# Chapter 4 Trigonometric Functions

## Section 4.1 Angles and Their Measures

### Exploration 1

1. $2\pi r$
2. $2\pi$ radians ($2\pi$ lengths of thread)
3. No, not quite, since the distance $\pi r$ would require a piece of thread $\pi$ times as long, and $\pi > 3$.
4. $\pi$ radians

### Quick Review 4.1

1. $C = 2\pi \cdot 2.5 = 5\pi$ in.
2. $C = 2\pi \cdot 4.6 = 9.2\pi$ m
3. $r = \dfrac{1}{2\pi} \cdot 12 = \dfrac{6}{\pi}$ m
4. $r = \dfrac{1}{2\pi} \cdot 8 = \dfrac{4}{\pi}$ ft
5. **(a)** $s = 47.52$ ft **(b)** $s = 39.77$ km
6. **(a)** $v = 26.1$ m/sec **(b)** $v = 8.06$ ft/sec
7. $60\,\dfrac{\text{mi}}{\text{hr}} \cdot 5280\,\dfrac{\text{ft}}{\text{mi}} \cdot \dfrac{1}{3600}\,\dfrac{\text{hr}}{\text{sec}} = 88$ ft/sec
8. $45\,\dfrac{\text{mi}}{\text{hr}} \cdot 5280\,\dfrac{\text{ft}}{\text{mi}} \cdot \dfrac{1}{3600}\,\dfrac{\text{hr}}{\text{sec}} = 66$ ft/sec
9. $8.8\,\dfrac{\text{ft}}{\text{sec}} \cdot \dfrac{1}{5280}\,\dfrac{\text{mi}}{\text{ft}} \cdot 3600\,\dfrac{\text{sec}}{\text{hr}} = 6$ mph
10. $132\,\dfrac{\text{ft}}{\text{sec}} \cdot \dfrac{1}{5280}\,\dfrac{\text{mi}}{\text{ft}} \cdot 3600\,\dfrac{\text{sec}}{\text{hr}} = 90$ mph

### Section 4.1 Exercises

1. $23°12' = \left(23 + \dfrac{12}{60}\right)^{\circ} = 23.2°$
2. $35°24' = \left(35 + \dfrac{24}{60}\right)^{\circ} = 35.4°$
3. $118°44'15'' = \left(118 + \dfrac{44}{60} + \dfrac{15}{3600}\right)^{\circ} = 118.7375°$
4. $48°30'36'' = \left(48 + \dfrac{30}{60} + \dfrac{36}{3600}\right)^{\circ} = 48.51°$
5. $21.2° = 21°(60 \cdot 0.2)' = 21°12'$
6. $49.7° = 49°(60 \cdot 0.7)' = 49°42'$
7. $118.32° = 118°(60 \cdot 0.32)' = 118°19.2'$
   $= 118°19'(60 \cdot 0.2)'' = 118°19'12''$
8. $99.37° = 99°(60 \cdot 0.37)' = 99°22.2'$
   $= 99°22'(60 \cdot 0.2)'' = 99°22'12''$

For #9–16, use the formula $s = r\theta$, and the equivalent forms $r = s/\theta$ and $\theta = s/r$.

9. $60° \cdot \dfrac{\pi}{180°} = \dfrac{\pi}{3}$ rad
10. $90° \cdot \dfrac{\pi}{180°} = \dfrac{\pi}{2}$ rad
11. $120° \cdot \dfrac{\pi}{180°} = \dfrac{2\pi}{3}$ rad
12. $150° \cdot \dfrac{\pi}{180°} = \dfrac{5\pi}{6}$ rad
13. $71.72° \cdot \dfrac{\pi}{180°} \approx 1.2518$ rad
14. $11.83° \cdot \dfrac{\pi}{180°} \approx 0.2065$ rad
15. $61°24' = \left(61 + \dfrac{24}{60}\right)^{\circ} = 61.4° \cdot \dfrac{\pi}{180°} \approx 1.0716$ rad
16. $75°30' = \left(75 + \dfrac{30}{60}\right)^{\circ} = 75.5° \cdot \dfrac{\pi}{180°} \approx 1.3177$ rad
17. $\dfrac{\pi}{6} \cdot \dfrac{180°}{\pi} = 30°$
18. $\dfrac{\pi}{4} \cdot \dfrac{180°}{\pi} = 45°$
19. $\dfrac{\pi}{10} \cdot \dfrac{180°}{\pi} = 18°$
20. $\dfrac{3\pi}{5} \cdot \dfrac{180°}{\pi} = 108°$
21. $\dfrac{7\pi}{9} \cdot \dfrac{180°}{\pi} = 140°$
22. $\dfrac{13\pi}{20} \cdot \dfrac{180°}{\pi} = 117°$
23. $2 \cdot \dfrac{180}{\pi} \approx 114.59°$
24. $1.3 \cdot \dfrac{180}{\pi} \approx 74.48°$
25. $s = 50$ in.
26. $s = 70$ cm
27. $r = 6/\pi$ ft
28. $r = 7.5/\pi$ cm
29. $\theta = 3$ radians
30. $\theta = \dfrac{4}{7}$ radians
31. $r = \dfrac{360}{\pi}$ cm

**32.** $s = (5 \text{ ft})(18°)\left(\frac{2\pi}{360°}\right) = \frac{\pi}{2}$ ft

**33.** $\theta = s_1/r_1 = \frac{9}{11}$ rad and $s_2 = r_2\theta = 36$

**34.** $\theta = s_1/r_1 = 4.5$ rad and $r_2 = s_2/\theta = 16$ km

**35.** The angle is $10° \cdot \frac{\pi}{180°} = \frac{\pi}{18}$ rad, so the curved side measures $\frac{11\pi}{18}$ in. The two straight sides measures 11 in. each, so the perimeter is $11 + 11 + \frac{11\pi}{18} \approx 24$ inches.

**36.** The angle is $100° \cdot \frac{\pi}{180°} = \frac{5\pi}{9}$ rad, so

$$7 = \frac{5\pi}{9}r.$$

Then

$$r = \frac{63}{5\pi} \approx 4 \text{ cm}.$$

**37.** Five pieces of track form a semicircle, so each arc has a central angle of $\pi/5$ radians. The inside arc length is $r_i(\pi/5)$ and the outside arc length is $r_o(\pi/5)$. Since $r_o(\pi/5) - r_i(\pi/5) = 3.4$ inches, we conclude that $r_o - r_i = 3.4(5/\pi) \approx 5.4$ inches.

**38.** Let the diameter of the inner (red) circle be $d$. The inner circle's perimeter is 37.7 inches, which equals $\pi d$. Then the next-largest (yellow) circle has a perimeter of $\pi(d + 6 + 6) = \pi d + 12\pi = 37.7 + 12\pi \approx 75.4$ inches.

**39.** **(a)** NE is 45°. **(b)** NNE is 22.5°. **(c)** WSW is 247.5°.

**40.** **(a)** SSW is 202.5°. **(b)** WNW is 292.5°. **(c)** NNW is 337.5°.

**41.** ESE is closest at 112.5°.

**42.** SW is closest at 225°.

**43.** The angle between them is $\theta = 9°42' = 9.7° \approx 0.1693$ radians, so the distance is about $s = r\theta = (25)(0.1693) \approx 4.23$ statute miles.

