\frac{1-\cos^2 x}{\sin^2 x}=\frac{\sin^2 x}{\sin^2 x}=1$

68. $\cos x$; $\frac{\sin x}{\tan x}=\frac{\sin x}{\sin x/\cos x}=\cos x.$

69. 1; $(\sec^2 x)(1-\sin^2 x)=\left(\frac{1}{\cos x}\right)^2(\cos^2 x)=1$

70. Since the sum of the logarithms is the logarithm of the product, and since the product of the absolute values of all six basic trig functions is 1, the logarithms sum to ln 1, which is 0.

71. If A and B are complementary angles, then
$$\begin{aligned}\sin^2 A + \sin^2 B &= \sin^2 A + \sin^2(\pi/2 - A)\\ &= \sin^2 A + \cos^2 A\\ &= 1\end{aligned}$$

72. Check Exercises 11–51 for correct identities.

73. Multiply and divide by $1 - \sin t$ under the radical:
$$\sqrt{\frac{1-\sin t}{1+\sin t}\cdot\frac{1-\sin t}{1-\sin t}} = \sqrt{\frac{(1-\sin t)^2}{1-\sin^2 t}}$$
$$= \sqrt{\frac{(1-\sin t)^2}{\cos^2 t}} = \frac{|1-\sin t|}{|\cos t|} \text{ since } \sqrt{a^2} = |a|.$$
Now, since $1 - \sin t \geq 0$, we can dispense with the absolute value in the numerator, but it must stay in the denominator.

74. Multiply and divide by $1 + \cos t$ under the radical:
$$\sqrt{\frac{1+\cos t}{1-\cos t}\cdot\frac{1+\cos t}{1+\cos t}} = \sqrt{\frac{(1+\cos t)^2}{1-\cos^2 t}}$$
$$= \sqrt{\frac{(1+\cos t)^2}{\sin^2 t}} = \frac{|1+\cos t|}{|\sin t|} \text{ since } \sqrt{a^2} = |a|.$$
Now, since $1 + \cos t \geq 0$, we can dispense with the absolute value in the numerator, but it must stay in the denominator.

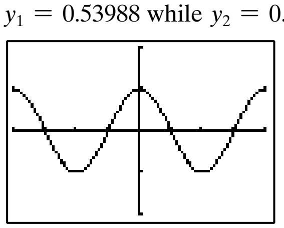
75. $\sin^6 x + \cos^6 x = (\sin^2 x)^3 + \cos^6 x$
$= (1 - \cos^2 x)^3 + \cos^6 x$
$= (1 - 3\cos^2 x + 3\cos^4 x - \cos^6 x) + \cos^6 x$
$= 1 - 3\cos^2 x(1 - \cos^2 x) = 1 - 3\cos^2 x \sin^2 x.$

76. Note that $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$. Also note that $a^2 + ab + b^2 = a^2 + 2ab + b^2 - ab$ $= (a + b)^2 - ab$. Taking $a = \cos^2 x$ and $b = \sin^2 x$, we have $\cos^6 x - \sin^6 x$
$= (\cos^2 x - \sin^2 x)(\cos^4 x + \cos^2 x \sin^2 x + \sin^4 x)$
$= (\cos^2 x - \sin^2 x)[(\cos^2 x + \sin^2 x)^2 - \cos^2 x \sin^2 x]$
$= (\cos^2 x - \sin^2 x)(1 - \cos^2 x \sin^2 x).$

77. One possible proof: $\ln|\tan x| = \ln\dfrac{|\sin x|}{|\cos x|}$
$= \ln|\sin x| - \ln|\cos x|.$

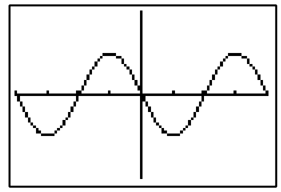
78. One possible proof:
$$\begin{aligned}\ln|\sec\theta + \tan\theta| + \ln|\sec\theta - \tan\theta| &= \ln|\sec^2\theta - \tan^2\theta|\\ &= \ln 1\\ &= 0\end{aligned}$$

79. (a) They are not equal. Shown is the window $[-2\pi, 2\pi,]$ by $[-2, 2]$; graphing on nearly any viewing window does not show any apparent difference — but using TRACE, one finds that the y coordinates are not identical. Likewise, a table of values will show slight differences; for example, when $x = 1$, $y_1 = 0.53988$ while $y_2 = 0.54030$



$[-2\pi, 2\pi]$ by $[-2, 2]$

(b) One choice for h is 0.001 (shown). The function y_3 is a combination of three sinusoidal functions ($1000\sin(x + 0.001)$, $1000\sin x$, and $\cos x$), all with period 2π.



$[-2\pi, 2\pi]$ by $[-0.001, 0.001]$

80. (a) $\cosh^2 x - \sinh^2 x = \dfrac{1}{4}(e^x + e^{-x})^2 - \dfrac{1}{4}(e^x - e^{-x})^2$
$= \dfrac{1}{4}[e^{2x} + 2 + e^{-2x} - (e^{2x} - 2 + e^{-2x})]$
$= \dfrac{1}{4}(4) = 1.$

(b) $1 - \tanh^2 x = 1 - \dfrac{\sinh^2 x}{\cosh^2 x} = \dfrac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}$
$= \dfrac{1}{\cosh^2 x}$, using the result from (a). This equals $\operatorname{sech}^2 x$.

(c) $\coth^2 x - 1 = \dfrac{\cosh^2 x}{\sinh^2 x} - 1 = \dfrac{\cosh^2 x - \sinh^2 x}{\sinh^2 x}$
$= \dfrac{1}{\sinh^2 x}$, using the result from (a). This equals $\operatorname{csch}^2 x$.

81. In the decimal window, the x coordinates used to plot the graph on the calculator are (e.g.) 0, 0.1, 0.2, 0.3, etc. — that is, $x = n/10$, where n is an integer. Then $10\pi x = \pi n$, and the sine of integer multiples of π is 0; therefore, $\cos x + \sin 10\pi x = \cos x + \sin \pi n = \cos x + 0$ $= \cos x$. However, for other choices of x, such as $x = \dfrac{1}{\pi}$, we have $\cos x + \sin 10\pi x = \cos x + \sin 10 \neq \cos x$.

■ Section 5.3 Sum and Difference Identities

Exploration 1

1. $\sin(u + v) = -1$, $\sin u + \sin v = 1$. No.
2. $\cos(u + v) = 1$, $\cos u + \cos v = 2$. No.
3. $\tan(\pi/3 + \pi/3) = -\sqrt{3}$, $\tan\pi/3 + \tan\pi/3 = 2\sqrt{3}$. (Many other answers are possible.)

Quick Review 5.3

1. $15° = 45° - 30°$
2. $75° = 45° + 30°$
3. $165° = 180° - 15° = 180° + 30° - 45° = 210° - 45°$
4. $\dfrac{\pi}{12} = 2\cdot\dfrac{\pi}{6} - \dfrac{\pi}{4} = \dfrac{\pi}{3} - \dfrac{\pi}{4}$
5. $\dfrac{5\pi}{12} = 4\cdot\dfrac{\pi}{6} - \dfrac{\pi}{4} = \dfrac{2\pi}{3} - \dfrac{\pi}{4}$
6. $\dfrac{7\pi}{12} = \dfrac{4\pi}{12} + \dfrac{3\pi}{12} = \dfrac{\pi}{3} + \dfrac{\pi}{4}$
7. No. $(f(x) + f(y) = \ln x + \ln y = \ln(xy)$ $= f(xy) \neq f(x + y))$

8. No. $(f(x+y) = e^{x+y} = e^x e^y$
$= f(x)f(y) \neq f(x+y))$

9. Yes. $(f(x+y) = 32(x+y) = 32x + 32y$
$= f(x) + f(y))$

10. No. $(f(x+y) = x + y + 10$
$= f(x) + y \neq f(x) + f(y))$

Section 5.3 Exercises

1. $\sin 15° = \sin(45° - 30°)$
$= \sin 45° \cos 30° - \cos 45° \sin 30°$
$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$

2. $\tan 15° = \tan(45° - 30°) = \frac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°}$
$= \frac{1 - \sqrt{3}/3}{1 + \sqrt{3}/3} = \frac{3 - \sqrt{3}}{3 + \sqrt{3}} = \frac{(3 - \sqrt{3})^2}{9 - 3} = 2 - \sqrt{3}$

3. $\sin 75° = \sin(45° + 30°)$
$= \sin 45° \cos 30° + \cos 45° \sin 30°$
$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$

4. $\cos 75° = \cos(45° + 30°)$
$= \cos 45° \cos 30° - \sin 45° \sin 30°$
$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$

5. $\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}$
$= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}+\sqrt{6}}{4}$

6. $\sin\frac{7\pi}{12} = \sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right) = \sin\frac{\pi}{3}\cos\frac{\pi}{4} + \cos\frac{\pi}{3}\sin\frac{\pi}{4}$
$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} + \frac{1}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$

7. $\tan\frac{5\pi}{12} = \tan\left(\frac{2\pi}{3} - \frac{\pi}{4}\right) = \frac{\tan(2\pi/3) - \tan(\pi/4)}{1 + \tan(2\pi/3)\tan(\pi/4)}$
$= \frac{-\sqrt{3}-1}{1-\sqrt{3}} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{(\sqrt{3}+1)^2}{3-1} = 2 + \sqrt{3}$

8. $\tan\frac{11\pi}{12} = \tan\left(\frac{2\pi}{3} + \frac{\pi}{4}\right) = \frac{\tan(2\pi/3) + \tan(\pi/4)}{1 - \tan(2\pi/3)\tan(\pi/4)}$
$= \frac{-\sqrt{3}+1}{1+\sqrt{3}} = \frac{1-\sqrt{3}}{1+\sqrt{3}} = \frac{(1-\sqrt{3})^2}{1-3} = \sqrt{3} - 2$

9. $\cos\frac{7\pi}{12} = \cos\left(\frac{5\pi}{6} - \frac{\pi}{4}\right)$
$= \cos\frac{5\pi}{6}\cos\frac{\pi}{4} + \sin\frac{5\pi}{6}\sin\frac{\pi}{4} = -\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} + \frac{1}{2}\cdot\frac{\sqrt{2}}{2}$
$= \frac{\sqrt{2}-\sqrt{6}}{4}$

10. $\sin\left(-\frac{\pi}{12}\right) = \sin\left(\frac{\pi}{6} - \frac{\pi}{4}\right)$
$= \sin\frac{\pi}{6}\cos\frac{\pi}{4} - \cos\frac{\pi}{6}\sin\frac{\pi}{4} = \frac{1}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}$
$= \frac{\sqrt{2}-\sqrt{6}}{4}$

In #11–22, match the given expression with the sum and difference identities.

11. $\sin(42° - 17°) = \sin 25°$

12. $\cos(94° - 18°) = \cos 76°$

13. $\sin\left(\frac{\pi}{5} + \frac{\pi}{2}\right) = \sin\frac{7\pi}{10}$

14. $\sin\left(\frac{\pi}{3} - \frac{\pi}{7}\right) = \sin\frac{4\pi}{21}$

15. $\tan(19° + 47°) = \tan 66°$

16. $\tan\left(\frac{\pi}{5} - \frac{\pi}{3}\right) = \tan -\frac{2\pi}{15}$

17. $\cos\left(\frac{\pi}{7} - x\right) = \cos\left(x - \frac{\pi}{7}\right)$

18. $\cos\left(x + \frac{\pi}{7}\right)$

19. $\sin(3x - x) = \sin 2x$

20. $\cos(7y + 3y) = \cos 10y$

21. $\tan(2y + 3x)$

22. $\tan(3\alpha - 2\beta)$

23. $\sin\left(x - \frac{\pi}{2}\right) = \sin x \cos\frac{\pi}{2} - \cos x \sin\frac{\pi}{2}$
$= \sin x \cdot 0 - \cos x \cdot 1 = -\cos x$

24. Using the difference identity for the tangent function, we encounter $\tan\frac{\pi}{2}$, which is undefined. However, we can compute $\tan\left(x - \frac{\pi}{2}\right) = \frac{\sin(x - \pi/2)}{\cos(x - \pi/2)}$. From #23, $\sin\left(x - \frac{\pi}{2}\right) = -\cos x$. Since the cosine function is even, $\cos\left(x - \frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2} - x\right) = \sin x$ (see Example 2, or #25). Therefore this simplifies to $\frac{-\cos x}{\sin x} = -\cot x$.

25. $\cos\left(x - \frac{\pi}{2}\right) = \cos x \cos\frac{\pi}{2} + \sin x \sin\frac{\pi}{2}$
$= \cos x \cdot 0 + \sin x \cdot 1 = \sin x$

26. The simplest way is to note that $\left(\frac{\pi}{2} - x\right) - y = \frac{\pi}{2} - x - y = \frac{\pi}{2} - (x + y)$, so that $\cos\left[\left(\frac{\pi}{2} - x\right) - y\right] = \cos\left[\frac{\pi}{2} - (x + y)\right]$. Now use Example 2 to conclude that $\cos\left[\frac{\pi}{2} - (x + y)\right]$
$= \sin(x + y)$.

27. $\sin\left(x + \frac{\pi}{6}\right) = \sin x \cos\frac{\pi}{6} + \cos x \sin\frac{\pi}{6}$
$= \sin x \cdot \frac{\sqrt{3}}{2} + \cos x \cdot \frac{1}{2}$

28. $\cos\left(x - \frac{\pi}{4}\right) = \cos x \cos\frac{\pi}{4} + \sin x \sin\frac{\pi}{4}$
$= \cos x \cdot \frac{\sqrt{2}}{2} + \sin x \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}(\cos x + \sin x)$

29. $\tan\left(\theta + \frac{\pi}{4}\right) = \frac{\tan\theta + \tan(\pi/4)}{1 - \tan\theta\tan(\pi/4)} = \frac{\tan\theta + 1}{1 - \tan\theta \cdot 1}$
$= \frac{1 + \tan\theta}{1 - \tan\theta}$

30. $\cos\left(\theta + \frac{\pi}{2}\right) = \cos\theta\cos\frac{\pi}{2} - \sin\theta\sin\frac{\pi}{2}$
$= \cos\theta \cdot 0 - \sin\theta \cdot 1 = -\sin\theta$

31. Equations B and F.

32. Equations C and E.

33. Equations D and H.

34. Equations A and G.

35. Rewrite as $\sin 2x \cos x - \cos 2x \sin x = 0$; the left side equals $\sin(2x - x) = \sin x$, so $x = n\pi$, n an integer.

36. Rewrite as $\cos 3x \cos x - \sin 3x \sin x = 0$; the left side equals $\cos(3x + x) = \cos 4x$, so $4x = \frac{\pi}{2} + n\pi$; then $x = \frac{\pi}{8} + n\frac{\pi}{4}$, n an integer.

37. $\sin\left(\frac{\pi}{2} - u\right) = \sin\frac{\pi}{2}\cos u - \cos\frac{\pi}{2}\sin u$
$= 1 \cdot \cos u - 0 \cdot \sin u = \cos u.$

38. Using the difference identity for the tangent function, we encounter $\tan\frac{\pi}{2}$, which is undefined. However, we can compute $\tan\left(\frac{\pi}{2} - u\right) = \frac{\sin(\pi/2 - u)}{\cos(\pi/2 - u)} = \frac{\cos u}{\sin u} = \cot u.$ Or, use #24, and the fact that the tangent function is odd.

39. $\cot\left(\frac{\pi}{2} - u\right) = \frac{\cos(\pi/2 - u)}{\sin(\pi/2 - u)} = \frac{\sin u}{\cos u} = \tan u$ using the first two cofunction identities.

40. $\sec\left(\frac{\pi}{2} - u\right) = \frac{1}{\cos(\pi/2 - u)} = \frac{1}{\sin u} = \csc u$ using the first cofunction identity.

41. $\csc\left(\frac{\pi}{2} - u\right) = \frac{1}{\sin(\pi/2 - u)} = \frac{1}{\cos u} = \sec u$ using the second cofunction identity.

42. $\cos\left(x + \frac{\pi}{2}\right) = \cos x \cos\left(\frac{\pi}{2}\right) - \sin x \sin\left(\frac{\pi}{2}\right)$
$= \cos x \cdot 0 - \sin x \cdot 1$
$= -\sin x$

43. To write $y = 3\sin x + 4\cos x$ in the form $y = a\sin(bx + c)$, rewrite the formula using the formula for the sine of a sum:
$y = a((\sin bx \cos c) + (\cos bx \sin c))$
$= a\sin bx \cos c + a\cos bx \sin c$
$= (a\cos c)\sin bx + (a\sin c)\cos bx.$
Then compare the coefficients: $a\cos c = 3$, $b = 1$, $a\sin c = 4$.

Solve for a as follows:
$(a\cos c)^2 + (a\sin c)^2 = 3^2 + 4^2$
$a^2\cos^2 c + a^2\sin^2 c = 25$
$a^2(\cos^2 c + \sin^2 c) = 25$
$a^2 = 25$
$a = \pm 5$

If we choose a to be positive, then $\cos c = 3/5$ and $\sin c = 4/5$. $c = \cos^{-1}(3/5) = \sin^{-1}(4/5)$. So the sinusoid is $y = 5\sin(x + \cos^{-1}(3/5)) \approx 5\sin(x + 0.9273)$.