**44.** Since $C = \pi d$, a tire travels a distance $\pi d$ with each revolution.

**(a)** Each tire travels at a speed of 800 $\pi d$ in. per minute, or

$$\left(\frac{800\pi d \text{ in.}}{1 \text{ min}}\right)\left(\frac{60 \text{ min}}{1 \text{ hr}}\right)\left(\frac{1 \text{ mi}}{63{,}360 \text{ in.}}\right) \approx 2.38d \text{ mi/hr}.$$

| Vehicle | $d$ | Speed $\approx 2.38d$ |
|---|---|---|
| Taurus | 26.16 | 62.26 mph |
| Charger | 28.63 | 68.14 mph |
| Mariner | 28.95 | 68.90 mph |

**(b)** $\left(\frac{\pi d \text{ in.}}{1 \text{ rev}}\right)\left(\frac{1 \text{ mi}}{63{,}360 \text{ in.}}\right) = \frac{\pi d}{63{,}360}$ mi/rev, so each mile requires $\frac{63{,}360}{\pi d} \approx \frac{20{,}168}{d}$ revolutions.

Taurus: $\frac{20{,}168}{26.16} \approx 770.95$ revolutions

Mariner: $\frac{20{,}168}{28.95} \approx 696.65$ revolutions

The Taurus must make just over 74 more revolutions.

**(c)** In each revolution, the tire would cover a distance of $\pi d_{new}$ rather than $\pi d_{old}$, so that the car would travel $(\pi d_{new})/(\pi d_{old}) = d_{new}/d_{old} = 28/26.16 \approx 1.07$ miles for every mile the car's instruments would show. Both the odometer and speedometer readings would be low.

**45.** $v = 44$ ft/sec and $r = 13$ in., so

$$\omega = v/r = \left(44\frac{\text{ft}}{\text{sec}} \cdot 60\frac{\text{sec}}{\text{min}}\right) \div \left(13 \text{ in.} \cdot \frac{1}{12}\frac{\text{ft}}{\text{in.}} \cdot 2\pi\frac{\text{rad}}{\text{rev}}\right)$$

$\approx 387.85$ rpm.

**46.** **(a)** $\frac{S}{W} = \frac{R}{100} \Rightarrow S = \frac{WR}{100}$ mm.

25.4 mm = 1 in., so

$$S = \frac{WR}{100} \cdot \frac{1}{25.4} = \frac{WR}{2540} \text{ in.}$$

**(b)** $D + 2S = D + 2\left(\frac{WR}{2540}\right) = D + \frac{WR}{1270}$ in.

**(c)** Taurus: $D = 16 + \frac{215 \cdot 60}{1270} \approx 26.16$ in.

Charger: $D = 18 + \frac{225 \cdot 60}{1270} \approx 28.63$ in.

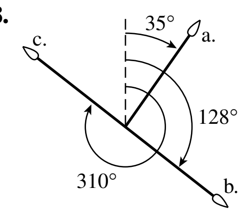
Mariner: $D = 16 + \frac{235 \cdot 70}{1270} \approx 28.95$ in.

Ridgeline: $D = 17 + \frac{245 \cdot 65}{1270} \approx 29.54$ in.

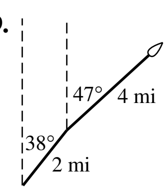
**47.** $\omega = 2000$ rpm and $r = 5$ in., so

$$v = r\omega = \left(5 \text{ in.} \cdot 12\frac{\text{teeth}}{\text{in.}}\right) \cdot \left(2000\frac{\text{rev}}{\text{min}} \cdot 2\pi\frac{\text{rad}}{\text{rev}} \cdot \frac{1}{60}\frac{\text{min}}{\text{sec}}\right) \approx 12{,}566.37 \text{ teeth per second.}$$

**48.**




**49.**




**50.** 257 naut mi $\cdot \frac{3956\pi \text{ stat mi}}{10{,}800 \text{ naut mi}} \approx 296$ statute miles

**51.** 895 stat mi $\cdot \frac{10{,}800 \text{ naut mi}}{3956\pi \text{ stat mi}} \approx 778$ nautical miles

**52.** **(a)** Lane 5 has inside radius 37 m, while the inside radius of lane 6 is 38 m, so over the whole semicircle, the difference is $38\pi - 37\pi = \pi \approx 3.142$ m. (This would be the answer for any two adjacent lanes.)

**(b)** $38\pi - 33\pi = 5\pi \approx 15.708$ m.

**53.** **(a)** $s = r\theta = (4)(4\pi) = 16\pi \approx 50.265$ in., or $\frac{4}{3}\pi \approx 4.189$ ft.

**(b)** $r\theta = 2\pi \approx 6.283$ ft.

**54.** $s = r\theta = (52)\left(\frac{\pi}{180}\right) = \frac{13}{45}\pi \approx 0.908$ ft

**55.** **(a)** $\omega_1 = 120\,\frac{\text{rev}}{\text{min}} \cdot 2\pi\,\frac{\text{rad}}{\text{rev}} \cdot \frac{1}{60}\,\frac{\text{min}}{\text{sec}} = 4\pi$ rad/sec

**(b)** $v = R\omega_1 = (7\text{ cm})\left(4\pi\,\frac{\text{rad}}{\text{sec}}\right) = 28\pi$ cm/sec

**(c)** $\omega_2 = v/r = \left(28\pi\,\frac{\text{cm}}{\text{sec}}\right) \div (4\text{ cm}) = 7\pi$ rad/sec

**56.** **(a)** $\omega = 135\,\frac{\text{rev}}{\text{min}} \cdot 2\pi\,\frac{\text{rad}}{\text{rev}} \cdot \frac{1}{60}\,\frac{\text{min}}{\text{sec}} = 4.5\pi$ rad/sec

**(b)** $v = r\omega = (1.2\text{ m})\left(4.5\pi\,\frac{\text{rad}}{\text{sec}}\right) = 5.4\pi$ m/sec

**(c)** The radius to this halfway point is $r^* = \frac{1}{2}r = 0.6$ m, so $v = r^*\omega = (0.6\text{ m})\left(4.5\pi\,\frac{\text{rad}}{\text{sec}}\right) = 2.7\pi$ m/sec.

**57.** True. In the amount of time it takes for the merry-go-round to complete one revolution, horse $B$ travels a distance of $2\pi r$, where $r$ is $B$'s distance from the center. In the same time, horse $A$ travels a distance of $2\pi(2r) = 2(2\pi r)$ — twice as far as $B$.

**58.** False. If all three radian measures were integers, their sum would be an integer. But the sum must equal $\pi$, which is not an integer.

**59.** $x° = x°\left(\frac{\pi\text{ rad}}{180°}\right) = \frac{\pi x}{180}$. The answer is C.

**60.** If the perimeter is 4 times the radius, the arc is two radii long, which implies an angle of 2 radians. The answer is A.

**61.** Let $n$ be the number of revolutions per minute.
$\left(\frac{26\pi\text{ in.}}{1\text{ rev}}\right)\left(\frac{n\text{ rev}}{1\text{ min}}\right)\left(\frac{60\text{ min}}{1\text{ hr}}\right)\left(\frac{1\text{ mi}}{63{,}360\text{ in.}}\right) \approx 0.07735\,n$ mph.
Solving $0.07735\,n = 10$ yields $n \approx 129$.
The answer is B.

**62.** The size of the circle does not affect the size of the angle. The radius and the subtended arc length both double, so that their ratio stays the same.
The answer is C.

In #63–66, we need to "borrow" 1° and change it to 60′ in order to complete the subtraction.

**63.** 122°25′ − 84°23′ = 38°02′

**64.** 117°09′ − 74°0′ = 43°09′

**65.** 93°16′ − 87°39′ = 92°76′ − 87°39′ = 5°37′

**66.** 122°20′ − 80°12′ = 42°08′

In #67–70, find the difference in the latitude. Convert this difference to minutes; this is the distance in nautical miles. The Earth's diameter is not needed.

**67.** The difference in latitude is 34°03′ − 32°43′ = 1°20′ = 80 minutes of arc, which is 80 naut mi.

**68.** The difference in latitude is 47°36′ − 37°47′ = 9°49′ = 589 minutes of arc, which is 589 naut mi.

**69.** The difference in latitude is 44°59′ − 29°57′ = 15°02′ = 902 minutes of arc, which is 902 naut mi.

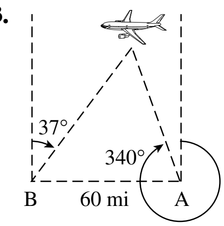
**70.** The difference in latitude is 42°20′ − 33°45′ = 8°35′ = 515 minutes of arc, which is 515 naut mi.

**71.** The whole circle's area is $\pi r^2$; the sector with central angle $\theta$ makes up $\theta/2\pi$ of that area, or $\frac{\theta}{2\pi} \cdot \pi r^2 = \frac{1}{2}\theta r^2$.

**72.** **(a)** $A = \frac{1}{2}(5.9)^2\left(\frac{\pi}{5}\right) \approx 3.481\pi = 10.936$ ft$^2$.

**(b)** $A = \frac{1}{2}(1.6)^2(3.7) = 4.736$ km$^2$.

**73.**




**74.** Bike wheels: $\omega_1 = v_1/r = (66\text{ ft/sec} \cdot 12\text{ in./ft}) \div (14\text{ in.}) \approx 56.5714$ rad/sec. The wheel sprocket must have the same angular velocity: $\omega_2 = \omega_1 \approx 56.5714$ rad/sec. For the pedal sprocket, we first need the velocity of the chain, using the wheel sprocket: $v_2 \approx (\frac{3}{2}\text{ in.})(56.5714\text{ rad/sec}) \approx 84.8571$ in./sec. Then the pedal sprocket's angular velocity is $\omega_3 = (84.8571\text{ in./sec}) \div (4.5\text{ in.}) \approx 18.9$ rad/sec.

## ■ Section 4.2 Trigonometric Functions of Acute Angles

### Exploration 1

1. sin and csc, cos and sec, and tan and cot.
2. tan $\theta$
3. sec $\theta$
4. 1
5. sin $\theta$ and cos $\theta$

### Exploration 2

1. Let $\theta = 60°$. Then

$\sin\theta = \frac{\sqrt{3}}{2} \approx 0.866$ $\qquad$ $\csc\theta = \frac{2}{\sqrt{3}} \approx 1.155$

$\cos\theta = \frac{1}{2}$ $\qquad$ $\sec\theta = 2$

$\tan\theta = \sqrt{3} \approx 1.732$ $\qquad$ $\cot\theta = \frac{1}{\sqrt{3}} \approx 0.577$

2. The values are the same, but for different functions. For example, sin 30° is the same as cos 60°, cot 30° is the same as tan 60°, etc.
3. The value of a trig function at $\theta$ is the same as the value of its co-function at $90° - \theta$.

### Quick Review 4.2

1. $x = \sqrt{5^2 + 5^2} = \sqrt{50} = 5\sqrt{2}$
2. $x = \sqrt{8^2 + 12^2} = \sqrt{208} = 4\sqrt{13}$
3. $x = \sqrt{10^2 - 8^2} = 6$
4. $x = \sqrt{4^2 - 2^2} = \sqrt{12} = 2\sqrt{3}$
5. $8.4 \text{ ft} \cdot 12 \dfrac{\text{in}}{\text{ft}} = 100.8 \text{ in.}$
6. $940 \text{ ft} \cdot \dfrac{1}{5280} \dfrac{\text{mi}}{\text{ft}} = \dfrac{47}{264} \approx 0.17803 \text{ mi}$
7. $a = (0.388)(20.4) = 7.9152 \text{ km}$
8. $b = \dfrac{23.9}{1.72} \approx 13.895 \text{ ft}$
9. $\alpha = 13.3 \cdot \dfrac{2.4}{31.6} \approx 1.0101$ (no units)
10. $\beta = 5.9 \cdot \dfrac{6.15}{8.66} \approx 4.18995$ (no units)