44. Follow the steps shown in Exercise 43 (using the formula for the sine of a difference) to compare the coefficients in $y = (a\cos c)\sin bx - (a\sin c)\cos bx$ to the coefficients in $y = 5\sin x - 12\cos x$: $a\cos c = 5$, $b = 1$, $a\sin c = 12$.

Solve for a as follows:
$(a\cos c)^2 + (a\sin c)^2 = 5^2 + 12^2$
$a^2(\cos^2 c + \sin^2 c) = 169$
$a = \pm 13$
If we choose a to be positive, then $\cos c = 5/13$ and $\sin c = 12/13$. So the sinusoid is
$y = 13\sin(x - \cos^{-1}(5/13)) \approx 13\sin(x - 1.176)$.

45. Follow the steps shown in Exercise 43 to compare the coefficients in $y = (a\cos c)\sin bx + (a\sin c)\cos bx$ to the coefficients in $y = \cos 3x + 2\sin 3x$: $a\cos c = 2$, $b = 3$, $a\sin c = 1$.
Solve for a as follows:
$(a\cos c)^2 + (a\sin c)^2 = 1^2 + 2^2$
$a^2(\cos^2 c + \sin^2 c) = 5$
$a = \pm\sqrt{5}$
If we choose a to be positive, then $\cos c = 2/\sqrt{5}$ and $\sin c = 1/\sqrt{5}$. So the sinusoid is
$y = \sqrt{5}\sin(3x - \cos^{-1}(2/\sqrt{5})) \approx 2.236\sin(3x - 0.4636)$.

46. Follow the steps shown in Exercise 43 to compare the coefficients in $y = (a\cos c)\sin bx + (a\sin c)\cos bx$ to the coefficients in $y = 3\cos 2x - 2\sin 2x$ $= -2\sin 2x + 3\cos 2x$: $a\cos c = -2$, $b = 2$, $a\sin c = 3$.
Solve for a as follows:
$(a\cos c)^2 + (a\sin c)^2 = (-2)^2 + 3^2$
$a^2(\cos^2 c + \sin^2 c) = 13$
$a = \pm\sqrt{13}$
If we choose a to be negative, then $\cos c = 2/\sqrt{13}$ and $\sin c = -3/\sqrt{13}$. So the sinusoid is
$y = -\sqrt{13}\sin(2x - \cos^{-1}(2/\sqrt{13}))$
$\approx -3.606\sin(2x - 0.9828)$.

47. $\sin(x - y) + \sin(x + y)$
$= (\sin x\cos y - \cos x\sin y) + (\sin x\cos y + \cos x\sin y)$
$= 2\sin x\cos y$

48. $\cos(x - y) + \cos(x + y)$
$= (\cos x\cos y + \sin x\sin y) + (\cos x\cos y - \sin x\sin y)$
$= 2\cos x\cos y$

49. $\cos 3x = \cos[(x + x) + x]$
$= \cos(x + x)\cos x - \sin(x + x)\sin x$
$= (\cos x\cos x - \sin x\sin x)\cos x$
$- (\sin x\cos x + \cos x\sin x)\sin x$
$= \cos^3 x - \sin^2 x\cos x - 2\cos x\sin^2 x$
$= \cos^3 x - 3\sin^2 x\cos x$

50. $\sin 3u = \sin[(u + u) + u] = \sin(u + u)\cos u + \cos(u + u)\sin u = (\sin u\cos u + \cos u\sin u)\cos u + (\cos u\cos u - \sin u\sin u)\sin u = 2\cos^2 u\sin u + \cos^2 u\sin u - \sin^3 u = 3\cos^2 u\sin u - \sin^3 u$

51. $\cos 3x + \cos x = \cos(2x + x) + \cos(2x - x)$; use #48 with x replaced with $2x$ and y replaced with x.

52. $\sin 4x + \sin 2x = \sin(3x + x) + \sin(3x - x)$; use #47 with x replaced with $3x$ and y replaced with x.

53. $\tan(x+y)\tan(x-y)$

$$= \left(\frac{\tan x + \tan y}{1 - \tan x \tan y}\right)\cdot\left(\frac{\tan x - \tan y}{1 + \tan x \tan y}\right)$$

$$= \frac{\tan^2 x - \tan^2 y}{1 - \tan^2 x \tan^2 y}$$ since both the numerator and denominator are factored forms for differences of squares.

54. $\tan 5u \tan 3u = \tan(4u+u)\tan(4u-u)$; use #53 with $x = 4u$ and $y = u$.

55. $$\frac{\sin(x+y)}{\sin(x-y)}$$

$$= \frac{\sin x \cos y + \cos x \sin y}{\sin x \cos y - \cos x \sin y}$$

$$= \frac{\sin x \cos y + \cos x \sin y}{\sin x \cos y - \cos x \sin y}\cdot\frac{1/(\cos x \cos y)}{1/(\cos x \cos y)}$$

$$= \frac{(\sin x \cos y)/(\cos x \cos y) + (\cos x \sin y)/(\cos x \cos y)}{(\sin x \cos y)/(\cos x \cos y) - (\cos x \sin y)/(\cos x \cos y)}$$

$$= \frac{(\sin x/\cos x) + (\sin y/\cos y)}{(\sin x/\cos x) - (\sin y/\cos y)}$$

$$= \frac{\tan x + \tan y}{\tan x - \tan y}$$

56. True. If $B = \pi - A$, then $\cos A + \cos B$
$= \cos A + \cos(\pi - A)$
$= \cos A + \cos\pi\cos A + \sin\pi\sin A$
$= \cos A + (-1)\cos A + (0)\sin A = 0.$

57. False. For example, $\cos 3\pi + \cos 4\pi = 0$, but 3π and 4π are not supplementary. And even though $\cos(3\pi/2) + \cos(3\pi/2) = 0$, $3\pi/2$ is not supplementary with itself.

58. If $\cos A \cos B = \sin A \sin B$, then $\cos(A+B) = \cos A \cos B - \sin A \sin B = 0$. The answer is A.

59. $y = \sin x \cos 2x + \cos x \sin 2x = \sin(x+2x) = \sin 3x$. The answer is A.

60. $\text{Sin } 15° = \sin(45° - 30°)$

$$= \sin 45° \cos 30° - \cos 45° \sin 30°$$

$$= \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{6} - \sqrt{2}}{4}$$

The answer is D.

61. For all u, v, $\tan(u+v) = \dfrac{\tan u + \tan v}{1 - \tan u \tan v}$. The answer is B.

62. $$\tan(u+v) = \frac{\sin(u+v)}{\cos(u+v)}$$

$$= \frac{\sin u \cos v + \cos u \sin v}{\cos u \cos v - \sin u \sin v}$$

$$= \frac{\frac{\sin u \cos v}{\cos u \cos v} + \frac{\cos u \sin v}{\cos u \cos v}}{\frac{\cos u \cos v}{\cos u \cos v} - \frac{\sin u \sin v}{\cos u \cos v}}$$

$$= \frac{\frac{\sin u}{\cos u} + \frac{\sin v}{\cos v}}{1 - \frac{\sin u \sin v}{\cos u \cos v}}$$

$$= \frac{\tan u + \tan v}{1 - \tan u \tan v}$$

63. $$\tan(u-v) = \frac{\sin(u-v)}{\cos(u-v)}$$

$$= \frac{\sin u \cos v - \cos u \sin v}{\cos u \cos v + \sin u \sin v}$$

$$= \frac{\frac{\sin u \cos v}{\cos u \cos v} - \frac{\cos u \sin v}{\cos u \cos v}}{\frac{\cos u \cos v}{\cos u \cos v} + \frac{\sin u \sin v}{\cos u \cos v}}$$

$$= \frac{\frac{\sin u}{\cos u} - \frac{\sin v}{\cos v}}{1 + \frac{\sin u \sin v}{\cos u \cos v}}$$

$$= \frac{\tan u - \tan v}{1 + \tan u \tan v}$$

64. The identity would involve $\tan\left(\frac{\pi}{2}\right)$, which does not exist.

$$\tan\left(x + \frac{\pi}{2}\right) = \frac{\sin\left(x + \frac{\pi}{2}\right)}{\cos\left(x + \frac{\pi}{2}\right)}$$

$$= \frac{\sin x \cos\frac{\pi}{2} + \cos x \sin\frac{\pi}{2}}{\cos x \cos\frac{\pi}{2} - \sin x \sin\frac{\pi}{2}}$$

$$= \frac{\sin x \cdot 0 + \cos x \cdot 1}{\cos x \cdot 0 - \sin x \cdot 1}$$

$$= -\cot x$$

65. The identity would involve $\tan\left(\frac{3\pi}{2}\right)$, which does not exit.

$$\tan\left(x - \frac{3\pi}{2}\right) = \frac{\sin\left(x - \frac{3\pi}{2}\right)}{\cos\left(x - \frac{3\pi}{2}\right)}$$

$$= \frac{\sin x \cos\frac{3\pi}{2} - \cos x \sin\frac{3\pi}{2}}{\cos x \cos\frac{3\pi}{2} + \sin x \sin\frac{3\pi}{2}}$$

$$= \frac{\sin x \cdot 0 - \cos x \cdot (-1)}{\cos x \cdot 0 + \sin x \cdot (-1)}$$

$$= -\cot x$$

66. $$\frac{\sin(x+h) - \sin x}{h} = \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

$$= \sin x\left(\frac{\cos h - 1}{h}\right) + \cos x \frac{\sin h}{h}$$

67. $\dfrac{\cos(x+h)-\cos x}{h} = \dfrac{\cos x \cos h - \sin x \sin h - \cos x}{h}$
$= \dfrac{\cos x(\cos h - 1) - \sin x \sin h}{h}$
$= \cos x\left(\dfrac{\cos h - 1}{h}\right) - \sin x \dfrac{\sin h}{h}$

68. The coordinates of all 24 points must be $\left(\cos\left(\frac{k\pi}{12}\right), \sin\left(\frac{k\pi}{12}\right)\right)$ for $k = 0, 1, 2, \ldots, 23$. We only need to find the coordinates of those points in Quadrant I, because the remaining points are symmetric. We already know the coordinates for the cases when $k = 0, 2, 3, 4, 6$ since these correspond to the special angles.

$k = 1$: $\cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)$
$+ \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) = \frac{1}{2}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}$
$= \dfrac{\sqrt{2}+\sqrt{6}}{4}$

$\sin\left(\frac{\pi}{12}\right) = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)$
$- \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2}$
$= \dfrac{\sqrt{6}-\sqrt{2}}{4}$

$k = 5$: $\cos\left(\frac{5\pi}{12}\right) = \cos\left(\frac{3\pi}{4} - \frac{\pi}{3}\right)$
$= \cos\left(\frac{3\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) + \sin\left(\frac{3\pi}{4}\right)\sin\left(\frac{\pi}{3}\right)$
$= -\frac{\sqrt{2}}{2}\cdot\frac{1}{2} + \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} = \dfrac{\sqrt{6}-\sqrt{2}}{4}$

$\sin\left(\frac{5\pi}{12}\right) = \sin\left(\frac{3\pi}{4} - \frac{\pi}{3}\right)$
$= \sin\left(\frac{3\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{3\pi}{4}\right)$
$= \frac{\sqrt{2}}{2}\cdot\frac{1}{2} - \frac{\sqrt{3}}{2}\cdot\left(-\frac{\sqrt{2}}{2}\right) = \dfrac{\sqrt{2}+\sqrt{6}}{4}$

Coordinates in the first quadrant are $(1, 0)$, $\left(\frac{\sqrt{2}+\sqrt{6}}{4}, \frac{\sqrt{6}-\sqrt{2}}{4}\right)$, $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $\left(\frac{\sqrt{6}-\sqrt{2}}{4}, \frac{\sqrt{2}+\sqrt{6}}{4}\right)$, $(0, 1)$

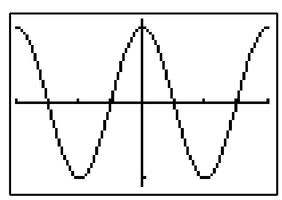
69. $\sin(A+B) = \sin(\pi - C)$
$= \sin\pi\cos C - \cos\pi\sin C$
$= 0\cdot\cos C - (-1)\sin C$
$= \sin C$

70. $\cos C = \cos(\pi - (A+B))$
$= \cos\pi\cos(A+B) + \sin\pi\sin(A+B)$
$= (-1)(\cos A\cos B - \sin A\sin B)$
$+ 0\cdot\sin(A+B)$
$= \sin A\sin B - \cos A\cos B$

71. $\tan A + \tan B + \tan C = \dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B} + \dfrac{\sin C}{\cos C}$
$= \dfrac{\sin A(\cos B\cos C) + \sin B(\cos A\cos C)}{\cos A\cos B\cos C}$
$+ \dfrac{\sin C(\cos A\cos B)}{\cos A\cos B\cos C}$
$= \dfrac{\cos C(\sin A\cos B + \cos A\sin B) + \sin C(\cos A\cos B)}{\cos A\cos B\cos C}$
$= \dfrac{\cos C\sin(A+B) + \sin C(\cos(A+B) + \sin A\sin B)}{\cos A\cos B\cos C}$
$= \dfrac{\cos C\sin(\pi - C) + \sin C(\cos(\pi - C) + \sin A\sin B)}{\cos A\cos B\cos C}$
$= \dfrac{\cos C\sin C + \sin C(-\cos C) + \sin C\sin A\sin B}{\cos A\cos B\cos C}$
$= \dfrac{\sin A\sin B\sin C}{\cos A\cos B\cos C}$
$= \tan A\tan B\tan C$

72. $\cos A\cos B\cos C - \sin A\sin B\cos C$
$- \sin A\cos B\sin C - \cos A\sin B\sin C$
$= \cos A(\cos B\cos C - \sin B\sin C)$
$- \sin A(\sin B\cos C + \cos B\sin C)$
$= \cos A\cos(B+C) - \sin A\sin(B+C)$
$= \cos(A+B+C)$
$= \cos\pi$
$= -1$

73. This equation is easier to deal with after rewriting it as $\cos 5x\cos 4x + \sin 5x\sin 4x = 0$. The left side of this equation is the expanded form of $\cos(5x - 4x)$, which of course equals $\cos x$; the graph shown is simply $y = \cos x$. The equation $\cos x = 0$ is easily solved on the interval $[-2\pi, 2\pi]$: $x = \pm\frac{\pi}{2}$ or $x = \pm\frac{3\pi}{2}$. The original graph is so crowded that one cannot see where crossings occur.



$[-2\pi, 2\pi]$ by $[-1.1, 1.1]$

74. $x = a\cos\left(\frac{2\pi t}{T} + \delta\right)$
$= a\left[\cos\left(\frac{2\pi t}{T}\right)\cos\delta - \sin\left(\frac{2\pi t}{T}\right)\sin\delta\right]$
$= (a\cos\delta)\cos\left(\frac{2\pi t}{T}\right) + (-a\sin\delta)\sin\left(\frac{2\pi t}{T}\right)$

75. $B = B_{in} + B_{ref}$
$= \frac{E_0}{c}\cos\left(\omega t - \frac{\omega x}{c}\right) + \frac{E_0}{c}\cos\left(\omega t + \frac{\omega x}{c}\right)$
$= \frac{E_0}{c}\left(\cos\omega t\cos\frac{\omega x}{c} + \sin\omega t\sin\frac{\omega x}{c}\right.$
$\left. + \cos\omega t\cos\frac{\omega x}{c} - \sin\omega t\sin\frac{\omega x}{c}\right)$
$= \frac{E_0}{c}\left(2\cos\omega t\cos\frac{\omega x}{c}\right) = 2\frac{E_0}{c}\cos\omega t\cos\frac{\omega x}{c}$

■ Section 5.4 Multiple-Angle Identities

Exploration 1

1. $\sin^2 \frac{\pi}{8} = \frac{1 - \cos(\pi/4)}{2}$
$= \frac{1 - (\sqrt{2}/2)}{2} \cdot \frac{2}{2}$
$= \frac{2 - \sqrt{2}}{4}$

2. $\sin \frac{\pi}{8} = \pm\sqrt{\frac{2 - \sqrt{2}}{4}} = \frac{\sqrt{2 - \sqrt{2}}}{2}.$

We take the positive square root because $\frac{\pi}{8}$ is a first-quadrant angle.

3. $\sin^2 \frac{9\pi}{8} = \frac{1 - \cos(9\pi/4)}{2}$
$= \frac{1 - (\sqrt{2}/2)}{2} \cdot \frac{2}{2}$
$= \frac{2 - \sqrt{2}}{4}$

4. $\sin \frac{9\pi}{8} = \pm\sqrt{\frac{2 - \sqrt{2}}{4}} = \frac{-\sqrt{2 - \sqrt{2}}}{2}.$

We take the negative square root because $\frac{\pi}{8}$ is a third-quadrant angle.