### Section 4.2 Exercises

1. $\sin\theta = \dfrac{4}{5}, \cos\theta = \dfrac{3}{5}, \tan\theta = \dfrac{4}{3}, \csc\theta = \dfrac{5}{4}, \sec\theta = \dfrac{5}{3}, \cot\theta = \dfrac{3}{4}.$
2. $\sin\theta = \dfrac{8}{\sqrt{113}}, \cos\theta = \dfrac{7}{\sqrt{113}}, \tan\theta = \dfrac{8}{7}; \csc\theta = \dfrac{\sqrt{113}}{8}, \sec\theta = \dfrac{\sqrt{113}}{7}, \cot\theta = \dfrac{7}{8}.$
3. $\sin\theta = \dfrac{12}{13}, \cos\theta = \dfrac{5}{13}, \tan\theta = \dfrac{12}{5}; \csc\theta = \dfrac{13}{12}, \sec\theta = \dfrac{13}{5}, \cot\theta = \dfrac{5}{12}.$
4. $\sin\theta = \dfrac{8}{17}, \cos\theta = \dfrac{15}{17}, \tan\theta = \dfrac{8}{15}; \csc\theta = \dfrac{17}{8}, \sec\theta = \dfrac{17}{15}, \cot\theta = \dfrac{15}{8}.$
5. The hypotenuse length is $\sqrt{7^2 + 11^2} = \sqrt{170}$, so $\sin\theta = \dfrac{7}{\sqrt{170}}, \cos\theta = \dfrac{11}{\sqrt{170}}, \tan\theta = \dfrac{7}{11}; \csc\theta = \dfrac{\sqrt{170}}{7}, \sec\theta = \dfrac{\sqrt{170}}{11}, \cot\theta = \dfrac{11}{7}.$
6. The adjacent side length is $\sqrt{8^2 - 6^2} = \sqrt{28} = 2\sqrt{7}$, so $\sin\theta = \dfrac{3}{4}, \cos\theta = \dfrac{\sqrt{7}}{4}, \tan\theta = \dfrac{3}{\sqrt{7}}; \csc\theta = \dfrac{4}{3}, \sec\theta = \dfrac{4}{\sqrt{7}}, \cot\theta = \dfrac{\sqrt{7}}{3}.$
7. The opposite side length is $\sqrt{11^2 - 8^2} = \sqrt{57}$, so $\sin\theta = \dfrac{\sqrt{57}}{11}, \cos\theta = \dfrac{8}{11}, \tan\theta = \dfrac{\sqrt{57}}{8}; \csc\theta = \dfrac{11}{\sqrt{57}}, \sec\theta = \dfrac{11}{8}, \cot\theta = \dfrac{8}{\sqrt{57}}.$
8. The adjacent side length is $\sqrt{13^2 - 9^2} = \sqrt{88} = 2\sqrt{22}$, so $\sin\theta = \dfrac{9}{13}, \cos\theta = \dfrac{2\sqrt{22}}{13}, \tan\theta = \dfrac{9}{2\sqrt{22}}; \csc\theta = \dfrac{13}{9}, \sec\theta = \dfrac{13}{2\sqrt{22}}, \cot\theta = \dfrac{2\sqrt{22}}{9}.$
9. Using a right triangle with hypotenuse 7 and legs 3 (opposite) and $\sqrt{7^2 - 3^2} = \sqrt{40} = 2\sqrt{10}$ (adjacent), we have $\sin\theta = \dfrac{3}{7}, \cos\theta = \dfrac{2\sqrt{10}}{7}, \tan\theta = \dfrac{3}{2\sqrt{10}}; \csc\theta = \dfrac{7}{3}, \sec\theta = \dfrac{7}{2\sqrt{10}}, \cot\theta = \dfrac{2\sqrt{10}}{3}.$
10. Using a right triangle with hypotenuse 3 and legs 2 (opposite) and $\sqrt{3^2 - 2^2} = \sqrt{5}$ (adjacent), we have $\sin\theta = \dfrac{2}{3}, \cos\theta = \dfrac{\sqrt{5}}{3}, \tan\theta = \dfrac{2}{\sqrt{5}}; \csc\theta = \dfrac{3}{2}, \sec\theta = \dfrac{3}{\sqrt{5}}, \cot\theta = \dfrac{\sqrt{5}}{2}.$
11. Using a right triangle with hypotenuse 11 and legs 5 (adjacent) and $\sqrt{11^2 - 5^2} = \sqrt{96} = 4\sqrt{6}$ (opposite), we have $\sin\theta = \dfrac{4\sqrt{6}}{11}, \cos\theta = \dfrac{5}{11}, \tan\theta = \dfrac{4\sqrt{6}}{5}; \csc\theta = \dfrac{11}{4\sqrt{6}}, \sec\theta = \dfrac{11}{5}, \cot\theta = \dfrac{5}{4\sqrt{6}}.$
12. Using a right triangle with hypotenuse 8 and legs 5 (adjacent) and $\sqrt{8^2 - 5^2} = \sqrt{39}$ (opposite), we have $\sin\theta = \dfrac{\sqrt{39}}{8}, \cos\theta = \dfrac{5}{8}, \tan\theta = \dfrac{\sqrt{39}}{5}; \csc\theta = \dfrac{8}{\sqrt{39}}, \sec\theta = \dfrac{8}{5}, \cot\theta = \dfrac{5}{\sqrt{39}}.$
13. Using a right triangle with legs 5 (opposite) and 9 (adjacent) and hypotenuse $\sqrt{5^2 + 9^2} = \sqrt{106}$, we have $\sin\theta = \dfrac{5}{\sqrt{106}}, \cos\theta = \dfrac{9}{\sqrt{106}}, \tan\theta = \dfrac{5}{9}; \csc\theta = \dfrac{\sqrt{106}}{5}, \sec\theta = \dfrac{\sqrt{106}}{9}, \cot\theta = \dfrac{9}{5}.$
14. Using a right triangle with legs 12 (opposite) and 13 (adjacent) and hypotenuse $\sqrt{12^2 + 13^2} = \sqrt{313}$, we have $\sin\theta = \dfrac{12}{\sqrt{313}}, \cos\theta = \dfrac{13}{\sqrt{313}}, \tan\theta = \dfrac{12}{13}; \csc\theta = \dfrac{\sqrt{313}}{12}, \sec\theta = \dfrac{\sqrt{313}}{13}, \cot\theta = \dfrac{13}{12}.$
15. Using a right triangle with legs 3 (opposite) and 11 (adjacent) and hypotenuse $\sqrt{3^2 + 11^2} = \sqrt{130}$, we have $\sin\theta = \dfrac{3}{\sqrt{130}}, \cos\theta = \dfrac{11}{\sqrt{130}}, \tan\theta = \dfrac{3}{11}; \csc\theta = \dfrac{\sqrt{130}}{3}, \sec\theta = \dfrac{\sqrt{130}}{11}, \cot\theta = \dfrac{11}{3}.$

**16.** Using a right triangle with hypotenuse 12 and legs 5 (opposite) and $\sqrt{12^2 - 5^2} = \sqrt{119}$ (adjacent), we have $\sin\theta = \frac{5}{12}$, $\cos\theta = \frac{\sqrt{119}}{12}$, $\tan\theta = \frac{5}{\sqrt{119}}$; $\csc\theta = \frac{12}{5}$, $\sec\theta = \frac{12}{\sqrt{119}}$, $\cot\theta = \frac{\sqrt{119}}{5}$.

**17.** Using a right triangle with hypotenuse 23 and legs 9 (opposite) and $\sqrt{23^2 - 9^2} = \sqrt{448} = 8\sqrt{7}$ (adjacent), we have $\sin\theta = \frac{9}{23}$, $\cos\theta = \frac{8\sqrt{7}}{23}$, $\tan\theta = \frac{9}{8\sqrt{7}}$; $\csc\theta = \frac{23}{9}$, $\sec\theta = \frac{23}{8\sqrt{7}}$, $\cot\theta = \frac{8\sqrt{7}}{9}$.

**18.** Using a right triangle with hypotenuse 17 and legs 5 (adjacent) and $\sqrt{17^2 - 5^2} = \sqrt{264} = 2\sqrt{66}$ (opposite), we have $\sin\theta = \frac{2\sqrt{66}}{17}$, $\cos\theta = \frac{5}{17}$, $\tan\theta = \frac{2\sqrt{66}}{5}$; $\csc\theta = \frac{17}{2\sqrt{66}}$, $\sec\theta = \frac{17}{5}$, $\cot\theta = \frac{5}{2\sqrt{66}}$.

**19.** $\frac{\sqrt{3}}{2}$

**20.** 1

**21.** $\sqrt{3}$

**22.** 2

**23.** $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

**24.** $\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

**25.** $\sec 45^\circ = 1/\cos 45^\circ \approx 1.4142$. Squaring this result yields 2.0000, so $\sec 45^\circ = \sqrt{2}$.

**26.** $\sin 60^\circ \approx 0.8660$. Squaring this result yields $0.7500 = 3/4$, so $\sin 60^\circ = \sqrt{3/4} = \sqrt{3}/2$.