Quick Review 5.4

1. $\tan x = 1$ when $x = \frac{\pi}{4} + n\pi$, n an integer

2. $\tan x = -1$ when $x = -\frac{\pi}{4} + n\pi$, n an integer

3. Either $\cos x = 0$ or $\sin x = 1$. The latter implies the former, so $x = \frac{\pi}{2} + n\pi$, n an integer.

4. Either $\sin x = 0$ or $\cos x = -1$. The latter implies the former, so $x = n\pi$, n an integer.

5. $\sin x = -\cos x$ when $x = -\frac{\pi}{4} + n\pi$, n an integer

6. $\sin x = \cos x$ when $x = \frac{\pi}{4} + n\pi$, n an integer

7. Either $\sin x = \frac{1}{2}$ or $\cos x = -\frac{1}{2}$. Then $x = \frac{\pi}{6} + 2n\pi$ or $x = \frac{5\pi}{6} + 2n\pi$ or $x = \pm\frac{2\pi}{3} + 2n\pi$, n an integer.

8. Either $\sin x = -1$ or $\cos x = \frac{\sqrt{2}}{2}$. Then $x = \frac{3\pi}{2} + 2n\pi$ or $x = \pm\frac{\pi}{4} + 2n\pi$, n an integer.

9. The trapezoid can be viewed as a rectangle and two triangles; the area is then $A = (2)(3) + \frac{1}{2}(1)(3) + \frac{1}{2}(2)(3)$ $= 10.5$ square units.

10. View the triangle as two right triangles with hypotenuse 3, one leg 1, and the other leg — the height — equal to $\sqrt{3^2 - 1^2} = \sqrt{8} = 2\sqrt{2}$

Section 5.4 Exercises

1. $\cos 2u = \cos(u + u) = \cos u \cos u - \sin u \sin u$
$= \cos^2 u - \sin^2 u$

2. Starting with the result of #1: $\cos 2u = \cos^2 u - \sin^2 u$
$= \cos^2 u - (1 - \cos^2 u) = 2\cos^2 u - 1$

3. Starting with the result of #1: $\cos 2u = \cos^2 u - \sin^2 u$
$= (1 - \sin^2 u) - \sin^2 u = 1 - 2\sin^2 u$

4. $\tan 2u = \tan(u + u) = \frac{\tan u + \tan u}{1 - \tan u \tan u} = \frac{2\tan u}{1 - \tan^2 u}$

5. $2\sin x \cos x - 2\sin x = 0$, so $2\sin x(\cos x - 1) = 0$; $\sin x = 0$ or $\cos x = 1$ when $x = 0$ or $x = \pi$.

6. $2\sin x \cos x - \sin x = 0$
$\sin x(2\cos x - 1) = 0,$
So $\sin x = 0$ or $\cos x = \frac{1}{2}$
when $x = 0, \pi, \frac{\pi}{3}$, or $\frac{5\pi}{3}$.

7. $2\sin^2 x + \sin x - 1 = 0$, so $(2\sin x - 1)(\sin x + 1)$ $= 0$; $\sin x = \frac{1}{2}$ or $\sin x = -1$ when $x = \frac{\pi}{6}$, $x = \frac{5\pi}{6}$ or $x = \frac{3\pi}{2}$.

8. $2\cos^2 x - \cos x - 1 = 0$, so $(2\cos x + 1)(\cos x - 1)$ $= 0$; $\cos x = -\frac{1}{2}$ or $\cos x = 1$ when $x = 0$, $x = \frac{2\pi}{3}$ or $x = \frac{4\pi}{3}$

9. $2\sin x \cos x - \frac{\sin x}{\cos x} = 0$, so $\frac{\sin x}{\cos x}(2\cos^2 x - 1) = 0$, or $\frac{\sin x \cos 2x}{\cos x} = 0$. Then $\sin x = 0$ or $\cos 2x = 0$ (but $\cos x \neq 0$), so $x = 0$, $x = \frac{\pi}{4}$, $x = \frac{3\pi}{4}$, $x = \pi$, $x = \frac{5\pi}{4}$ or $x = \frac{7\pi}{4}$.

10. $\cos^2 x - \cos x - 1 = 0$, so $\cos x = \frac{1 \pm \sqrt{5}}{2}$. Only $\frac{1 - \sqrt{5}}{2}$ is in $[-1, 1]$, so $x = \cos^{-1}\left(\frac{1 - \sqrt{5}}{2}\right) \approx 2.2370$ or $x = 2\pi - \cos^{-1}\left(\frac{1 - \sqrt{5}}{2}\right) \approx 4.0461$

For #11–14, any one of the last several expressions given is an answer to the question. In some cases, other answers are possible, as well.

11. $\sin 2\theta + \cos\theta = 2\sin\theta\cos\theta + \cos\theta$
$= (\cos\theta)(2\sin\theta + 1)$

12. $\sin 2\theta + \cos 2\theta = 2\sin\theta\cos\theta + \cos^2\theta - \sin^2\theta$
$= 2\sin\theta\cos\theta + 2\cos^2\theta - 1$
$= 2\sin\theta\cos\theta + 1 - 2\sin^2\theta$

13. $\sin 2\theta + \cos 3\theta$
$= 2\sin\theta\cos\theta + \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$
$= 2\sin\theta\cos\theta + (\cos^2\theta - \sin^2\theta)\cos\theta - 2\sin^2\theta\cos\theta$
$= 2\sin\theta\cos\theta + \cos^3\theta - 3\sin^2\theta\cos\theta$
$= 2\sin\theta\cos\theta + 4\cos^3\theta - 3\cos\theta$

14. $\sin 3\theta + \cos 2\theta$
$= \sin 2\theta \cos\theta + \cos 2\theta \sin\theta + \cos^2\theta - \sin^2\theta$
$= 2\sin\theta\cos^2\theta + (\cos^2\theta - \sin^2\theta)\sin\theta + \cos^2\theta - \sin^2\theta$
$= 3\sin\theta\cos^2\theta - \sin^3\theta + \cos^2\theta - \sin^2\theta$

15. $\sin 4x = \sin 2(2x) = 2\sin 2x\cos 2x$

16. $\cos 6x = \cos 2(3x) = 2\cos^2 3x - 1$

17. $2\csc 2x = \dfrac{2}{\sin 2x} = \dfrac{2}{2\sin x\cos x}$
$= \dfrac{1}{\sin^2 x}\cdot\dfrac{\sin x}{\cos x} = \csc^2 x\tan x$

18. $2\cot 2x = \dfrac{2}{\tan 2x} = \dfrac{2(1-\tan^2 x)}{2\tan x} = \dfrac{1}{\tan x} - \tan x$
$= \cot x - \tan x$

19. $\sin 3x = \sin 2x\cos x + \cos 2x\sin x = 2\sin x\cos^2 x$
$+ (2\cos^2 x - 1)\sin x = (\sin x)(4\cos^2 x - 1)$

20. $\sin 3x = \sin 2x\cos x + \cos 2x\sin x$
$= 2\sin x\cos^2 x + (1 - 2\sin^2 x)\sin x$
$= (\sin x)(2\cos^2 x + 1 - 2\sin^2 x)$
$= (\sin x)(3 - 4\sin^2 x)$

21. $\cos 4x = \cos 2(2x) = 1 - 2\sin^2 2x$
$= 1 - 2(2\sin x\cos x)^2 = 1 - 8\sin^2 x\cos^2 x$

22. $\sin 4x = \sin 2(2x) = 2\sin 2x\cos 2x$
$= 2(2\sin x\cos x)(2\cos^2 x - 1)$
$= (4\sin x\cos x)(2\cos^2 x - 1)$

23. $2\cos^2 x + \cos x - 1 = 0$, so $\cos x = -1$ or $\cos x = \dfrac{1}{2}$,
$x = \dfrac{\pi}{3}, x = \pi$ or $x = \dfrac{5\pi}{3}$

24. $\cos 2x + \sin x = 1 - 2\sin^2 x + \sin x = 0$, so
$\sin x = 1$ or $\sin x = -\dfrac{1}{2}, x = \dfrac{\pi}{2}, x = \dfrac{7\pi}{6}$, or $x = \dfrac{11\pi}{6}$.

25. $\cos 3x = \cos 2x\cos x - \sin 2x\sin x$
$= (1 - 2\sin^2 x)\cos x$
$- (2\sin x\cos x)\sin x$
$= \cos x - 2\sin^2 x\cos x$
$- 2\sin^2 x\cos x$
$= \cos x - 4\sin^2 x\cos x$
Thus the left side can be written as $2(\cos x)(1 - 2\sin^2 x)$
$= 2\cos x\cos 2x$. This equals 0 in $[0, 2\pi)$ when
$x = \dfrac{\pi}{4}, x = \dfrac{\pi}{2}, x = \dfrac{3\pi}{4}, x = \dfrac{5\pi}{4}, x = \dfrac{3\pi}{2}$, or $x = \dfrac{7\pi}{4}$.

26. Using #19, this become $4\sin x\cos^2 x = 0$, so $x = 0$,
$x = \dfrac{\pi}{2}, x = \pi$, or $x = \dfrac{3\pi}{2}$.

27. $\sin 2x + \sin 4x = \sin 2x + 2\sin 2x\cos 2x$
$= (\sin 2x)(1 + 2\cos 2x) = 0$. Then $\sin 2x = 0$ or
$\cos 2x = -\dfrac{1}{2}$; the solutions in $[0, 2\pi)$ are
$x = 0, x = \dfrac{\pi}{3}, x = \dfrac{\pi}{2}$,
$x = \dfrac{2\pi}{3}, x = \pi, x = \dfrac{4\pi}{3}, x = \dfrac{3\pi}{2}$, or $x = \dfrac{5\pi}{3}$.

28. With $u = 2x$, this becomes $\cos u + \cos 2u = 0$, the same as #23. This means $u = \dfrac{\pi}{3}, u = \pi, u = \dfrac{5\pi}{3}$, etc. — i.e., $2x = \dfrac{\pi}{3} + n\dfrac{2\pi}{3}$. Then $x = \dfrac{\pi}{6}, x = \dfrac{\pi}{2}, x = \dfrac{5\pi}{6}$,
$x = \dfrac{7\pi}{6}, x = \dfrac{3\pi}{2}, x = \dfrac{11\pi}{6}$.

29. Using results from #25, $\sin 2x - \cos 3x$
$= (2\sin x\cos x) - (\cos x - 4\sin^2 x\cos x)$
$= (\cos x)(4\sin^2 x + 2\sin x - 1) = 0$.
$\cos x = 0$ when $x = \dfrac{\pi}{2}$ or $x = \dfrac{3\pi}{2}$, while the second factor equals zero when $\sin x = \dfrac{-1 \pm \sqrt{5}}{4}$. It turns out — as can be observed by noting, e.g., that
$\sin^{-1}\left(\dfrac{-1+\sqrt{5}}{4}\right) \approx 0.31415926$ — that this means
$x = 0.1\pi, x = 0.9\pi, x = 1.3\pi$, or $x = 1.7\pi$.

30. Using #14, the left side can be rewritten as
$3\sin x\cos^2 x - \sin^3 x + \cos^2 x - \sin^2 x$.
Replacing $\cos^2 x$ with $1 - \sin^2 x$ gives
$-4\sin^3 x - 2\sin^2 x + 3\sin x + 1$
$= (\sin x + 1)(-4\sin^2 x + 2\sin x + 1)$.
This equals 0 when $\sin x = -1\left(x = \dfrac{3\pi}{2}\right)$, and when $\sin x = \dfrac{1 \pm \sqrt{5}}{4}$. These values turn out to be
$x = 0.3\pi, x = 0.7\pi, x = 1.1\pi$, and $x = 1.9\pi$,
as can be observed by noting, e.g., that
$\sin^{-1}\left(\dfrac{1-\sqrt{5}}{4}\right) \approx -0.31415926$.

31. $\sin 15° = \pm\sqrt{\dfrac{1 - \cos 30°}{2}} = \pm\sqrt{\dfrac{1}{2}\left(1 - \dfrac{\sqrt{3}}{2}\right)}$
$= \pm\dfrac{1}{2}\sqrt{2 - \sqrt{3}}$. Since $\sin 15° > 0$, take the positive square root.

32. $\tan 195° = \dfrac{1 - \cos 390°}{\sin 390°} = \dfrac{1 - \sqrt{3}/2}{1/2} = 2 - \sqrt{3}$. Note that $\tan 195° = \tan 15°$.

33. $\cos 75° = \pm\sqrt{\dfrac{1 + \cos 150°}{2}} = \pm\sqrt{\dfrac{1}{2}\left(1 - \dfrac{\sqrt{3}}{2}\right)}$
$= \pm\dfrac{1}{2}\sqrt{2 - \sqrt{3}}$. Since $\cos 75° > 0$, take the positive square root.

34. $\sin\dfrac{5\pi}{12} = \pm\sqrt{\dfrac{1 - \cos(5\pi/6)}{2}} = \pm\sqrt{\dfrac{1}{2}\left(1 + \dfrac{\sqrt{3}}{2}\right)}$
$= \pm\dfrac{1}{2}\sqrt{2 + \sqrt{3}}$. Since $\sin\dfrac{5\pi}{12} > 0$, take the positive square root.

35. $\tan\dfrac{7\pi}{12} = \dfrac{1 - \cos(7\pi/6)}{\sin(7\pi/6)} = \dfrac{1 + \sqrt{3}/2}{-1/2} = -2 - \sqrt{3}$.

36. $\cos\dfrac{\pi}{8} = \pm\sqrt{\dfrac{1 + \cos(\pi/4)}{2}} = \pm\sqrt{\dfrac{1}{2}\left(1 + \dfrac{\sqrt{2}}{2}\right)}$
$= \pm\dfrac{1}{2}\sqrt{2 + \sqrt{2}}$. Since $\cos\dfrac{\pi}{8} > 0$, take the positive square root.

37. (a) Starting from the right side: $\frac{1}{2}(1 - \cos 2u)$

$= \frac{1}{2}[1 - (1 - 2\sin^2 u)] = \frac{1}{2}(2\sin^2 u) = \sin^2 u.$

(b) Starting from the right side: $\frac{1}{2}(1 + \cos 2u)$

$= \frac{1}{2}[1 + (2\cos^2 u - 1)] = \frac{1}{2}(2\cos^2 u) = \cos^2 u.$

38. (a) $\tan^2 u = \frac{\sin^2 u}{\cos^2 u} = \frac{(1 - \cos 2u)/2}{(1 + \cos 2u)/2} = \frac{1 - \cos 2u}{1 + \cos 2u}$

(b) The equation is false when $\tan u$ is a negative number. It would be an identity if it were written as $|\tan u| = \sqrt{\frac{1 - \cos u}{1 + \cos u}}$.

39. $\sin^4 x = (\sin^2 x)^2 = \left[\frac{1}{2}(1 - \cos 2x)\right]^2$

$= \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x)$

$= \frac{1}{4}\left[1 - 2\cos 2x + \frac{1}{2}(1 + \cos 4x)\right]$

$= \frac{1}{8}(2 - 4\cos 2x + 1 + \cos 4x)$

$= \frac{1}{8}(3 - 4\cos 2x + \cos 4x)$

40. $\cos^3 x = \cos x \cos^2 x = \cos x \cdot \frac{1}{2}(1 + \cos 2x)$

$= \frac{1}{2}(\cos x)(1 + \cos 2x)$

41. $\sin^3 2x = \sin 2x \sin^2 2x = \sin 2x \cdot \frac{1}{2}(1 - \cos 4x)$

$= \frac{1}{2}(\sin 2x)(1 - \cos 4x)$

42. $\sin^5 x = (\sin x)(\sin^2 x)^2 = (\sin x)\left[\frac{1}{2}(1 - \cos 2x)\right]^2$

$= \frac{1}{4}(\sin x)(1 - 2\cos 2x + \cos^2 2x)$

$= \frac{1}{4}(\sin x)\left[1 - 2\cos 2x + \frac{1}{2}(1 + \cos 4x)\right]$

$= \frac{1}{8}(\sin x)(2 - 4\cos 2x + 1 + \cos 4x)$

$= \frac{1}{8}(\sin x)(3 - 4\cos 2x + \cos 4x).$

Alternatively, take $\sin^5 x = \sin x \sin^4 x$ and apply the result of #39.

43. $\cos^2 x = \frac{1 - \cos x}{2}$, so $2\cos^2 x + \cos x - 1 = 0$. Then $\cos x = -1$ or $\cos x = \frac{1}{2}$. In the interval $[0, 2\pi)$, $x = \frac{\pi}{3}$, $x = \pi$, or $x = \frac{5\pi}{3}$. General solution: $= \pm\frac{\pi}{3} + 2n\pi$ or $x = \pi + 2n\pi$, n an integer.

44. $1 - \cos^2 x = \frac{1 + \cos x}{2}$, so $2\cos^2 x + \cos x - 1 = 0$. Then $\cos x = -1$ or $\cos x = \frac{1}{2}$. In the interval $[0, 2\pi)$, $x = \frac{\pi}{3}$, $x = \pi$, or $x = \frac{5\pi}{3}$. General solution: $x = \pm\frac{\pi}{3} + 2n\pi$ or $x = \pi + 2n\pi$, n an integer.

45. The right side equals $\tan^2(x/2)$; the only way that $\tan(x/2) = \tan^2(x/2)$ is if either $\tan(x/2) = 0$ or $\tan(x/2) = 1$. In $[0, 2\pi)$, this happens when $x = 0$ or $x = \frac{\pi}{2}$. The general solution is $x = 2n\pi$ or $x = \frac{\pi}{2} + 2n\pi$, n an integer.