**27.** $\csc(\pi/3) = 1/\sin(\pi/3) \approx 1.1547$. Squaring this result yields 1.3333 or essentially 4/3, so $\csc(\pi/3) = \sqrt{4/3} = 2/\sqrt{3} = 2\sqrt{3}/3$.

**28.** $\tan(\pi/3) \approx 1.73205$. Squaring this result yields 3.0000, so $\tan(\pi/3) = \sqrt{3}$.

For #29–40, the answers marked with an asterisk (*) should be found in DEGREE mode; the rest should be found in RADIAN mode. Since most calculators do not have the secant, cosecant, and cotangent functions built in, the reciprocal versions of these functions are shown.

**29.** $\approx 0.961$*

**30.** $\approx 0.141$*

**31.** $\approx 0.943$*

**32.** $\approx 0.439$*

**33.** $\approx 0.268$

**34.** $\approx 0.208$

**35.** $\frac{1}{\cos 49^\circ} \approx 1.524$*

**36.** $\frac{1}{\sin 19^\circ} \approx 3.072$*

**37.** $\frac{1}{\tan 0.89} \approx 0.810$

**38.** $\frac{1}{\cos 1.24} \approx 3.079$

**39.** $\frac{1}{\tan(\pi/8)} \approx 2.414$

**40.** $\frac{1}{\sin(\pi/10)} \approx 3.236$

**41.** $\theta = 30^\circ = \frac{\pi}{6}$

**42.** $\theta = 60^\circ = \frac{\pi}{3}$

**43.** $\theta = 60^\circ = \frac{\pi}{3}$

**44.** $\theta = 45^\circ = \frac{\pi}{4}$

**45.** $\theta = 60^\circ = \frac{\pi}{3}$

**46.** $\theta = 45^\circ = \frac{\pi}{4}$

**47.** $\theta = 30^\circ = \frac{\pi}{6}$

**48.** $\theta = 30^\circ = \frac{\pi}{6}$

**49.** $x = \frac{15}{\sin 34^\circ} \approx 26.82$

**50.** $z = \frac{23}{\cos 39^\circ} \approx 29.60$

**51.** $y = \frac{32}{\tan 57^\circ} \approx 20.78$

**52.** $x = 14 \sin 43^\circ \approx 9.55$

**53.** $y = 6/\sin 35^\circ \approx 10.46$

**54.** $x = 50 \cos 66^\circ \approx 20.34$

For #55–58, choose whichever of the following formulas is appropriate:

$$a = \sqrt{c^2 - b^2} = c\sin\alpha = c\cos\beta = b\tan\alpha = \frac{b}{\tan\beta}$$

$$b = \sqrt{c^2 - a^2} = c\cos\alpha = c\sin\beta = a\tan\beta = \frac{a}{\tan\alpha}$$

$$c = \sqrt{a^2 + b^2} = \frac{a}{\cos\beta} = \frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{b}{\cos\alpha}$$

If one angle is given, subtract from 90° to find the other angle.

**55.** $b = \frac{a}{\tan\alpha} = \frac{12.3}{\tan 20^\circ} \approx 33.79$,
$c = \frac{a}{\sin\alpha} = \frac{12.3}{\sin 20^\circ} \approx 35.96$, $\beta = 90^\circ - \alpha = 70^\circ$

**56.** $a = c\sin\alpha = 10\sin 41^\circ \approx 6.56$,
$b = c\cos\alpha = 10\cos 41^\circ \approx 7.55$, $\beta = 90^\circ - \alpha = 49^\circ$

**57.** $b = a\tan\beta = 15.58\tan 55^\circ \approx 22.25$,
$c = \frac{a}{\cos\beta} = \frac{15.58}{\cos 55^\circ} \approx 27.16$, $\alpha = 90^\circ - \beta = 35^\circ$

**58.** $b = a\tan\beta = 5\tan 59^\circ \approx 8.32$,
$c = \frac{a}{\cos\beta} = \frac{5}{\cos 59^\circ} \approx 9.71$, $\alpha = 90^\circ - \beta = 31^\circ$

**59.** 0. As $\theta$ gets smaller and smaller, the side opposite $\theta$ gets smaller and smaller, so its ratio to the hypotenuse approaches 0 as a limit.

**60.** 1. As $\theta$ gets smaller and smaller, the side adjacent to $\theta$ approaches the hypotenuse in length, so its ratio to the hypotenuse approaches 1 as a limit.

**61.** $h = 55\tan 75^\circ \approx 205.26$ ft

**62.** $h = 5 + 120\tan 8^\circ \approx 21.86$ ft

**63.** $A = 12 \cdot \frac{5}{\sin 54^\circ} \approx 74.16\text{ ft}^2$

**64.** $h = 130\tan 82.9^\circ \approx 1043.70$ ft

**65.** $AC = 100\tan 75^\circ 12' 42'' \approx 378.80$ ft

**66.** Connect the three points on the arc to the center of the circle, forming three triangles, each with hypotenuse 10 ft. The horizontal legs of the three triangles have lengths $10\cos 67.5^\circ \approx 3.827$, $10\cos 45^\circ \approx 7.071$, and $10\cos 22.5^\circ \approx 9.239$. The widths of the four strips are therefore,

$3.827 - 0 = 3.827$ (strip $A$)
$7.071 - 3.827 = 3.244$ (strip $B$)
$9.239 - 7.071 = 2.168$ (strip $C$)
$10 - 9.239 = 0.761$ (strip $D$)

Allen needs to correct his data for strips $B$ and $C$.

**67.** False. This is only true if $\theta$ is an acute angle in a right triangle. (Then it is true by definition.)

**68.** False. The larger the angle of a triangle, the smaller its cosine.

**69.** $\sec 90° = \frac{1}{\cos 90°} = \frac{1}{0}$ is undefined. The answer is E.

**70.** $\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{3}{5}$. The answer is A.

**71.** If the unknown slope is $m$, then $m \sin\theta = -1$, so $m = -\frac{1}{\sin\theta} = -\csc\theta$. The answer is D.

**72.** For all $\theta$, $-1 \le \cos\theta \le 1$. The answer is B.

**73.** For angles in the first quadrant, sine values will be increasing, cosine values will be decreasing and only tangent values can be greater than 1. Therefore, the first column is tangent, the second column is sine, and the third column is cosine.

**74.** For angles in the first quadrant, secant values will be increasing, and cosecant and cotangent values will be decreasing. We recognize that $\csc(30°) = 2$. Therefore, the first column is secant, the second column is cotangent, and the third column is cosecant.

**75.** The distance $d_A$ from $A$ to the mirror is $5 \cos 30°$; the distance from $B$ to the mirror is $d_B = d_A - 2$. Then

$$PB = \frac{d_B}{\cos\beta} = \frac{d_A - 2}{\cos 30°} = 5 - \frac{2}{\cos 30°}$$
$$= 5 - \frac{4}{\sqrt{3}} \approx 2.69 \text{ m.}$$

**76.** Let $P$ be the point at which we should aim; let $\alpha$ and $\beta$ be the angles as labeled in #73. Since $\alpha = \beta$, $\tan\alpha = \tan\beta$. $P$ should be $x$ inches to the right of $C$, where $x$ is chosen so that $\tan\alpha = \frac{x}{15} = \tan\beta = \frac{30 - x}{10}$. Then $10x = 15(30 - x)$, so $25x = 450$, which gives $x = 18$. Aim 18 in. to the right of $C$ (or 12 in. to the left of $D$).