46. $\frac{1 - \cos x}{2} = 2\cos^2 x - 1$, so $4\cos^2 x + \cos x - 3 = 0$, or $(4\cos x - 3)(\cos x + 1) = 0$. Then $\cos x = -1$ or $\cos x = \frac{3}{4}$. Let $\alpha = \cos^{-1}\left(\frac{3}{4}\right) \approx 0.7227$. In the interval $[0, 2\pi)$, $x = \alpha$, $x = \pi$, or $x = 2\pi - \alpha$. General solution: $x = \pm\alpha + 2n\pi$ or $x = \pi + 2n\pi$, n an integer.

47. False. For example, $f(x) = 2\sin x$ has period 2π and $g(x) = \cos x$ has period 2π, but the product $f(x)g(x) = 2\sin x \cos x = \sin 2x$ has period π.

48. True. $\cos^2 x = \frac{1 + \cos 2x}{2}$

$= \frac{1}{2} + \frac{1}{2}\cos 2x$

$= \frac{1}{2}\sin\left(\frac{\pi}{2} - 2x\right) + \frac{1}{2}$

$= \frac{1}{2}\sin\left(-2\left(x - \frac{\pi}{4}\right)\right) + \frac{1}{2}$

The last expression is in the form for a sinusoid.

49. $f(2x) = \sin 2x = 2\sin x \cos x = 2f(x)g(x)$. The answer is D.

50. $\sin 22.5° = \sin\left(\frac{45°}{2}\right)$

$= \sqrt{\frac{1 - \cos 45°}{2}}$

$= \sqrt{\frac{1 - \sqrt{2}/2}{2}}$

$= \sqrt{\frac{2 - \sqrt{2}}{4}}$

$= \frac{\sqrt{2 - \sqrt{2}}}{2}$

The answer is E.

51. $\sin 2x = \cos x$

$2\sin x \cos x = \cos x$

$2\sin x = 1$ or $\cos x = 0$

$\sin x = \frac{1}{2}$ or $\cos x = 0$

$x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ $\quad x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

The answer is E.

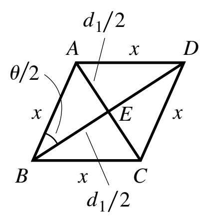
52. $\sin^2 x - \cos^2 x = 1 - 2\cos^2 x$, which has the same period as the function $\cos^2 x$, namely π. The answer is C.

53. (a) In the figure, the triangle with side lengths $x/2$ and R is a right triangle, since R is given as the perpendicular distance. Then the tangent of the angle $\theta/2$ is the ratio "opposite over adjacent": $\tan\frac{\theta}{2} = \frac{x/2}{R}$ Solving for x gives the desired equation. The central angle θ is $2\pi/n$ since one full revolution of 2π radians is divided evenly into n sections.

(b) $5.87 \approx 2R\tan\frac{\theta}{2}$, where $\theta = 2\pi/11$, so

$R \approx 5.87/(2\tan\frac{\pi}{11}) \approx 9.9957$. $R = 10$.

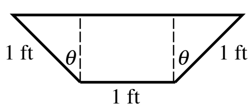
54. (a)



Call the center of the rhombus E. Consider right ΔABE, with legs $d_2/2$ and $d_1/2$, and hypotenuse length x. $\angle ABE$ has measure $\theta/2$, and using "sine equals $\frac{\text{opp}}{\text{hyp}}$," and "cosine equals $\frac{\text{adj}}{\text{hyp}}$," we have $\cos\frac{\theta}{2} = \frac{d_2/2}{x} = \frac{d_2}{2x}$ and $\sin\frac{\theta}{2} = \frac{d_1/2}{x} = \frac{d_1}{2x}$.

(b) Use the double angle formula for the sine function:
$\sin\theta = \sin 2\left(\frac{\theta}{2}\right) = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\frac{d_1}{2x}\cdot\frac{d_2}{2x} = \frac{d_1 d_2}{2x^2}$

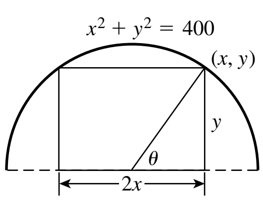
55. (a)



The volume is 10 ft times the area of the end. The end is made up of two identical triangles, with area $\frac{1}{2}(\sin\theta)(\cos\theta)$ each, and a rectangle with area $(1)(\cos\theta)$. The total volume is then $10\cdot(\sin\theta\cos\theta + \cos\theta) = 10(\cos\theta)(1 + \sin\theta)$.

Considering only $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, the maximum value occurs when $\theta \approx 0.52$ (in fact, it happens exactly at $\theta = \frac{\pi}{6}$). The maximum value is about 12.99 ft^3.

56. (a)



The height of the tunnel is y, and the width is $2x$, so the area is $2xy$. The x- and y-coordinates of the vertex are $20\cos\theta$ and $20\sin\theta$, so the area is $2(20\cos\theta)(20\sin\theta) = 400(2\cos\theta\sin\theta) = 400\sin 2\theta$.

(b) Considering $0 \le \theta \le \frac{\pi}{2}$, the maximum area occurs when $\theta = \frac{\pi}{4}$, or about 0.79. This gives $x = 20\cos\frac{\pi}{4} = 10\sqrt{2}$, or about 14.14, for a width of about 28.28, and a height of $y = 10\sqrt{2} \approx 14.14$

57. $\csc 2u = \frac{1}{\sin 2u} = \frac{1}{2\sin u\cos u} = \frac{1}{2}\cdot\frac{1}{\sin u}\cdot\frac{1}{\cos u}$
$= \frac{1}{2}\csc u \sec u$

58. $\cot 2u = \frac{1}{\tan 2u} = \frac{1 - \tan^2 u}{2\tan u}$
$= \left(\frac{1 - \tan^2 u}{2\tan u}\right)\left(\frac{\cot^2 u}{\cot^2 u}\right) = \frac{\cot^2 u - 1}{2\cot u}$

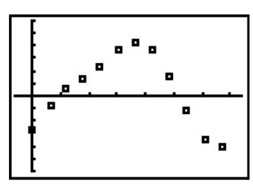
59. $\sec 2u = \frac{1}{\cos 2u} = \frac{1}{1 - 2\sin^2 u}$
$= \left(\frac{1}{1 - 2\sin^2 u}\right)\left(\frac{\csc^2 u}{\csc^2 u}\right) = \frac{\csc^2 u}{\csc^2 u - 2}$

60. $\sec 2u = \frac{1}{\cos 2u} = \frac{1}{2\cos^2 u - 1}$
$= \left(\frac{1}{2\cos^2 u - 1}\right)\left(\frac{\sec^2 u}{\sec^2 u}\right) = \frac{\sec^2 u}{2 - \sec^2 u}$

61. $\sec 2u = \frac{1}{\cos 2u} = \frac{1}{\cos^2 u - \sin^2 u}$
$= \left(\frac{1}{\cos^2 u - \sin^2 u}\right)\left(\frac{\sec^2 u\csc^2 u}{\sec^2 u\csc^2 u}\right)$
$= \frac{\sec^2 u\csc^2 u}{\csc^2 u - \sec^2 u}$

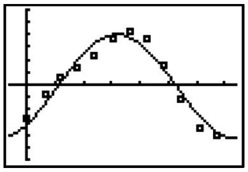
62. The second equation cannot work for any values of x for which $\sin x < 0$, since the square root cannot be negative. The first is correct since a double angle identity for the cosine gives $\cos 2x = 1 - 2\sin^2 x$; solving for $\sin x$ gives $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, so that $\sin x = \pm\sqrt{\frac{1}{2}(1 - \cos 2x)}$. The absolute value of both sides removes the "$\pm$."

63. (a) The following is a scatter plot of the days past January 1 as x-coordinates (L1) and the time (in 24 hour mode) as y-coordinates (L2) for the time of day that astronomical twilight began in northeastern Mali in 2005.



[–30, 370] by [–60, 60]

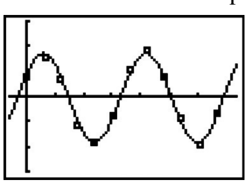
(b) The sine regression curve through the points defined by L1 and L2 is $y = 41.656 \sin(0.015x - 0.825) - 1.473$. This is a fairly good fit, but not really as good as one might expect from data generated by a sinusoidal physical model.



[–30, 370] by [–60, 60]

(c) Using the formula L2 − Y1(L1) (where Y1 is the sine regression curve), the residual list is: {3.64, 7.56, 3.35, −5.94, −9.35, −3.90, 5.12, 9.43, 3.90, −4.57, −9.72, −3.22}.

(d) The following is a scatter plot of the days past January 1 as x-coordinates (L1) and the residuals (the difference between the actual number of minutes (L2) and the number of minutes predicted by the regression curve (Y1)) as y-coordinates (L3) for the time of day that astronomical twilight began in northeastern Mali in 2005.
The sine regression curve through the points defined by L1 and L3 is $y = 8.856 \sin(0.0346x + 0.576) - 0.331$. (Note: Round L3 to 2 decimal places to obtain this answer.) This is another fairly good fit, which indicates that the residuals are not due to chance. There is a periodic variation that is most probably due to physical causes.



[–30, 370] by [–15, 15]

(e) The first regression indicates that the data are periodic and nearly sinusoidal. The second regression indicates that the *variation* of the data around the predicted values is also periodic and nearly sinusoidal. Periodic variation around periodic models is a predictable consequence of bodies orbiting bodies, but ancient astronomers had a difficult time reconciling the data with their simpler models of the universe.

■ Section 5.5 The Law of Sines

Exploration 1

1. If $BC \le AB$, the segment will not reach from point B to the dotted line. On the other hand, if $BC > AB$, then a circle of radius BC will intersect the dotted line in a unique point. (Note that the line only extends to the left of point A.)
2. A circle of radius BC will be tangent to the dotted line at C if $BC = h$, thus determining a unique triangle. It will miss the dotted line entirely if $BC < h$, thus determining zero triangles.
3. The second point (C_2) is the reflection of the first point (C_1) on the other side of the altitude.
4. $\sin C_2 = \sin(\pi - C_1) = \sin\pi\cos C_1 - \cos\pi\sin C_1 = \sin C_1$.
5. If $BC \ge AB$, then BC can only extend to the right of the altitude, thus determining a unique triangle.

Quick Review 5.5

1. $a = bc/d$
2. $b = ad/c$
3. $c = ad/b$
4. $d = bc/a$
5. $\dfrac{7\sin 48°}{\sin 23°} \approx 13.314°$
6. $\dfrac{9\sin 121°}{\sin 14°} \approx 31.888°$
7. $x = \sin^{-1} 0.3 \approx 17.458°$
8. $x = 180° - \sin^{-1} 0.3 \approx 162.542°$
9. $x = 180° - \sin^{-1}(-0.7) \approx 224.427°$
10. $x = 360° + \sin^{-1}(-0.7) \approx 315.573°$

Section 5.5 Exercises

1. Given: $b = 3.7, B = 45°, A = 60°$ — an AAS case.
$C = 180° - (A + B) = 75°$;
$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} \Rightarrow a = \dfrac{b\sin A}{\sin B} = \dfrac{3.7\sin 60°}{\sin 45°} \approx 4.5$;
$\dfrac{b}{\sin B} = \dfrac{c}{\sin C} \Rightarrow c = \dfrac{b\sin C}{\sin B} = \dfrac{3.7\sin 75°}{\sin 45°} \approx 5.1$
2. Given: $c = 17, B = 15°, C = 120°$ — an AAS case.
$A = 180° - (B + C) = 45°$;
$\dfrac{a}{\sin A} = \dfrac{c}{\sin C} \Rightarrow a = \dfrac{c\sin A}{\sin C} = \dfrac{17\sin 45°}{\sin 120°} \approx 13.9$;
$\dfrac{b}{\sin B} = \dfrac{c}{\sin C} \Rightarrow b = \dfrac{c\sin B}{\sin C} = \dfrac{17\sin 15°}{\sin 120°} \approx 5.1$
3. Given: $A = 100°, C = 35°, a = 22$ — an AAS case.
$B = 180° - (A + C) = 45°$;
$b = \dfrac{a\sin B}{\sin A} = \dfrac{22\sin 45°}{\sin 100°} \approx 15.8$;
$c = \dfrac{a\sin C}{\sin A} = \dfrac{22\sin 35°}{\sin 100°} \approx 12.8$
4. Given: $A = 81°, B = 40°, b = 92$ — an AAS case.
$C = 180° - (A + B) = 59°$;
$a = \dfrac{b\sin A}{\sin B} = \dfrac{92\sin 81°}{\sin 40°} \approx 141.4$;
$c = \dfrac{b\sin C}{\sin B} = \dfrac{92\sin 59°}{\sin 40°} \approx 122.7$
5. Given: $A = 40°, B = 30°, b = 10$ — an AAS case.
$C = 180° - (A + B) = 110°$;
$a = \dfrac{b\sin A}{\sin B} = \dfrac{10\sin 40°}{\sin 30°} \approx 12.9$;
$c = \dfrac{b\sin C}{\sin B} = \dfrac{10\sin 110°}{\sin 30°} \approx 18.8$
6. Given: $A = 50°, B = 62°, a = 4$ — an AAS case.
$C = 180° - (A + B) = 68°$;
$b = \dfrac{a\sin B}{\sin A} = \dfrac{4\sin 62°}{\sin 50°} \approx 4.6$;
$c = \dfrac{a\sin C}{\sin A} = \dfrac{4\sin 68°}{\sin 50°} \approx 4.8$

7. Given: $A = 33°, B = 70°, b = 7$ — an AAS case.
$C = 180° - (A + B) = 77°$;
$a = \dfrac{b \sin A}{\sin B} = \dfrac{7 \sin 33°}{\sin 70°} \approx 4.1$;
$c = \dfrac{b \sin C}{\sin B} = \dfrac{7 \sin 77°}{\sin 70°} \approx 7.3$

8. Given: $B = 16°, C = 103°, c = 12$ — an AAS case.
$A = 180° - (B + C) = 61°$;
$a = \dfrac{c \sin A}{\sin C} = \dfrac{12 \sin 61°}{\sin 103°} \approx 10.8$;
$b = \dfrac{c \sin B}{\sin C} = \dfrac{12 \sin 16°}{\sin 103°} \approx 3.4$

9. Given: $A = 32°, a = 17, b = 11$ — an SSA case.
$h = b \sin A \approx 5.8$; $h < b < a$, so there is one triangle.
$B = \sin^{-1}\left(\dfrac{b \sin A}{a}\right) = \sin^{-1}(0.342\ldots) \approx 20.1°$
$C = 180° - (A + B) \approx 127.9°$;
$c = \dfrac{a \sin C}{\sin A} = \dfrac{17 \sin 127.9°}{\sin 32°} \approx 25.3$

10. Given: $A = 49°, a = 32, b = 28$ — an SSA case.
$h = b \sin A \approx 21.1$; $h < b < a$, so there is one triangle.
$B = \sin^{-1}\left(\dfrac{b \sin A}{a}\right) = \sin^{-1}(0.660\ldots) \approx 41.3°$
$C = 180° - (A + B) = 89.7°$;
$c = \dfrac{a \sin C}{\sin A} = \dfrac{32 \sin 89.7°}{\sin 49°} \approx 42.4$

11. Given: $B = 70°, b = 14, c = 9$ — an SSA case.
$h = c \sin B \approx 8.5$; $h < c < b$, so there is one triangle.
$C = \sin^{-1}\left(\dfrac{c \sin B}{b}\right) = \sin^{-1}(0.604\ldots) \approx 37.2°$
$A = 180° - (B + C) \approx 72.8°$;
$a = \dfrac{b \sin A}{\sin B} = \dfrac{14 \sin 72.8°}{\sin 70°} \approx 14.2$

12. Given: $C = 103°, b = 46, c = 61$ — an SSA case.
$h = b \sin C \approx 44.8$; $h < b < c$, so there is one triangle.
$B = \sin^{-1}\left(\dfrac{b \sin C}{c}\right) = \sin^{-1}(0.734\ldots) \approx 47.3°$
$A = 180° - (B + C) = 29.7°$;
$a = \dfrac{c \sin A}{\sin C} = \dfrac{61 \sin 29.7°}{\sin 103°} \approx 31.0$

13. Given: $A = 36°, a = 2, b = 7$. $h = b \sin A \approx 4.1$;
$a < h$, so no triangle is formed.

14. Given: $B = 82°, b = 17, c = 15$. $h = c \sin B \approx 14.9$;
$h < c < b$, so there is one triangle.

15. Given: $C = 36°, a = 17, c = 16$. $h = a \sin C \approx 10.0$;
$h < c < a$, so there are two triangles.

16. Given: $A = 73°, a = 24, b = 28$. $h = b \sin A \approx 26.8$;
$a < h$, so no triangle is formed.

17. Given: $C = 30°, a = 18, c = 9$. $h = a \sin C = 9$;
$h = c$, so there is one triangle.

18. Given: $B = 88°, b = 14, c = 62$. $h = c \sin B \approx 62.0$;
$b < h$, so no triangle is formed.

19. Given: $A = 64°, a = 16, b = 17$. $h = b \sin A \approx 15.3$;
$h < a < b$, so there are two triangles.
$B_1 = \sin^{-1}\left(\dfrac{b \sin A}{a}\right) = \sin^{-1}(0.954\ldots) \approx 72.7°$
$C_1 = 180° - (A + B_1) \approx 43.3°$;
$c_1 = \dfrac{a \sin C_1}{\sin A} = \dfrac{16 \sin 43.3°}{\sin 64°} \approx 12.2$
Or (with B obtuse):
$B_2 = 180° - B_1 \approx 107.3°$;
$C_2 = 180° - (A + B_2) \approx 8.7°$;
$c_2 = \dfrac{a \sin C_2}{\sin A} \approx 2.7$

20. Given: $B = 38°, b = 21, c = 25$. $h = c \sin B \approx 15.4$;
$h < b < c$, so there are two triangles.
$C_1 = \sin^{-1}\left(\dfrac{c \sin B}{b}\right) = \sin^{-1}(0.732\ldots) \approx 47.1°$
$A_1 = 180° - (B + C_1) \approx 94.9°$;
$a_1 = \dfrac{b \sin A_1}{\sin B} = \dfrac{21 \sin 94.9°}{\sin 38°} \approx 34.0$
Or (with C obtuse):
$C_2 = 180° - C_1 \approx 132.9°$;
$A_2 = 180° - (B + C_2) \approx 9.1°$;
$a_2 = \dfrac{b \sin A_2}{\sin B} \approx 5.4$

21. Given: $C = 68°, a = 19, c = 18$. $h = a \sin C \approx 17.6$;
$h < c < a$, so there are two triangles.
$A_1 = \sin^{-1}\left(\dfrac{a \sin C}{c}\right) = \sin^{-1}(0.978\ldots) \approx 78.2°$
$B_1 = 180° - (A + C) \approx 33.8°$;
$b_1 = \dfrac{c \sin B_1}{\sin C} = \dfrac{18 \sin 33.8°}{\sin 68°} \approx 10.8$
Or (with A obtuse):
$A_2 = 180° - A_1 \approx 101.8°$;
$B_2 = 180° - (A_2 + C) \approx 10.2°$;
$b_2 = \dfrac{c \sin B_2}{\sin C} \approx 3.4$

22. Given: $B = 57°, a = 11, b = 10$. $h = a \sin B \approx 9.2$;
$h < b < a$, so there are two triangles.
$A_1 = \sin^{-1}\left(\dfrac{a \sin B}{b}\right) = \sin^{-1}(0.922\ldots) \approx 67.3°$
$C_1 = 180° - (A_1 + B) \approx 55.7°$;
$c_1 = \dfrac{b \sin C_1}{\sin B} = \dfrac{10 \sin 55.7°}{\sin 57°} \approx 9.9$
Or (with A obtuse):
$A_2 = 180° - A_1 \approx 112.7°$;
$C_2 = 180° - (A_2 + B) \approx 10.3°$;
$c_2 = \dfrac{b \sin C_2}{\sin B} \approx 2.1$

23. $h = 10 \sin 42° \approx 6.691$, so:
(a) $6.691 < b < 10$.
(b) $b \approx 6.691$ or $b \geq 10$.
(c) $b < 6.691$

24. $h = 12 \sin 53° \approx 9.584$, so:
(a) $9.584 < c < 12$.
(b) $c \approx 9.584$ or $c \geq 12$.
(c) $c < 9.584$

25. **(a)** No: this is an SAS case
(b) No: only two pieces of information given.

26. (a) Yes: this is an AAS case.
$B = 180° - (A + C) = 32°$;
$b = \dfrac{a \sin B}{\sin A} = \dfrac{81° \sin 32°}{\sin 29°} \approx 88.5$;
$c = \dfrac{a \sin C}{\sin A} = \dfrac{81 \sin 119°}{\sin 29°} \approx 146.1$

(b) No: this is an SAS case.

27. Given: $A = 61°$, $a = 8$, $b = 21$ — an SSA case.
$h = b \sin A = 18.4$; $a < h$, so no triangle is formed.

28. Given: $B = 47°$, $a = 8$, $b = 21$ — an SSA case.
$h = a \sin B \approx 5.9$; $h < a < b$, so there is one triangle.
$A = \sin^{-1}\left(\dfrac{a \sin B}{b}\right) = \sin^{-1}(0.278\ldots) \approx 16.2°$
$C = 180° - (A + B) = 116.8°$;
$c = \dfrac{b \sin C}{\sin B} = \dfrac{21 \sin 116.8°}{\sin 47°} \approx 25.6$

29. Given: $A = 136°$, $a = 15$, $b = 28$ — an SSA case.
$h = b \sin A \approx 19.5$; $a < h$, so no triangle is formed.

30. Given: $C = 115°$, $b = 12$, $c = 7$ — an SSA case.
$h = b \sin C \approx 10.9$; $c < h$, so no triangle is formed.

31. Given: $B = 42°$, $c = 18$, $C = 39°$ — an AAS case.
$A = 180° - (B + C) = 99°$;
$a = \dfrac{c \sin A}{\sin C} = \dfrac{18 \sin 99°}{\sin 39°} \approx 28.3$;
$b = \dfrac{c \sin B}{\sin C} = \dfrac{18 \sin 42°}{\sin 39°} \approx 19.1$

32. Given: $A = 19°$, $b = 22$, $B = 47°$ — an AAS case.
$C = 180° - (A + B) = 114°$;
$a = \dfrac{b \sin A}{\sin B} = \dfrac{22 \sin 19°}{\sin 47°} \approx 9.8$;
$c = \dfrac{b \sin C}{\sin B} = \dfrac{22 \sin 114°}{\sin 47°} \approx 27.5$

33. Given: $C = 75°$, $b = 49$, $c = 48$. — an SSA case.
$h = b \sin C \approx 47.3$; $h < c < b$, so there are two triangles.
$B_1 = \sin^{-1}\left(\dfrac{b \sin C}{c}\right) = \sin^{-1}(0.986\ldots) \approx 80.4°$
$A_1 = 180° - (B + C) \approx 24.6°$;
$a_1 = \dfrac{c \sin A_1}{\sin C} = \dfrac{48 \sin 24.6°}{\sin 75°} \approx 20.7$
Or (with B obtuse):
$B_2 = 180° - B_1 \approx 99.6°$;
$A_2 = 180° - (B_2 + C) \approx 5.4°$;
$a_2 = \dfrac{c \sin A_2}{\sin C} \approx 4.7$

34. Given: $A = 54°$, $a = 13$, $b = 15$. — an SSA case.
$h = b \sin A \approx 12.1$; $h < a < b$, so there are two triangles.
$B_1 = \sin^{-1}\left(\dfrac{b \sin A}{a}\right) = \sin^{-1}(0.933\ldots) \approx 69.0°$
$C_1 = 180° - (A + B_1) \approx 57.0°$;
$c_1 = \dfrac{a \sin C_1}{\sin A} = \dfrac{13 \sin 57.0°}{\sin 54°} \approx 13.5$
Or (with B obtuse):
$B_2 = 180° - B_1 \approx 111.0°$;
$C_2 = 180° - (A + B_2) \approx 15.0°$;
$c_2 = \dfrac{a \sin C_2}{\sin A} \approx 4.2$

35. Cannot be solved by law of sines (an SAS case).

36. Cannot be solved by law of sines (an SAS case).

37. Given: $c = AB = 56$, $A = 72°$, $B = 53°$ — an ASA case, so $C = 180° - (A + B) = 55°$

(a) $AC = b = \dfrac{c \sin B}{\sin C} = \dfrac{56 \sin 53°}{\text{in } 55°} \approx 54.6$ ft.

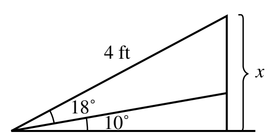
(b) $h = b \sin A (= a \sin B) \approx 51.9$ ft.

38. Given: $c = 25$, $A = 90° - 38° = 52°$,
$B = 90° - 53° = 37°$ — an ASA case, so
$C = 180° - (A + B) = 91°$ and
$a = \dfrac{c \sin A}{\sin C} = \dfrac{25 \sin 52°}{\sin 91°} \approx 19.7$ mi.
$b = \dfrac{c \sin B}{\sin C} = \dfrac{25 \sin 37°}{\sin 91°} \approx 15.0$ mi,
and finally $h = b \sin A = a \sin B \approx 11.9$ mi.

39. Given: $c = 16$, $C = 90° - 62 = 28°$,
$B = 90° + 15° = 105°$ — an AAS case.
$A = 180° - (B + C) = 47°$, so
$a = \dfrac{c \sin A}{\sin C} = \dfrac{16 \sin 47°}{\sin 28°} \approx 24.9$ ft.

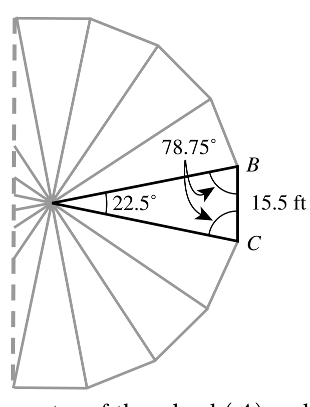
40. Given: $c = 2.32$, $A = 28°$, $B = 37°$ — an ASA case.
$C = 180° - (A + B) = 115°$;
$a = \dfrac{c \sin A}{\sin C} = \dfrac{2.32 \sin 28°}{\sin 115°} \approx 1.2$ mi.
$b = \dfrac{c \sin B}{\sin C} = \dfrac{2.32 \sin 37°}{\sin 115°} \approx 1.5$ mi.
Therefore, the altitude is $h = b \sin A \approx (1.5) \sin 28°$ ≈ 0.7 mi — or $a \sin B \approx (1.2) \sin 37°$ mi ≈ 0.7 mi.

41.



The length of the brace is the leg of the larger triangle.
$\sin 28° = \dfrac{x}{4}$, so $x = 1.9$ ft.

42.



The center of the wheel (A) and two adjacent chairs $(B$ and $C)$ form a triangle with $a = 15.5$, $A = \dfrac{360°}{16}$ $= 22.5°$, and $B = C = 78.75°$. This is an ASA case, so the radius is $b = c = \dfrac{a \sin B}{\sin A} = \dfrac{15.5 \sin 78.75°}{\sin 22.5°} \approx 39.7$ ft.
Alternatively, let D be the midpoint of $\overline{BC}$, and consider right $\triangle ABD$, with $m\angle BAD = 11.25°$ and $BD = 7.75$ ft; then r is the hypotenuse of this triangle, so
$r = \dfrac{7.75}{\sin 11.25°} \approx 39.7$ ft.

43. Consider the triangle with vertices at the top of the flagpole (A) and the two observers $(B$ and $C)$. Then $a = 600$, $B = 19°$, and $C = 21°$ (an ASA case), so
$A = 180° - (B + C) = 140°$;
$$b = \frac{a \sin B}{\sin A} = \frac{600 \sin 19°}{\sin 140°} \approx 303.9;$$
$$c = \frac{a \sin C}{\sin A} = \frac{600 \sin 21°}{\sin 140°} \approx 334.5$$
and finally $h = b \sin C = c \sin B \approx 108.9$ ft.

44. Consider the triangle with vertices at the top of the tree (A) and the two observers $(B$ and $C)$. Then $a = 400$, $B = 15°$, and $C = 20°$ (an ASA case), so
$A = 180° - (B + C) = 145°$;
$$b = \frac{a \sin B}{\sin A} = \frac{400 \sin 15°}{\sin 145°} \approx 180.5;$$
$$c = \frac{a \sin C}{\sin A} = \frac{400 \sin 20°}{\sin 145°} \approx 238.5;$$
and finally $h = b \sin C = c \sin B \approx 61.7$ ft.

45. Given: $c = 20$, $B = 52°$, $C = 33°$ — an AAS case.
$A = 180° - (B + C) = 95°$, so
$$a = \frac{c \sin A}{\sin C} = \frac{20 \sin 95°}{\sin 33°} \approx 36.6 \text{ mi, and}$$
$$b = \frac{c \sin B}{\sin C} = \frac{20 \sin 52°}{\sin 33°} \approx 28.9 \text{ mi.}$$

46. We use the mean (average) measurements for *A, B,* and *AB,* which are 79.7°, 83.9°, and 25.9 feet, respectively. This gives 16.4° for angle *C.* By the Law of Sines,
$$AC = \frac{25.9 \sin 83.9°}{\sin 16.4°} \approx 91.2 \text{ feet.}$$

47. True. By the law of sines, $\frac{\sin A}{a} = \frac{\sin B}{b}$,
which is equivalent to $\frac{\sin A}{\sin B} = \frac{a}{b}$ (since $\sin A, \sin B \neq 0$).

48. False. By the law of sines, the third side of the triangle measures $\frac{10 \sin 100°}{\sin 40°}$, which is about 15.32 inches. That makes the perimeter about $10 + 10 + 15.32 = 35.32$, which is less than 36 inches.

49. The third angle is 32°. By the Law of Sines,
$\frac{\sin 32°}{12.0} = \frac{\sin 53°}{x}$, which can be solved for *x*.
The answer is C.

50. With SSA, the known side opposite the known angle sometimes has two different possible positions. The answer is D.

51. The longest side is opposite the largest angle, while the shortest side is opposite the smallest angle. By the Law of Sines, $\frac{\sin 50°}{9.0} = \frac{\sin 70°}{x}$, which can be solved for *x*.
The answer is A.

52. Because $BC > AB$, only one triangle is possible. The answer is B.

53. (a) Given any triangle with side lengths a, b, and c, the law of sines says that $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.
But we can also find another triangle(using ASA) with two angles the same as the first (in which case the third angle is also the same) and a different side length — say, a'. Suppose that $a' = ka$ for some constant k. Then for this new triangle, we have
$\frac{\sin A}{a'} = \frac{\sin B}{b'} = \frac{\sin C}{c'}$. Since $\frac{\sin A}{a'} = \frac{\sin A}{ka} = \frac{1}{k} \cdot \frac{\sin A}{a}$, we can see that $\frac{\sin B}{b'} = \frac{1}{k} \cdot \frac{\sin B}{b}$,
so that $b' = kb$ and similarly, $c' = kc$. So for any choice of a positive constant k, we can create a triangle with angles A, B, and C.

(b) Possible answers: $a = 1, b = \sqrt{3}, c = 2$ (or any set of three numbers proportional to these).

(c) Any set of three identical numbers.

54. In each proof, assume that sides a, b, and c are opposite angles A, B, and C, and that c is the hypotenuse.

(a) $\frac{\sin A}{a} = \frac{\sin 90°}{c}$
$\frac{\sin A}{a} = \frac{1}{c}$
$\sin A = \frac{a}{c} = \frac{opp}{hyp}$

(b) $\frac{\sin B}{b} = \frac{\sin 90°}{c}$
$\frac{\cos(\pi/2 - B)}{b} = \frac{1}{c}$
$\cos A = \frac{b}{c} = \frac{adj}{hyp}$

(c) $\frac{\sin A}{a} = \frac{\sin B}{b}$
$\frac{\sin A}{\sin B} = \frac{a}{b}$
$\frac{\sin A}{\cos A} = \frac{a}{b}$
$\tan A = \frac{a}{b} = \frac{opp}{adj}$

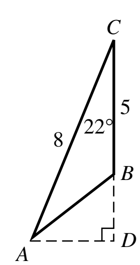
55. (a) $h = AB \sin A$

(b) $BC < AB \sin A$

(c) $BC \geq AB$ or $BC = AB \sin A$

(d) $AB \sin A < BC < AB$

56. Drawing the line suggested in the hint, and extending $\overline{BC}$ to meet that line at, say, D, gives right $\triangle ADC$ and right $\triangle ADB$.



Then $AD = 8 \sin 22° \approx 3.0$ and $DC = 8 \cos 22° \approx 7.4$, so $DB = DC - 5$ and
$c = AB = \sqrt{AD^2 + DB^2} \approx 3.9$. Finally,
$A = (90° - 22°) - \sin^{-1}\left(\frac{DB}{AB}\right) \approx 29.1°$ and
$B = 180° - A - C \approx 128.9°$.

57. Given: $c = 4.1, B = 25°, C = 36.5° - 25° = 11.5°$. An AAS case: $A = 180° - (B + C) = 143.5°$, so
$$AC = b = \frac{c \sin B}{\sin C} = \frac{4.1 \sin 25°}{\sin 11.5°} \approx 8.7 \text{ mi, and}$$
$$BC = a = \frac{c \sin A}{\sin C} = \frac{4.1 \sin 143.5°}{\sin 11.5°} \approx 12.2 \text{ mi.}$$
The height is $h = a \sin 25° = b \sin 36.5° \approx 5.2$ mi.

■ Section 5.6 The Law of Cosines

Exploration 1

1. The semiperimeters are 154 and 150.
$$A = \sqrt{154(154-115)(154-81)(154-112)} + \sqrt{150(150-112)(150-102)(150-86)} = 8475.742818 \text{ paces}^2$$
2. 41,022.59524 square feet
3. 0.0014714831 square miles
4. 0.94175 acres
5. The estimate of "a little over an acre" seems questionable, but the roughness of their measurement system does not provide firm evidence that it is incorrect. If Jim and Barbara wish to make an issue of it with the owner, they would be well-advised to get some more reliable data.
6. Yes. In fact, any polygonal region can be subdivided into triangles.