**77.** One possible proof:

$$(\sin\theta)^2 + (\cos\theta)^2 = \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2$$
$$= \frac{a^2}{c^2} + \frac{b^2}{c^2}$$
$$= \frac{a^2 + b^2}{c^2}$$
$$= \frac{c^2}{c^2} \quad \text{(Pythagorean theorem: } a^2 + b^2 = c^2.\text{)}$$
$$= 1$$

**78.** Let $h$ be the length of the altitude to base $b$ and denote the area of the triangle by $A$. Then

$$\frac{h}{a} = \sin\theta$$
$$\therefore h = a\sin\theta$$

Since $A = \frac{1}{2}bh$, we can substitute $h = a\sin\theta$ to get $A = \frac{1}{2}ab\sin\theta$.

## ■ Section 4.3 Trigonometry Extended: The Circular Functions

### Exploration 1

**1.** The side opposite $\theta$ in the triangle has length $y$ and the hypotenuse has length $r$. Therefore $\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{r}$.

**2.** $\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{x}{r}$

**3.** $\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{y}{x}$

**4.** $\cot\theta = \frac{x}{y}$; $\sec\theta = \frac{r}{x}$; $\csc\theta = \frac{r}{y}$

### Exploration 2

**1.** The $x$-coordinates on the unit circle lie between $-1$ and 1, and $\cos t$ is always an $x$-coordinate on the unit circle.

**2.** The $y$-coordinates on the unit circle lie between $-1$ and 1, and $\sin t$ is always a $y$-coordinate on the unit circle.

**3.** The points corresponding to $t$ and $-t$ on the number line are wrapped to points above and below the $x$-axis with the same $x$-coordinates. Therefore $\cos t$ and $\cos(-t)$ are equal.

**4.** The points corresponding to $t$ and $-t$ on the number line are wrapped to points above and below the $x$-axis with exactly opposite $y$-coordinates. Therefore $\sin t$ and $\sin(-t)$ are opposites.

**5.** Since $2\pi$ is the distance around the unit circle, both $t$ and $t + 2\pi$ get wrapped to the same point.

**6.** The points corresponding to $t$ and $t + \pi$ get wrapped to points on either end of a diameter on the unit circle. These points are symmetric with respect to the origin and therefore have coordinates $(x, y)$ and $(-x, -y)$. Therefore $\sin t$ and $\sin(t + \pi)$ are opposites, as are $\cos t$ and $\cos(t + \pi)$.

**7.** By the observation in (6), $\tan t$ and $\tan(t + \pi)$ are ratios of the form $\frac{y}{x}$ and $\frac{-y}{-x}$, which are either equal to each other or both undefined.

**8.** The sum is always of the form $x^2 + y^2$ for some $(x, y)$ on the unit circle. Since the equation of the unit circle is $x^2 + y^2 = 1$, the sum is always 1.

**9.** Answers will vary. For example, there are similar statements that can be made about the functions cot, sec, and csc.

### Quick Review 4.3

**1.** $-30°$

**2.** $-150°$

**3.** $45°$

**4.** $240°$

**5.** $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

**6.** $\cot\frac{\pi}{4} = 1$

**7.** $\csc\frac{\pi}{4} = \sqrt{2}$

**8.** $\sec\frac{\pi}{3} = 2$

**9.** Using a right triangle with hypotenuse 13 and legs 5 (opposite) and $\sqrt{13^2 - 5^2} = 12$ (adjacent), we have $\sin\theta = \frac{5}{13}$, $\cos\theta = \frac{12}{13}$, $\tan\theta = \frac{5}{12}$; $\csc\theta = \frac{13}{5}$, $\sec\theta = \frac{13}{12}$, $\cot\theta = \frac{12}{5}$.

**10.** Using a right triangle with hypotenuse 17 and legs 15 (adjacent) and $\sqrt{17^2 - 15^2} = 8$ (opposite), we have $\sin\theta = \frac{8}{17}$, $\cos\theta = \frac{15}{17}$, $\tan\theta = \frac{8}{15}$; $\csc\theta = \frac{17}{8}$, $\sec\theta = \frac{17}{15}$, $\cot\theta = \frac{15}{8}$.

## Section 4.3 Exercises

**1.** The 450° angle lies on the positive–$y$ axis ($450° - 360° = 90°$), while the others are all coterminal in Quadrant II.

**2.** The $-\frac{5\pi}{3}$ angle lies in Quadrant I $\left(-\frac{5\pi}{3} + 2\pi = \frac{\pi}{3}\right)$, while the others are all coterminal in Quadrant IV.

In #3–12, recall that the distance from the origin is $r = \sqrt{x^2 + y^2}$.

**3.** $\sin\theta = \frac{2}{\sqrt{5}}$, $\cos\theta = -\frac{1}{\sqrt{5}}$, $\tan\theta = -2$; $\csc\theta = \frac{\sqrt{5}}{2}$, $\sec\theta = -\sqrt{5}$, $\cot\theta = -\frac{1}{2}$.

**4.** $\sin\theta = -\frac{3}{5}$, $\cos\theta = \frac{4}{5}$, $\tan\theta = -\frac{3}{4}$; $\csc\theta = -\frac{5}{3}$, $\sec\theta = \frac{5}{4}$, $\cot\theta = -\frac{4}{3}$.

**5.** $\sin\theta = -\frac{1}{\sqrt{2}}$, $\cos\theta = -\frac{1}{\sqrt{2}}$, $\tan\theta = 1$; $\csc\theta = -\sqrt{2}$, $\sec\theta = -\sqrt{2}$, $\cot\theta = 1$.

**6.** $\sin\theta = -\frac{5}{\sqrt{34}}$, $\cos\theta = \frac{3}{\sqrt{34}}$, $\tan\theta = -\frac{5}{3}$; $\csc\theta = -\frac{\sqrt{34}}{5}$, $\sec\theta = \frac{\sqrt{34}}{3}$, $\cot\theta = -\frac{3}{5}$.

**7.** $\sin\theta = \frac{4}{5}$, $\cos\theta = \frac{3}{5}$, $\tan\theta = \frac{4}{3}$; $\csc\theta = \frac{5}{4}$, $\sec\theta = \frac{5}{3}$, $\cot\theta = \frac{3}{4}$.

**8.** $\sin\theta = -\frac{3}{\sqrt{13}}$, $\cos\theta = -\frac{2}{\sqrt{13}}$, $\tan\theta = \frac{3}{2}$; $\csc\theta = -\frac{\sqrt{13}}{3}$, $\sec\theta = -\frac{\sqrt{13}}{2}$, $\cot\theta = \frac{2}{3}$.

**9.** $\sin\theta = 1$, $\cos\theta = 0$, $\tan\theta$ undefined; $\csc\theta = 1$, $\sec\theta$ undefined, $\cot\theta = 0$.