Quick Review 5.6

1. $A = \cos^{-1}\left(\frac{3}{5}\right) \approx 53.130°$
2. $C = \cos^{-1}(-0.23) \approx 103.297°$
3. $A = \cos^{-1}(-0.68) \approx 132.844°$
4. $C = \cos^{-1}\left(\frac{1.92}{3}\right) \approx 50.208°$
5. **(a)** $\cos A = \frac{81 - x^2 - y^2}{-2xy} = \frac{x^2 + y^2 - 81}{2xy}$.
   (b) $A = \cos^{-1}\left(\frac{x^2 + y^2 - 81}{2xy}\right)$
6. **(a)** $\cos A = \frac{y^2 - x^2 - 25}{-10} = \frac{x^2 - y^2 + 25}{10}$
   (b) $A = \cos^{-1}\left(\frac{x^2 - y^2 + 25}{10}\right)$
7. One answer: $(x - 1)(x - 2) = x^2 - 3x + 2$. Generally: $(x - a)(x - b) = x^2 - (a + b)x + ab$ for any two positive numbers a and b.
8. One answer: $(x - 1)(x + 1) = x^2 - 1$. Generally, $(x - a)(x + b) = x^2 - (a - b)x - ab$ for any two positive numbers a and b.
9. One answer: $(x - i)(x + i) = x^2 + 1$
10. One answer: $(x - 1)^2 = x^2 - 2x + 1$. Generally: $(x - a)^2 = x^2 - 2ax + a^2$ for any positive number a.

Section 5.6 Exercises

1. Given: $B = 131°, c = 8, a = 13$ — an SAS case.
$$b = \sqrt{a^2 + c^2 - 2ac \cos B} \approx \sqrt{369.460} \approx 19.2;$$
$$C = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right) \approx \cos^{-1}(0.949) \approx 18.3°;$$
$$A = 180° - (B + C) \approx 30.7°.$$
2. Given: $C = 42°, b = 12, a = 14$ — an SAS case.
$$c = \sqrt{a^2 + b^2 - 2ab \cos C} \approx \sqrt{90.303} \approx 9.5;$$
$$A = \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2bc}\right) \approx \cos^{-1}(0.167) \approx 80.3°;$$
$$B = 180° - (A + C) \approx 57.7°.$$
3. Given: $a = 27, b = 19, c = 24$ — an SSS case.
$$A = \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2bc}\right) \approx \cos^{-1}(0.228) \approx 76.8°;$$
$$B = \cos^{-1}\left(\frac{a^2 + c^2 - b^2}{2ac}\right) \approx \cos^{-1}(0.728) \approx 43.2°;$$
$$C = 180° - (A + B) \approx 60°.$$
4. Given: $a = 28, b = 35, c = 17$ — an SSS case.
$$A = \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2bc}\right) \approx \cos^{-1}(0.613) \approx 52.2°;$$
$$B = \cos^{-1}\left(\frac{a^2 + c^2 - b^2}{2ac}\right) \approx \cos^{-1}(-0.159) \approx 99.2°;$$
$$C = 180° - (A + B) \approx 28.6°.$$
5. Given: $A = 55°, b = 12, c = 7$ — an SAS case.
$$a = \sqrt{b^2 + c^2 - 2bc \cos A} \approx \sqrt{96.639} \approx 9.8;$$
$$B = \cos^{-1}\left(\frac{a^2 + c^2 - b^2}{2ac}\right) \approx \cos^{-1}(0.011) \approx 89.3°;$$
$$C = 180° - (A + B) \approx 35.7°.$$
6. Given: $B = 35°, a = 43, c = 19$ — an SAS case.
$$b = \sqrt{a^2 + c^2 - 2ac \cos B} \approx \sqrt{871.505} \approx 29.5;$$
$$C = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right) \approx \cos^{-1}(0.929) \approx 21.7°;$$
$$A = 180° - (B + C) \approx 123.3°.$$
7. Given: $a = 12, b = 21, C = 95°$ — an SAS case.
$$c = \sqrt{a^2 + b^2 - 2ab \cos C} \approx \sqrt{628.926} \approx 25.1;$$
$$A = \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2bc}\right) \approx \cos^{-1}(0.879) \approx 28.5°;$$
$$B = 180° - (A + C) \approx 56.5°.$$
8. Given: $b = 22, c = 31, A = 82°$ — an SAS case.
$$a = \sqrt{b^2 + c^2 - 2bc \cos A} \approx \sqrt{1255.167} \approx 35.4;$$
$$B = \cos^{-1}\left(\frac{a^2 + c^2 - b^2}{2ac}\right) \approx \cos^{-1}(0.788) \approx 37.9°;$$
$$C = 180° - (A + B) \approx 60.1°.$$
9. No triangles possible $(a + c = b)$
10. No triangles possible $(a + b < c)$
11. Given: $a = 3.2, b = 7.6, c = 6.4$ — an SSS case.
$$A = \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2bc}\right) \approx \cos^{-1}(0.909) \approx 24.6°;$$
$$B = \cos^{-1}\left(\frac{a^2 + c^2 - b^2}{2ac}\right) \approx \cos^{-1}(^-0.160) \approx 99.2°;$$
$$C = 180° - (A + B) \approx 56.2°.$$

12. No triangles possible $(a + b < c)$

Exercises 13–16 are SSA cases, and can be solved with either the Law of Sines or the Law of Cosines. The law of cosines solution is shown.

13. Given: $A = 42°$, $a = 7$, $b = 10$ — an SSA case. Solve the quadratic equation $7^2 = 10^2 + c^2 - 2(10)c \cos 42°$, or $c^2 - (14.862\ldots)c + 51 = 0$; there are two positive solutions: ≈ 9.487 *or* 5.376. Since $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$:
$c_1 \approx 9.487$, $B_1 \approx \cos^{-1}(0.294) \approx 72.9°$, and
$C_1 = 180° - (A + B_1) \approx 65.1°$,
or
$c_2 \approx 5.376$, $B_2 \approx \cos^{-1}(-0.294) \approx 107.1°$, and
$C_2 = 180° - (A + B_2) \approx 30.9°$.

14. Given: $A = 57°$, $a = 11$, $b = 10$ — an SSA case. Solve the quadratic equation $11^2 = 10^2 + c^2 - 2(10)C \cos 57°$, or $c^2 - (10.893)c - 21 = 0$; there is one positive solution $c = 12.564$. Since $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$,
$B \approx \cos^{-1}(0.647) \approx 49.7°$ and $C = 180° - (A + B) \approx 73.3°$.

15. Given: $A = 63°$, $a = 8.6$, $b = 11.1$ — an SSA case. Solve the quadratic equation
$8.6^2 = 11.1^2 + c^2 - 2(11.1)c \cos 63°$, or
$c^2 - (10.079)c + 49.25 = 0$; there are no real solutions, so there is no triangle.

16. Given: $A = 71°$, $a = 9.3$, $b = 8.5$ — an SSA case. Solve the quadratic equation
$9.3^2 = 8.5^2 + c^2 - 2(8.5)c \cos 71°$, or
$c^2 - (5.535)c - 14.24 = 0$; there is one positive solution: $c \approx 7.447$. Since $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$,
$B \approx \cos^{-1}(0.503) \approx 59.8°$ and $C = 180° - (A + B) \approx 49.2°$.

17. Given: $A = 47°$, $b = 32$, $c = 19$ — an SAS case.
$a = \sqrt{b^2 + c^2 - 2bc \cos A} \approx \sqrt{555.689} \approx 23.573$,
so Area $\approx \sqrt{49431.307} \approx 222.33$ ft^2 (using Heron's formula). Or, use $A = \dfrac{1}{2}bc \sin A$.

18. Given: $A = 52°$, $b = 14$, $c = 21$ — an SAS case.
$a = \sqrt{b^2 + c^2 - 2bc \cos A} \approx \sqrt{274.991} \approx 16.583$,
so Area $\approx \sqrt{13418.345} \approx 115.84$ m^2 (using Heron's formula). Or, use $A = \dfrac{1}{2}bc \sin A$.

19. Given: $B = 101°$, $a = 10$, $c = 22$ — an SAS case.
$b = \sqrt{a^2 + c^2 - 2ac \cos B} \approx \sqrt{667.955} \approx 25.845$,
so Area $\approx \sqrt{11659.462} \approx 107.98$ cm^2 (using Heron's formula). Or, use $A = \dfrac{1}{2}ac \sin B$.

20. Given: $C = 112°$, $a = 1.8$, $b = 5.1$ — an SAS case.
$c = \sqrt{a^2 + b^2 - 2ab \cos C} \approx \sqrt{36.128} \approx 6.011$,
so Area $\approx \sqrt{18.111} \approx 4.26$ in.^{2} (using Heron's formula). Or, use $A = \dfrac{1}{2}ab \sin C$.

For #21–28, a triangle can be formed if $a + b < c$, $a + c < b$, and $b + c < a$.

21. $s = \dfrac{17}{2}$; Area $= \sqrt{66.9375} \approx 8.18$

22. $s = \dfrac{21}{2}$; Area $= \sqrt{303.1875} \approx 17.41$

23. No triangle is formed $(a + b = c)$.

24. $s = 27$; Area $= \sqrt{12{,}960} = 36\sqrt{10} \approx 113.84$

25. $a = 36.4$; Area $= \sqrt{46{,}720.3464} \approx 216.15$

26. No triangle is formed $(a + b < c)$

27. $s = 42.1$; Area $= \sqrt{98{,}629.1856} \approx 314.05$

28. $s = 23.8$; Area $\sqrt{10{,}269.224} \approx 101.34$

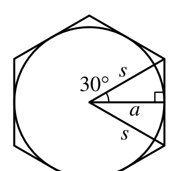
29. Let $a = 4$, $b = 5$, and $c = 6$. The largest angle is opposite the largest side, so we call it C. Since $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$, $C = \cos^{-1}\left(\dfrac{1}{8}\right) \approx 82.819°$ ≈ 1.445 radians.

30. The shorter diagonal splits the parallelogram into two (congruent) triangles with $a = 26$, $B = 39°$, and $c = 18$.
The diagonal has length $b = \sqrt{a^2 + c^2 - 2ac \cos B}$ $\approx \sqrt{272.591} \approx 16.5$ ft.

31. Following the method of Example 3, divide the hexagon into 6 triangles. Each has two 12-inch sides that form a 60° angle.
$6 \times \dfrac{1}{2}(12)(12)\sin 60° = 216\sqrt{3} \approx 374.1$ square inches

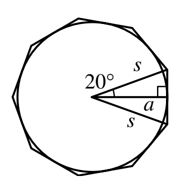
32. Following the method of Example 3, divide the nonagon into 9 triangles. Each has two 10-inch sides that form a 40° angle.
$9 \times \dfrac{1}{2}(10)(10)\sin 40° \approx 289.3$ square inches

33.



In the figure, $a = 12$ and so $s = 12 \sec 30° = 8\sqrt{3}$. The area of the hexagon is
$6 \times \dfrac{1}{2}(8\sqrt{3})(8\sqrt{3})\sin 60° = 288\sqrt{3}$
≈ 498.8 square inches.

34.



In the figure, $a = 10$ and so $s = 10 \sec 20°$. The area of the nonagon is
$9 \times \dfrac{1}{2}(10 \sec 20°)(10 \sec 20°)\sin 40°$
≈ 327.6 square inches.

35. Given: $C = 54°$, $BC = a = 160$, $AC = b = 110$ — an SAS case. $AB = c = \sqrt{a^2 + b^2 - 2ab\cos C}$ $\approx \sqrt{17{,}009.959} \approx 130.42$ ft.

36. **(a)** The home-to-second segment is the hypotenuse of a right triangle, so the distance from the pitcher's rubber to second base is $90\sqrt{2} - 60.5 \approx 66.8$ ft. This is a bit more than $c = \sqrt{60.5^2 + 90^2 - 2(60.5)(90)\cos 45°}$ $\approx \sqrt{4059.857} \approx 63.7$ ft.

(b) $B = \cos^{-1}\left(\frac{a^2 + c^2 - b^2}{2ac}\right) \approx \cos^{-1}(-0.049)$ $\approx 92.8°$.

37. **(a)** $c = \sqrt{40^2 + 60^2 - 2(40)(60)\cos 45°}$ $\approx \sqrt{1805.887} \approx 42.5$ ft.

(b) The home-to-second segment is the hypotenuse of a right triangle, so the distance from the pitcher's rubber to second base is $60\sqrt{2} - 40 \approx 44.9$ ft.

(c) $B = \cos^{-1}\left(\frac{a^2 + c^2 - b^2}{2ac}\right) \approx \cos^{-1}(-0.057)$ $\approx 93.3°$.

38. Given: $a = 175$, $b = 860$, and $C = 78°$. An SAS case, so $AB = c = \sqrt{a^2 + b^2 - 2ab\cos C} \approx \sqrt{707{,}643.581}$ ≈ 841.2 ft.

39. **(a)** Using right ΔACE, $m\angle CAE = \tan^{-1}\left(\frac{6}{18}\right)$ $= \tan^{-1}\left(\frac{1}{3}\right) \approx 18.435°$.

(b) Using $A \approx 18.435°$, we have an SAS case, so $DF = \sqrt{9^2 + 12^2 - 2(9)(12)\cos A} \approx \sqrt{20.084}$ ≈ 4.5 ft.

(c) $EF = \sqrt{18^2 + 12^2 - 2(18)(12)\cos A} \approx \sqrt{58.168}$ ≈ 7.6 ft.

40. After two hours, the planes have traveled 700 and 760 miles, and the angle between them is 22.5°, so the distance is $\sqrt{700^2 + 760^2 - 2(700)(760)\cos 22.5°}$ $\approx \sqrt{84{,}592.177} \approx 290.8$ mi.

41. $AB = \sqrt{73^2 + 65^2 - 2(73)(65)\cos 8°}$ $\approx \sqrt{156.356} \approx 12.5$ yd.

42. $m\angle HAB = 135°$, so

$HB = \sqrt{20^2 + 20^2 - 2(20)(20)\cos 135°}$ $\approx \sqrt{1365.685} \approx 37.0$ ft.

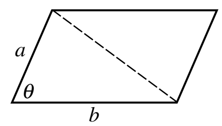
Note that $\overline{AB}$ is the hypotenuse of an equilateral right triangle with leg length $\frac{20}{\sqrt{2}} = 10\sqrt{2}$, and $\overline{HC}$ is the hypotenuse of an equilateral right triangle with leg length $20 + 10\sqrt{2}$, so $HC = \sqrt{2(20 + 10\sqrt{2})^2} \approx 48.3$ ft. Finally, using right ΔHAD with leg lengths $HA = 20$ ft and $AD = HC \approx 48.3$ ft, we have $HD = \sqrt{HA^2 + AD^2} \approx 52.3$ ft.

43. $AB = c = \sqrt{2^2 + 3^2} = \sqrt{13}$, $AC = b = \sqrt{1^2 + 3^2}$ $= \sqrt{10}$, and $BC = a = \sqrt{1^2 + 2^2} = \sqrt{5}$, so $m\angle CAB = A = \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2bc}\right)$ $= \cos^{-1}\left(\frac{9}{\sqrt{130}}\right) \approx 37.9°$.

44. ΔABC is a right triangle ($C = 90°$), with $BC = a$ $= \sqrt{2^2 + 2^2} = 2\sqrt{2}$ and $AC = b = 1$, so $AB = c$ $= \sqrt{a^2 + b^2} = 3$ and $B = m\angle ABC = \sin^{-1}\left(\frac{1}{3}\right)$ $\approx 19.5°$.

45. True. By the Law of Cosines, $b^2 + c^2 - 2bc\cos A = a^2$, which is a positive number. Since $b^2 + c^2 - 2bc\cos A > 0$, it follows that $b^2 + c^2 > 2bc\cos A$.

46. True. The diagonal opposite angle θ splits the parallelogram into two congruent triangles, each with area $\frac{1}{2}ab\sin\theta$.



47. Following the method of Example 3, divide the dodecagon into 12 triangles. Each has two 12-inch sides that form a 30° angle.

$12 \times \frac{1}{2}(12)(12)\sin 30° = 432$

The answer is B.

48. The semiperimeter is $s = (7 + 8 + 9)/2 = 12$. Then by Heron's Formula, $A = \sqrt{12(12 - 7)(12 - 8)(12 - 9)}$ $= 12\sqrt{5}$. The answer is B.

49. After 30 minutes, the first boat has traveled 12 miles and the second has traveled 16 miles. By the Law of Cosines, the two boats are $\sqrt{12^2 + 16^2 - 2(12)(16)\cos 110°} \approx 23.05$ miles apart. The answer is C.

50. By the Law of Cosines, $12^2 = 17^2 + 25^2 - 2(17)(25)$ $\cos\theta$, so $\theta = \cos^{-1}\left(\frac{17^2 + 25^2 - 12^2}{2(17)(25)}\right) \approx 25.06°$.

The answer is E.

51. Consider that a n-sided regular polygon inscribed within a circle can divide into n equilateral triangles, each with equal area of $\frac{r^2}{2}\sin\frac{360°}{n}$. (The two equal sides of the equilateral triangle are of length r, the radius of the circle.) Then, the area of the polygon is exactly $\frac{nr^2}{2}\sin\frac{360°}{n}$.

52. **(a)**

$$\frac{b^2 + c^2 - a^2}{2abc} = \frac{b^2 + c^2 - (b^2 + c^2 - 2bc\cos A)}{2abc} \quad \text{Law of Cosines}$$

$$= \frac{2bc\cos A}{2abc}$$

$$= \frac{\cos A}{a}$$

(b) The identity in (a) has two other equivalent forms:

$$\frac{\cos B}{b} = \frac{a^2 + c^2 - b^2}{2abc}$$

$$\frac{\cos C}{c} = \frac{a^2 + b^2 - c^2}{2abc}$$

We use them all in the proof:

$$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$$
$$= \frac{b^2 + c^2 - a^2}{2abc} + \frac{a^2 + c^2 - b^2}{2abc} + \frac{a^2 + b^2 - c^2}{2abc}$$
$$= \frac{b^2 + c^2 - a^2 + a^2 + c^2 - b^2 + a^2 + b^2 - c^2}{2abc}$$
$$= \frac{a^2 + b^2 + c^2}{2abc}$$

53. **(a)** Ship A: $\frac{30.2 - 15.1}{1 \text{ hr}} = 15.1$ knots;

Ship B: $\frac{37.2 - 12.4}{2 \text{ hrs}} = 12.4$ knots

(b) $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$
$= \frac{(15.1)^2 + (12.4)^2 - (8.7)^2}{2(15.1)(12.4)}$
$A = 35.18°$

(c) $c^2 = a^2 + b^2 - 2ab\cos C$
$= (49.6)^2 + (60.4)^2 - 2(49.6)(60.4)\cos(35.18°)$
≈ 1211.04, so the boats are 34.8 nautical miles apart at noon.

54. Use the area formula and the Law of Sines:

$$A_\Delta = \frac{1}{2}ab\sin C$$
$$= \frac{1}{2}a\left(\frac{a\sin B}{\sin A}\right)\sin C \quad \left(\text{Law of Sines} \Rightarrow b = \frac{a\sin B}{\sin A}\right)$$
$$= \frac{a^2\sin B\sin C}{2\sin a}$$

55. Let P be the center of the circle. Then, $\cos P = \frac{5^2 + 5^2 - 7^2}{2(5)(5)} = 0.02$, so $P \approx 88.9°$. The area of the segment is $\pi r^2 \cdot \frac{88.9°}{360°} \approx 25\pi \cdot (0.247) \approx 19.39 \text{ in}^2$.

The area of the triangle, however, is $\frac{1}{2}(5)(5)\sin(88.9)$ $\approx 12.50 \text{ in}^2$, so the area of the shaded region is approx. 6.9 in^2.

■ Chapter 5 Review

1. $2\sin 100°\cos 100° = \sin 200°$

2. $\frac{2\tan 40°}{1 - \tan^2 40°} = \tan 80°$

3. 1; the expression simplifies to $(\cos 2\theta)^2 + (2\sin\theta\cos\theta)^2 = (\cos 2\theta)^2 + (\sin 2\theta)^2 = 1$.

4. $\cos^2 2x$; the expression can be rewritten $1 - (2\sin x\cos x)^2 = 1 - (\sin 2x)^2 = \cos^2 2x$.

5. $\cos 3x = \cos(2x + x) = \cos 2x\cos x - \sin 2x\sin x$
$= (\cos^2 x - \sin^2 x)\cos x - (2\sin x\cos x)\sin x$
$= \cos^3 x - 3\sin^2 x\cos x$
$= \cos^3 x - 3(1 - \cos^2 x)\cos x$
$= \cos^3 x - 3\cos x + 3\cos^3 x$
$= 4\cos^3 x - 3\cos x$

6. $\cos^2 2x - \cos^2 x = (1 - \sin^2 2x) - (1 - \sin^2 x)$
$= \sin^2 x - \sin^2 2x$

7. $\tan^2 x - \sin^2 x = \sin^2 x\left(\frac{1 - \cos^2 x}{\cos^2 x}\right)$
$= \sin^2 x \cdot \frac{\sin^2 x}{\cos^2 x} = \sin^2 x\tan^2 x$

8. $2\sin\theta\cos^3\theta + 2\sin^3\theta\cos\theta$
$= (2\sin\theta\cos\theta)(\cos^2\theta + \sin^2\theta)$
$= (2\sin\theta\cos\theta)(1) = \sin 2\theta$.

9. $\csc x - \cos x\cot x = \frac{1}{\sin x} - \cos x \cdot \frac{\cos x}{\sin x}$
$= \frac{1 - \cos^2 x}{\sin x} = \frac{\sin^2 x}{\sin x} = \sin x$

10. $\frac{\tan\theta + \sin\theta}{2\tan\theta} = \frac{1 + \cos\theta}{2} = \left(\pm\sqrt{\frac{1 + \cos\theta}{2}}\right)^2$
$= \left(\cos\frac{\theta}{2}\right)^2$

11. Recall that $\tan\theta\cot\theta = 1$. $\frac{1 + \tan\theta}{1 - \tan\theta} + \frac{1 + \cot\theta}{1 - \cot\theta}$
$$= \frac{(1 + \tan\theta)(1 - \cot\theta) + (1 + \cot\theta)(1 - \tan\theta)}{(1 - \tan\theta)(1 - \cot\theta)}$$
$$= \frac{(1 + \tan\theta - \cot\theta - 1) + (1 + \cot\theta - \tan\theta - 1)}{(1 - \tan\theta)(1 - \cot\theta)}$$
$$= \frac{0}{(1 - \tan\theta)(1 - \cot\theta)} = 0$$

12. $\sin 3\theta = \sin(2\theta + \theta) = \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$
$= 2\sin\theta\cos^2\theta + (\cos^2\theta - \sin^2\theta)\sin\theta$
$= 3\sin\theta\cos^2\theta - \sin^3\theta$

13. $\cos^2\frac{t}{2} = \left[\pm\sqrt{\frac{1}{2}(1 + \cos t)}\right]^2 = \frac{1}{2}(1 + \cos t)$
$= \left(\frac{1 + \cos t}{2}\right)\left(\frac{\sec t}{\sec t}\right) = \frac{1 + \sec t}{2\sec t}$

14. $\frac{\tan^3\gamma - \cot^3\gamma}{\tan^2\gamma + \csc^2\gamma}$
$$= \frac{(\tan\gamma - \cot\gamma)(\tan^2\gamma + \tan\gamma\cot\gamma + \cot^2\gamma)}{\tan^2\gamma + \csc^2\gamma}$$
$$= \frac{(\tan\gamma - \cot\gamma)(\tan^2\gamma + 1 + \cot^2\gamma)}{\tan^2\gamma + \csc^2\gamma}$$
$$= \frac{(\tan\gamma - \cot\gamma)(\tan^2\gamma + \csc^2\gamma)}{\tan^2\gamma + \csc^2\gamma} = \tan\gamma - \cot\gamma$$

15. $\frac{\cos\phi}{1 - \tan\phi} + \frac{\sin\phi}{1 - \cot\phi}$
$$= \left(\frac{\cos\phi}{1 - \tan\phi}\right)\left(\frac{\cos\phi}{\cos\phi}\right) + \left(\frac{\sin\phi}{1 - \cot\phi}\right)\left(\frac{\sin\phi}{\sin\phi}\right)$$
$$= \frac{\cos^2\phi}{\cos\phi - \sin\phi} + \frac{\sin^2\phi}{\sin\phi - \cos\phi} = \frac{\cos^2\phi - \sin^2\phi}{\cos\phi - \sin\phi}$$
$= \cos\phi + \sin\phi$

16. $\frac{\cos(-z)}{\sec(-z) + \tan(-z)} = \frac{\cos(-z)}{[1 + \sin(-z)]/\cos(-z)}$
$$= \frac{\cos^2(-z)}{1 + \sin(-z)} = \frac{\cos^2 z}{1 - \sin z} = \frac{1 - \sin^2 z}{1 - \sin z} = 1 + \sin z$$

17. $\sqrt{\dfrac{1-\cos y}{1+\cos y}} = \sqrt{\dfrac{(1-\cos y)^2}{(1+\cos y)(1-\cos y)}}$
$= \sqrt{\dfrac{(1-\cos y)^2}{1-\cos^2 y}} = \sqrt{\dfrac{(1-\cos y)^2}{\sin^2 y}}$
$= \dfrac{|1-\cos y|}{|\sin y|} = \dfrac{1-\cos y}{|\sin y|}$ — since $1-\cos y \ge 0$, we can drop that absolute value.

18. $\sqrt{\dfrac{1-\sin\gamma}{1+\sin\gamma}} = \sqrt{\dfrac{(1-\sin\gamma)(1+\sin\gamma)}{(1+\sin\gamma)^2}}$
$= \sqrt{\dfrac{1-\sin^2\gamma}{(1+\sin\gamma)^2}} = \sqrt{\dfrac{\cos^2\gamma}{(1+\sin\gamma)^2}}$
$= \dfrac{|\cos\gamma|}{|1+\sin\gamma|} = \dfrac{|\cos\gamma|}{1+\sin\gamma}$ — since $1+\sin\gamma \ge 0$, we can drop that absolute value.

19. $\tan\left(u+\dfrac{3\pi}{4}\right) = \dfrac{\tan u + \tan(3\pi/4)}{1-\tan u\tan(3\pi/4)}$
$= \dfrac{\tan u + (-1)}{1-\tan u(-1)} = \dfrac{\tan u - 1}{1+\tan u}$

20. $\dfrac{1}{4}\sin 4\gamma = \dfrac{1}{4}\sin 2(2\gamma) = \dfrac{1}{4}(2\sin 2\gamma\cos 2\gamma)$
$= \dfrac{1}{2}(2\sin\gamma\cos\gamma)(\cos^2\gamma - \sin^2\gamma)$
$= \sin\gamma\cos^3\gamma - \cos\gamma\sin^3\gamma$

21. $\tan\dfrac{1}{2}\beta = \dfrac{1-\cos\beta}{\sin\beta} = \dfrac{1}{\sin\beta} - \dfrac{\cos\beta}{\sin\beta} = \csc\beta - \cot\beta$

22. Let $\theta = \arctan t$, so that $\tan\theta = t$. Then $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta} = \dfrac{2t}{1-t^2}$. Note also that since $-1 < t < 1$, $-\dfrac{\pi}{4} < \theta < \dfrac{\pi}{4}$, and therefore $-\dfrac{\pi}{2} < 2\theta < \dfrac{\pi}{2}$. That means that 2θ is in the range of the arctan function, and so $2\theta = \arctan\dfrac{2t}{1-t^2}$, or equivalently
$\theta = \dfrac{1}{2}\arctan\dfrac{2t}{1-t^2}$ — and of course, $\theta = \arctan t$.

23. Yes: $\sec x - \sin x\tan x = \dfrac{1}{\cos x} - \dfrac{\sin^2 x}{\cos x}$
$= \dfrac{1-\sin^2 x}{\cos x} = \dfrac{\cos^2 x}{\cos x} = \cos x$.

24. Yes: $(\sin^2\alpha - \cos^2 a)(\tan^2\alpha + 1)$
$= (\sin^2\alpha - \cos^2\alpha)(\sec^2\alpha)$
$= \dfrac{\sin^2\alpha - \cos^2\alpha}{\cos^2\alpha} = \dfrac{\sin^2\alpha}{\cos^2\alpha} - 1 = \tan^2\alpha - 1$.

25. Many answers are possible, for example,
$\sin 3x + \cos 3x$
$= (3\sin x - 4\sin^3 x) + (4\cos^3 x - 3\cos x)$
$= 3(\sin x - \cos x) - 4(\sin^3 x - \cos^3 x)$
$= (\sin x - \cos x)[3 - 4(\sin^2 x + \sin x\cos x + \cos^2 x)]$
$= (\sin x - \cos x)\ (3 - 4 - 4\sin x\cos x)$
$= (\cos x - \sin x)(1 + 4\sin x\cos x)$. Check other answers with a grapher.

26. Many answers are possible, for example,
$\sin 2x + \cos 3x = 2\sin x\cos x + 4\cos^3 x - 3\cos x$
$= \cos x(2\sin x + 4\cos^2 x - 3)$
$= \cos x(2\sin x + 1 - 4\sin^2 x)$. Check other answers with a grapher.

27. Many answers are possible, for example,
$\cos^2 2x - \sin 2x = 1 - \sin^2 2x - \sin 2x$
$= 1 - 4\sin^2 x\cos^2 x - 2\sin x\cos x$. Check other answers with a grapher.

28. Many answers are possible, for example (using Review Exercise #12), $\sin 3x - 3\sin 2x$
$= 3\cos^2 x\sin - \sin^3 x - 6\sin x\cos x$
$= \sin x(3\cos^2 x - \sin^2 x - 6\cos x)$
$= \sin x(4\cos^2 x - 1 - 6\cos x)$. Check other answers with a grapher.

In #29–33, n represents any integer.

29. $\sin 2x = 0.5$ when $2x = \dfrac{\pi}{6} + 2n\pi$ or $2x = \dfrac{5\pi}{6} + 2n\pi$,
so $x = \dfrac{\pi}{12} + n\pi$ or $x = \dfrac{5\pi}{12} + n\pi$.

30. $\cos x = \dfrac{\sqrt{3}}{2}$ when $x = \pm\dfrac{\pi}{6} + 2n\pi$

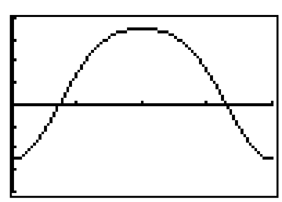
31. $\tan x = -1$ when $x = -\dfrac{\pi}{4} + n\pi$

32. If $\sin^{-1} x = \dfrac{\sqrt{2}}{2}$, then $x = \sin\dfrac{\sqrt{2}}{2}$.

33. If $\tan^{-1} x = 1$, then $x = \tan 1$.

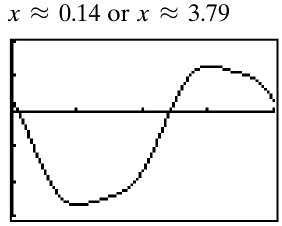
34.
$$2\cos 2x = 1$$
$$\cos 2x = \frac{1}{2}$$
$$2\cos^2 x - 1 = \frac{1}{2}$$
$$\cos^2 x = \frac{3}{4}$$
$$\cos x = \pm\frac{\sqrt{3}}{2}$$
So $x = \dfrac{\pi}{6} + 2n\pi$ or $x = \dfrac{5\pi}{6} + 2n\pi$ for n any integer.

35. $x \approx 1.12$ or $x \approx 5.16$



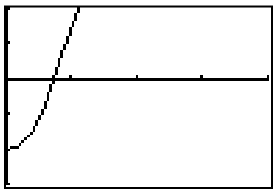
[0, 2π] by [–4, 4]

36. $x \approx 0.14$ or $x \approx 3.79$



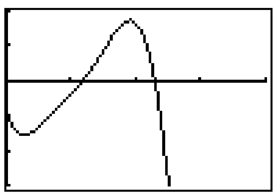
[0, 2π] by [–3, 2]

37. $x \approx 1.15$



$[0, 2\pi]$ by $[-3, 2]$

38. $x \approx 1.85$ or $x \approx 3.59$



$[0, 2\pi]$ by $[-3, 2]$

39. $\cos x = \frac{1}{2}$, so $x = \frac{\pi}{3}$ or $x = \frac{5\pi}{3}$.

40. $\sin 3x = (\sin x)(4\cos^2 x - 1)$. This equation becomes $(\sin x)(4\cos^2 x - 1) = \sin x$, or $2(\sin x)(2\cos^2 x - 1) = 0$, so $\sin x = 0$ or $\cos x = \pm\frac{\sqrt{2}}{2}$; $x = 0, x = \frac{\pi}{4}, x = \frac{3\pi}{4}, x = \pi$, $x = \frac{5\pi}{4}$, or $x = \frac{7\pi}{4}$.

41. The left side factors to $(\sin x - 3)(\sin x + 1) = 0$; only $\sin x = -1$ is possible, so $x = \frac{3\pi}{2}$.

42. $2\cos^2 t - 1 = \cos t$, or $2\cos^2 t - \cos t - 1 = 0$, or $(2\cos t + 1)(\cos t - 1) = 0$. Then $\cos t = -\frac{1}{2}$ or $\cos t = 1$: $t = 0, t = \frac{2\pi}{3}$ or $t = \frac{4\pi}{3}$.

43. $\sin(\cos x) = 1$ only if $\cos x = \frac{\pi}{2} + 2n\pi$. No choice of n gives a value in $[-1, 1]$, so there are no solutions.

44. $\cos(2x) + 5\cos x - 2 = 2\cos^2 x - 1 + 5\cos x - 2 = 2\cos^2 x + 5\cos x - 3 = 0$. $(2\cos x - 1)(\cos x + 3) = 0$, so $\cos(x) = \frac{1}{2}$ and $\cos(x) = -3$. The latter is extraneous so $x = \left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$

For #45–48, use graphs to suggest the intervals. To find the endpoints of the intervals, treat the inequalities as equations and solve.