**10.** $\sin\theta = 0$, $\cos\theta = -1$, $\tan\theta = 0$; $\csc\theta$ undefined, $\sec\theta = -1$, $\cot\theta$ undefined.

**11.** $\sin\theta = -\frac{2}{\sqrt{29}}$, $\cos\theta = \frac{5}{\sqrt{29}}$, $\tan\theta = -\frac{2}{5}$; $\csc\theta = -\frac{\sqrt{29}}{2}$, $\sec\theta = \frac{\sqrt{29}}{5}$, $\cot\theta = -\frac{5}{2}$.

**12.** $\sin\theta = -\frac{1}{\sqrt{2}}$, $\cos\theta = \frac{1}{\sqrt{2}}$, $\tan\theta = -1$; $\csc\theta = -\sqrt{2}$, $\sec\theta = \sqrt{2}$, $\cot\theta = -1$.

For #13–16, determine the quadrant(s) of angles with the given measures, and then use the fact that $\sin t$ is positive when the terminal side of the angle is above the $x$-axis (in Quadrants I and II) and $\cos t$ is positive when the terminal side of the angle is to the right of the $y$-axis (in quadrants I and IV). Note that since $\tan t = \sin t/\cos t$, the sign of $\tan t$ can be determined from the signs of $\sin t$ and $\cos t$: if $\sin t$ and $\cos t$ have the same sign, the answer to (c) will be '+'; otherwise it will be '−'. Thus $\tan t$ is positive in Quadrants I and III.

**13.** These angles are in Quadrant I. **(a)** + (i.e., $\sin t > 0$). **(b)** + (i.e., $\cos t > 0$). **(c)** + (i.e., $\tan t > 0$).

**14.** These angles are in Quadrant II. **(a)** +. **(b)** −. **(c)** −.

**15.** These angles are in Quadrant III. **(a)** −. **(b)** −. **(c)** +.

**16.** These angles are in Quadrant IV. **(a)** −. **(b)** +. **(c)** −.

For #17–20, use strategies similar to those for the previous problem set.

**17.** 143° is in Quadrant II, so cos 143° is negative.

**18.** 192° is in Quadrant III, so tan 192° is positive.

**19.** $\frac{7\pi}{8}$ rad is in Quadrant II, so $\cos\frac{7\pi}{8}$ is negative.

**20.** $\frac{4\pi}{5}$ rad is in Quadrant II, so $\tan\frac{4\pi}{5}$ is negative.

**21.** A $(2, 2)$; $\tan 45° = \frac{y}{x} = 1 \Rightarrow y = x$.

**22.** B $(-1, \sqrt{3})$; $\tan\frac{2\pi}{3} = \frac{y}{x} = -\sqrt{3}$. $\frac{2\pi}{3}$ is in Quadrant II, so $x$ is negative.

**23.** C $(-\sqrt{3}, -1)$; $\frac{7\pi}{6}$ is in Quadrant III, so $x$ and $y$ are both negative. $\tan\frac{7\pi}{6} = \frac{1}{\sqrt{3}}$.

**24.** D $(1, -\sqrt{3})$; −60° is in Quadrant IV, so $x$ is positive while $y$ is negative. $\tan(-60°) = -\sqrt{3}$.

For #25–36, recall that the reference angle is the acute angle formed by the terminal side of the angle in standard position and the $x$-axis.

**25.** The reference angle is 60°. A right triangle with a 60° angle at the origin has the point $P(-1, \sqrt{3})$ as one vertex, with hypotenuse length $r = 2$, so $\cos 120° = \frac{x}{r} = -\frac{1}{2}$.

**26.** The reference angle is 60°. A right triangle with a 60° angle at the origin has the point $P(1, -\sqrt{3})$ as one vertex, so $\tan 300° = \frac{y}{x} = -\sqrt{3}$.

**27.** The reference angle is the given angle, $\frac{\pi}{3}$. A right triangle with a $\frac{\pi}{3}$ radian angle at the origin has the point $P(1, \sqrt{3})$ as one vertex, with hypotenuse length $r = 2$, so $\sec\frac{\pi}{3} = \frac{r}{x} = 2$.

**28.** The reference angle is $\frac{\pi}{4}$. A right triangle with a $\frac{\pi}{4}$ radian angle at the origin has the point $P(1, 1)$ as one vertex, with hypotenuse length $r = \sqrt{2}$, so $\csc\frac{3\pi}{4} = \frac{r}{y} = \sqrt{2}$.

**29.** The reference angle is $\frac{\pi}{6}$ (in fact, the given angle is coterminal with $\frac{\pi}{6}$). A right triangle with a $\frac{\pi}{6}$ radian angle at the origin has the point $P(\sqrt{3}, 1)$ as one vertex, with hypotenuse length $r = 2$, so $\sin\frac{13\pi}{6} = \frac{y}{r} = \frac{1}{2}$.

**30.** The reference angle is $\frac{\pi}{3}$ (in fact, the given angle is coterminal with $\frac{\pi}{3}$). A right triangle with a $\frac{\pi}{3}$ radian angle at the origin has the point $P(1, \sqrt{3})$ as one vertex, with hypotenuse length $r = 2$, so $\cos\frac{7\pi}{3} = \frac{x}{r} = \frac{1}{2}$.

**31.** The reference angle is $\frac{\pi}{4}$ (in fact, the given angle is coterminal with $\frac{\pi}{4}$). A right triangle with a $\frac{\pi}{4}$ radian angle at the origin has the point $P(1, 1)$ as one vertex, so $\tan\frac{-15\pi}{4} = \frac{y}{x} = 1$.

**32.** The reference angle is $\frac{\pi}{4}$. A right triangle with a $\frac{\pi}{4}$ radian angle at the origin has the point $P(-1, -1)$ as one vertex, so $\cot\frac{13\pi}{4} = \frac{x}{y} = 1$.

**33.** $\cos\frac{23\pi}{6} = \cos\frac{11\pi}{6} = \frac{\sqrt{3}}{2}$

**34.** $\cos\frac{17\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

**35.** $\sin\frac{11\pi}{3} = \sin\frac{5\pi}{3} = -\frac{\sqrt{3}}{2}$

**36.** $\cot\frac{19\pi}{6} = \cot\frac{7\pi}{6} = \sqrt{3}$

**37.** $-450°$ is coterminal with $270°$, on the negative $y$-axis. **(a)** $-1$ **(b)** 0 **(c)** Undefined

**38.** $-270°$ is coterminal with $90°$, on the positive $y$-axis. **(a)** 1 **(b)** 0 **(c)** Undefined

**39.** $7\pi$ radians is coterminal with $\pi$ radians, on the negative $x$-axis. **(a)** 0 **(b)** $-1$ **(c)** 0

**40.** $\frac{11\pi}{2}$ radians is coterminal with $\frac{3\pi}{2}$ radians, on the negative $y$-axis. **(a)** $-1$ **(b)** 0 **(c)** Undefined

**41.** $\frac{-7\pi}{2}$ radians is coterminal with $\frac{\pi}{2}$ radians, on the positive $y$-axis. **(a)** 1 **(b)** 0 **(c)** Undefined

**42.** $-4\pi$ radians is coterminal with 0 radians, on the positive $x$-axis. **(a)** 0 **(b)** 1 **(c)** 0

**43.** Since $\cot\theta > 0$, $\sin\theta$ and $\cos\theta$ have the same sign, so $\sin\theta = +\sqrt{1 - \cos^2\theta} = \frac{\sqrt{5}}{3}$, and $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{5}}{2}$.