45. $\cos 2x = \frac{1}{2}$ has solutions $x = \frac{\pi}{6}, x = \frac{5\pi}{6}, x = \frac{7\pi}{6}$, and $x = \frac{11\pi}{6}$ in interval $[0, 2\pi)$. The solution set for the inequality is, $0 \le x < \frac{\pi}{6}$ or $\frac{5\pi}{6} < x < \frac{7\pi}{6}$ or $\frac{11\pi}{6} < x < 2\pi$; that is, $\left[0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, \frac{7\pi}{6}\right) \cup \left(\frac{11\pi}{6}, 2\pi\right)$.

46. $2\sin x\cos x = 2\cos x$ is equivalent to $(\cos x)(\sin x - 1) = 0$, so the solutions in $(0, 2\pi]$ are $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$. The solution set for the inequality is $\frac{\pi}{2} < x < \frac{3\pi}{2}$; that is, $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

47. $\cos x = \frac{1}{2}$ has solutions $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{3}$ in the interval $[0, 2\pi)$. The solution set for the inequality is $\frac{\pi}{3} < x < \frac{5\pi}{3}$; that is, $\left(\frac{\pi}{3}, \frac{5\pi}{3}\right)$.

48. $\tan x = \sin x$ is equivalent to $(\sin x)(\cos x - 1) = 0$, so the only solution in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is $x = 0$. The solution set for the inequality is $-\frac{\pi}{2} < x < 0$; that is, $\left(-\frac{\pi}{2}, 0\right)$.

49. $y = 5\sin(3x + \cos^{-1}(3/5)) \approx 5\sin(3x + 0.9273)$

50. $y = 13\sin(2x - \cos^{-1}(5/13)) \approx 13\sin(2x - 1.176)$

51. Given: $A = 79°, B = 33°, a = 7$ — an AAS case.
$C = 180° - (A + B) = 68°$
$b = \frac{a\sin B}{\sin A} = \frac{7\sin 33°}{\sin 79°} \approx 3.9$;
$c = \frac{a\sin C}{\sin A} = \frac{7\sin 68°}{\sin 79°} \approx 6.6$.

52. Given: $a = 5, b = 8, B = 110°$ — an SSA case. Using the Law of Sines: $h = a\sin B = 4.7$; $h < a < b$, so there is one triangle.
$A = \sin^{-1}\left(\frac{a\sin B}{b}\right) \approx \sin^{-1}(0.587) \approx 36.0°$;
$C = 180° - (A + B) \approx 34.0°$;
$c = \frac{b\sin C}{\sin B} \approx \frac{8\sin 34.0°}{\sin 110°} \approx 4.8$.
Using Law of Cosines: Solve the quadratic equation $8^2 = 5^2 + c^2 - 2(5)c\cos 110°$, or $c^2 + (3.420)c - 39 = 0$; there is one positive solution: $c \approx 4.8$. Since $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$:
$A \approx \cos^{-1}(0.809) \approx 36.0°$ and $C = 180° - (A + B) \approx 34.0°$.

53. Given: $a = 8, b = 3, B = 30°$ — an SSA case. Using the Law of Sines: $h = a\sin B = 4$; $b < h$, so no triangle is formed. Using the Law of Cosines: Solve the quadratic equation $3^2 = 8^2 + c^2 - 2(8)c\cos 30°$, or $c^2 - (8\sqrt{3})c + 55 = 0$; there are no real solutions.

54. Given: $a = 14.7, A = 29.3°, C = 33°$ — an AAS case.
$B = 180° - (A + C) = 117.7°$, and
$b = \frac{a\sin B}{\sin A} = \frac{14.7\sin 117.7°}{\sin 29.3°} \approx 26.6$;
$c = \frac{a\sin C}{\sin A} = \frac{14.7\sin 33°}{\sin 29.3°} \approx 16.4$.

55. Given: $A = 34°, B = 74°, c = 5$ — an ASA case.
$C = 180° - (A + B) = 72°$;
$a = \frac{c\sin A}{\sin C} = \frac{5\sin 34°}{\sin 72°} \approx 2.9$;
$b = \frac{c\sin B}{\sin C} = \frac{5\sin 74°}{\sin 72°} \approx 5.1$.

56. Given: $c = 41$, $A = 22.9°$, $C = 55.1°$ — an AAS case.
$B = 180° - (A + C) = 102°$;
$a = \dfrac{c \sin A}{\sin C} = \dfrac{41 \sin 22.9°}{\sin 55.1°} \approx 19.5$;
$b = \dfrac{c \sin B}{\sin C} = \dfrac{41 \sin 102°}{\sin 55.1°} \approx 48.9$.

57. Given: $a = 5$, $b = 7$, $c = 6$ — an SSS case.
$A = \cos^{-1}\left(\dfrac{b^2 + c^2 - a^2}{2bc}\right) \approx \cos^{-1}(0.714) \approx 44.4°$;
$B = \cos^{-1}\left(\dfrac{a^2 + c^2 - b^2}{2ac}\right) \approx \cos^{-1}(0.2) \approx 78.5°$;
$C = 180° - (A + B) \approx 57.1°$.

58. Given: $A = 85°$, $a = 6$, $b = 4$ — an SSA case. Using the Law of Sines: $h = b \sin A \approx 4.0$; $h < b < a$, so there is one triangle.
$B = \sin^{-1}\left(\dfrac{b \sin A}{a}\right) \approx \sin^{-1}(0.664) \approx 41.6°$;
$C = 180° - (A + B) \approx 53.4°$;
$c = \dfrac{a \sin C}{\sin A} = \dfrac{6 \sin 53.4°}{\sin 85°} \approx 4.8$.
Using the Law of Cosines: Solve the quadratic equation $6^2 = 4^2 + c^2 - 2(4)c \cos 85°$, or $c^2 - (0.697)c - 20 = 0$; there is one positive solution:
$c \approx 4.8$. Since $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$:
$B \approx \cos^{-1}(0.747) \approx 41.6°$ and $C = 180° - (A + B)$ $\approx 53.4°$.

59. $s = \dfrac{1}{2}(3 + 5 + 6) = 7$;
Area $= \sqrt{s(s - a)(s - b)(s - c)}$
$= \sqrt{7(7 - 3)(7 - 5)(7 - 6)}$
$= \sqrt{56} \approx 7.5$

60. $c \approx 7.672$ so area $\approx \sqrt{528.141} \approx 23.0$ (using Heron's formula). Or, use $A = \dfrac{1}{2}ab \sin C$.

61. $h = 12 \sin 28° \approx 5.6$, so:
(a) $\approx 5.6 < b < 12$.
(b) $b \approx 5.6$ or $b \geq 12$.
(c) $b < 5.6$.

62. **(a)** $C = 180° - (A + B) = 45°$, so
$AC = b = \dfrac{c \sin B}{\sin C} = \dfrac{80 \sin 65°}{\sin 45°} \approx 102.5$ ft.
(b) The distance across the canyon is $b \sin A \approx 96.4$ ft.

63. Given: $c = 1.75$, $A = 33°$, $B = 37°$ — an ASA case, so
$C = 180° - (A + B) = 110°$;
$a = \dfrac{c \sin A}{\sin C} = \dfrac{1.75 \sin 33°}{\sin 110°} \approx 1.0$;
$b = \dfrac{c \sin B}{\sin C} = \dfrac{1.75 \sin 37°}{\sin 110°} \approx 1.1$,
and finally, the height is $h = b \sin A = a \sin b \approx 0.6$ mi.

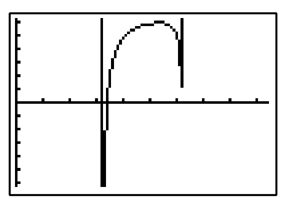
64. Given: $C = 70°$, $a = 225$, $b = 900$ — an SAS case, so
$AB = c = \sqrt{a^2 + b^2 - 2ab \cos C}$
$\approx \sqrt{722{,}106.841} \approx 849.77$ ft.

65. Let $a = 8$, $b = 9$, and $c = 10$. The largest angle is opposite the largest side, so we call it C.
Since $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$, $C = \cos^{-1}\left(\dfrac{5}{16}\right)$
$\approx 71.790°$, 1.253 rad.

66. The shorter diagonal splits the parallelogram into two (congruent) triangles with $a = 15$, $B = 40°$, and $c = 24$. The shorter diagonal has length $b = \sqrt{a^2 + b^2 - 2ac \cos B} \approx \sqrt{249.448} \approx 15.794$ ft. Since adjacent angles are supplementary, the other angle is 140°. The longer diagonal splits the parallelogram into (two) congruent triangles with $a = 15$, $B = 140°$, and $c = 24$, so the longer diagonal length is
$b = \sqrt{a^2 + c^2 - 2ac \cos B} \approx \sqrt{1352.552} \approx 36.777$ ft.

67. **(a)** The point (x, y) has coordinates $(\cos\theta, \sin\theta)$, so the bottom is $b_1 = 2$ units wide, the top is $b_2 = 2x = 2\cos\theta$ units wide, and the height is $h = y = \sin\theta$ units. Either use the formula for the area of a trapezoid, $A = \dfrac{1}{2}(b_1 + b_2)h$, or notice that the trapezoid can be split into two triangles and a rectangle. Either way:
$A(\theta) = \sin\theta + \sin\theta\cos\theta = \sin\theta(1 + \cos\theta)$
$= \sin\theta + \dfrac{1}{2}\sin 2\theta$.
(b) The maximizing angle is $\theta = \dfrac{\pi}{3} = 60°$; the maximum area is $\dfrac{3}{4}\sqrt{3} \approx 1.30$ square units.

68. **(a)** Substituting the values of a and b:
$S(\theta) = 6.825 + 0.63375(-\cot\theta + \sqrt{3} \cdot \csc\theta)$
$= 6.825 + \dfrac{0.63375(\sqrt{3} - \cos\theta)}{\sin\theta}$



[0, 9.4] by [−6.2, 6.2]

(b) Considering only angles between 0 and π, the minimum occurs when $\theta \approx 0.96 \approx 54.74°$.
(c) The minimum value of S is approximately $S(0.96) \approx 7.72$ in.^{2}

69. **(a)** Split the quadrilateral in half to leave two (identical) right triangles, with one leg 4000 mi, hypotenuse $4000 + h$ mi, and one acute angle $\theta/2$.
Then $\cos\dfrac{\theta}{2} = \dfrac{4000}{4000 + h}$; solve for h to leave
$h = \dfrac{4000}{\cos(\theta/2)} - 4000 = 4000 \sec\dfrac{\theta}{2} - 4000$ miles.
(b) $\cos\dfrac{\theta}{2} = \dfrac{4000}{4200}$, so $\theta = 2\cos^{-1}\left(\dfrac{20}{21}\right) \approx 0.62 \approx 35.51°$.

70. Rewrite the left side of the equation as follows:

$\sin x - \sin 2x + \sin 3x$
$= \sin x - 2\sin x\cos x + \sin(2x + x)$
$= \sin x - 2\sin x\cos x + \sin 2x\cos x + \cos 2x\sin x$
$= \sin x - 2\sin x\cos x + 2\sin x\cos^2 x + (\cos^2 x - \sin^2 x)\sin x$
$= \sin x - 2\sin x\cos x + 2\sin x\cos^2 x + \sin x\cos^2 x - \sin^3 x$
$= \sin x - 2\sin x\cos x + 3\sin x\cos^2 x - \sin^3 x$
$= \sin x\,(1 - 2\cos x + 3\cos^2 x - \sin^2 x)$
$= \sin x\,((1 - \sin^2 x) - 2\cos x + 3\cos^2 x)$
$= \sin x\,(\cos^2 x - 2\cos x + 3\cos^2 x)$
$= \sin x\,(4\cos^2 x - 2\cos x)$
$= 2\sin x\cos x\,(2\cos x - 1)$
$= \sin 2x\,(2\cos x - 1)$

So $\sin 2x = 0$ or $2\cos x - 1 = 0$. So $x = n\frac{\pi}{2}$ or $x = \pm\frac{\pi}{3} + 2n\pi$, n an integer.

71. The hexagon is made up of 6 equilateral triangles; using Heron's formula (or some other method), we find that each triangle has area $\sqrt{24(24 - 16)^3} = \sqrt{12{,}288} = 64\sqrt{3}$. The hexagon's area is therefore, $384\sqrt{3}$ cm^2, and the radius of the circle is 16 cm, so the area of the circle is 256π cm^2, and the area outside the hexagon is $256\pi - 384\sqrt{3} \approx 139.140$ cm^2.

72. The pentagon is made up of 5 triangles with base length 12 cm and height $\frac{6}{\tan 36°} \approx 8.258$ cm, so its area is about 247.749 cm^2. The radius of the circle is the height of those triangles, so the desired area is about $247.749 - \pi(8.258)^2 \approx 33.51$ cm^2.

73. The volume of a cylinder with radius r and height h is $V = \pi r^2 h$, so the wheel of cheese has volume $\pi(9^2)(5) = 405\pi$ cm^3; a 15° wedge would have fraction $\frac{15}{360} = \frac{1}{24}$ of that volume, or $\frac{405\pi}{24} \approx 53.01$ cm^3.

74. **(a)** $\frac{1}{2}(\cos(u - v) - \cos(u + v))$
$= \frac{1}{2}(\cos u\cos v + \sin u\sin v - (\cos u\cos v - \sin u\sin v))$
$= \frac{1}{2}(2\sin u\sin v)$
$= \sin u\sin v$

(b) $\frac{1}{2}(\cos(u - v) + \cos(u + v))$
$= \frac{1}{2}(\cos u\cos v + \sin u\sin v + \cos u\cos v - \sin u\sin v)$
$= \frac{1}{2}(2\cos u\cos v)$
$= \cos u\cos v$

(c) $\frac{1}{2}(\sin(u + v) + \sin(u - v))$
$= \frac{1}{2}(\sin u\cos v + \cos u\sin v + \sin u\cos v - \cos u\sin v)$
$= \frac{1}{2}(2\sin u\cos v)$
$= \sin u\cos v$

75. **(a)** By the product-to-sum formula in 74 (c),
$2\sin\frac{u+v}{2}\cos\frac{u-v}{2}$
$= 2\cdot\frac{1}{2}\left(\sin\frac{u+v+u-v}{2} + \sin\frac{u+v-(u-v)}{2}\right)$
$= \sin u + \sin v$

(b) By the product-to-sum formula in 74 (c),
$2\sin\frac{u-v}{2}\cos\frac{u+v}{2}$
$= 2\cdot\frac{1}{2}\left(\sin\frac{u-v+u+v}{2} + \sin\frac{u-v-(u+v)}{2}\right)$
$= \sin u + \sin(-v)$
$= \sin u - \sin v$

(c) By the product-to-sum formula in 74 (b),
$2\cos\frac{u+v}{2}\cos\frac{u-v}{2}$
$= 2\cdot\frac{1}{2}\left(\cos\frac{u+v-(u-v)}{2} + \cos\frac{u+v+u-v}{2}\right)$
$= \cos v + \cos u$
$= \cos u + \cos v$

(d) By the product-to-sum formula in 74 (a),
$-2\sin\frac{u+v}{2}\sin\frac{u-v}{2}$
$= -2\cdot\frac{1}{2}\left(\cos\frac{u+v-(u-v)}{2} - \cos\frac{u+v+u-v}{2}\right)$
$= -(\cos v - \cos u)$
$= \cos u - \cos v$

76. Pat faked the data. The Law of Cosines can be solved to show that $x = 12\sqrt{\frac{2}{1 - \cos\theta}}$. Only Carmen's values are consistent with the formula.

77. **(a)** Any inscribed angle that intercepts an arc of 180° is a right angle.

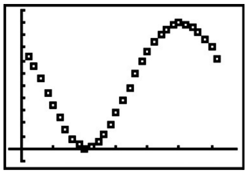
(b) Two inscribed angles that intercept the same arc are congruent.

(c) In right $\Delta A'BC$, $\sin A' = \frac{opp}{hyp} = \frac{a}{d}$.

(d) Because $\angle A'$ and $\angle A$ are congruent, $\frac{\sin A}{a} = \frac{\sin A'}{a} = \frac{a/d}{a} = \frac{1}{d}$.

(e) Of course. They both equal $\frac{\sin A}{a}$ by the Law of Sines.

Chapter 5 Project

1. 

[–2, 34] by [–0.1, 1.1]

2. We set the amplitude as half the difference between the maximum value, 1.00, and the minimum value, 0.00, so $a = 0.5$. And we set the average value as the average of the maximum and minimum values, so $k = 0.5$. Since $\cos(b(x - h))$ takes on its maximum value at h, we set $h = 29$. Experimenting with the graph suggests that b should be about $2\pi/30.5$. So the equation is

$$y \approx 0.5\cos\left(\frac{2\pi}{30.5}(x - 29)\right) + 0.5.$$

5. Using the identities from Questions 3 and 4 together,

$$\begin{aligned} y &\approx 0.5\cos\left(\frac{2\pi}{30.5}(x - 29)\right) + 0.5 \\ &= 0.5\cos\left(\frac{2\pi}{30.5}(29 - x)\right) + 0.5 \\ &= 0.5\sin\left(\frac{\pi}{2} - \frac{2\pi}{30.5}(29 - x)\right) + 0.5 \\ &= 0.5\sin\left(\frac{2\pi}{30.5}(x - 21.375)\right) + 0.5, \end{aligned}$$

which is equivalent to
$y \approx 0.5\sin(0.21x - 4.4) + 0.5$.
Sinusoidal regression yields
$y \approx 0.5\sin(0.21x + 1.87) + 0.49$.