**44.** Since $\tan\theta < 0$, $\sin\theta$ and $\cos\theta$ have opposite signs, so $\cos\theta = -\sqrt{1 - \sin^2\theta} = -\frac{\sqrt{15}}{4}$, and $\cot\theta = \frac{\cos\theta}{\sin\theta} = -\sqrt{15}$.

**45.** $\cos\theta = +\sqrt{1 - \sin^2\theta} = \frac{\sqrt{21}}{5}$, so $\tan\theta = \frac{\sin\theta}{\cos\theta} = -\frac{2}{\sqrt{21}}$ and $\sec\theta = \frac{1}{\cos\theta} = \frac{5}{\sqrt{21}}$.

**46.** $\sec\theta$ has the same sign as $\cos\theta$, and since $\cot\theta > 0$, $\sin\theta$ must also be negative. With $x = -3$, $y = -7$, and $r = \sqrt{3^2 + 7^2} = \sqrt{58}$, we have $\sin\theta = -\frac{7}{\sqrt{58}}$ and $\cos\theta = -\frac{3}{\sqrt{58}}$.

**47.** Since $\cos\theta < 0$ and $\cot\theta < 0$, $\sin\theta$ must be positive. With $x = -4$, $y = 3$, and $r = \sqrt{4^2 + 3^2} = 5$, we have $\sec\theta = -\frac{5}{4}$ and $\csc\theta = \frac{5}{3}$.

**48.** Since $\sin\theta > 0$ and $\tan\theta < 0$, $\cos\theta$ must be negative. With $x = -3$, $y = 4$, and $r = \sqrt{4^2 + 3^2} = 5$, we have $\csc\theta = \frac{5}{4}$ and $\cot\theta = -\frac{3}{4}$.

**49.** $\sin\left(\frac{\pi}{6} + 49{,}000\pi\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

**50.** $\tan(1{,}234{,}567\pi) - \tan(7{,}654{,}321\pi) = \tan(\pi) - \tan(\pi) = 0$

**51.** $\cos\left(\frac{5{,}555{,}555\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$

**52.** $\tan\left(\frac{3\pi - 70{,}000\pi}{2}\right) = \tan\left(\frac{3\pi}{2}\right) =$ undefined.

**53.** The calculator's value of the irrational number $\pi$ is necessarily an approximation. When multiplied by a very large number, the slight error of the original approximation is magnified sufficiently to throw the trigonometric functions off.

**54.** $\sin t$ is the $y$-coordinate of the point on the unit circle after measuring counterclockwise $t$ units from $(1, 0)$. This will repeat every $2\pi$ units (and not before), since the distance around the circle is $2\pi$.

**55.** $\mu = \frac{\sin 83°}{\sin 36°} \approx 1.69$

**56.** $\sin\theta_2 = \frac{\sin 42°}{1.52} \approx 0.44$

**57.** **(a)** When $t = 0$, $d = 0.4$ in.

**(b)** When $t = 3$, $d = 0.4e^{-0.6}\cos 12 \approx 0.1852$ in.

**58.** When $t = 0, \theta = 0.25$ (rad). When $t = 2.5$, $\theta = 0.25 \cos 2.5 \approx -0.2003$ rad.

**59.** The difference in the elevations is 600 ft, so $d = 600/\sin \theta$. Then:

**(a)** $d = 600\sqrt{2} \approx 848.53$ ft.

**(b)** $d = 600$ ft.

**(c)** $d \approx 933.43$ ft.

**60.** January ($t = 1$): $72.4 + 61.7 \sin \frac{\pi}{6} = 103.25$.

April ($t = 4$): $72.4 + 61.7 \sin \frac{2\pi}{3} \approx 125.83$.

June ($t = 6$): $72.4 + 61.7 \sin \pi = 72.4$.

October ($t = 10$): $72.4 + 61.7 \sin \frac{5\pi}{3} \approx 18.97$.

December ($t = 12$): $72.4 + 61.7 \sin 2\pi = 72.4$. June and December are the same; perhaps by June most people have suits for the summer, and by December they are beginning to purchase them for next summer (or as Christmas presents, or for mid-winter vacations).

**61.** True. Any angle in a triangle measures between 0° and 180°. Acute angles (<90°) determine reference triangles in Quadrant I, where the cosine is positive, while obtuse angles (>90°) determine reference triangles in Quadrant II, where the cosine is negative.

**62.** True. The point determines a reference triangle in Quadrant IV, with $r = \sqrt{8^2 + (-6)^2} = 10$. Thus $\sin \theta = y/x = -6/10 = -0.6$

**63.** If $\sin \theta = 0.4$, then $\sin(-\theta) + \csc \theta = -\sin \theta + \frac{1}{\sin \theta}$ $= -0.4 + \frac{1}{0.4} = 2.1$. The answer is E.

**64.** If $\cos \theta = 0.4$, then $\cos(\theta + \pi) = -\cos \theta = -0.4$. The answer is B.

**65.** $(\sin t)^2 + (\cos t)^2 = 1$ for all $t$. The answer is A.

**66.** $\sin \theta = -\sqrt{1 - \cos^2 \theta}$, because $\tan \theta$ $= (\sin \theta)/(\cos \theta) > 0$. So $\sin \theta = -\sqrt{1 - \frac{25}{169}} = -\frac{12}{13}$. The answer is A.

**67.** Since $\sin \theta > 0$ and $\tan \theta < 0$, the terminal side must be in Quadrant II, so $\theta = \frac{5\pi}{6}$.

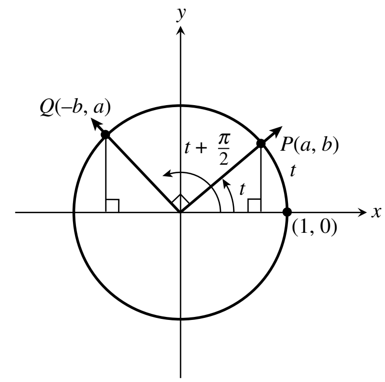
**68.** Since $\cos \theta > 0$ and $\sin \theta < 0$, the terminal side must be in Quadrant IV, so $\theta = \frac{11\pi}{6}$.

**69.** Since $\tan \theta < 0$ and $\sin \theta < 0$, the terminal side must be in Quadrant IV, so $\theta = \frac{7\pi}{4}$.

**70.** Since $\sin \theta < 0$ and $\tan \theta > 0$, the terminal side must be in Quadrant III, so $\theta = \frac{5\pi}{4}$.

**71.** The two triangles are congruent: both have hypotenuse 1, and the corresponding angles are congruent—the smaller acute angle has measure $t$ in both triangles, and the two acute angles in a right triangle add up to $\pi/2$.

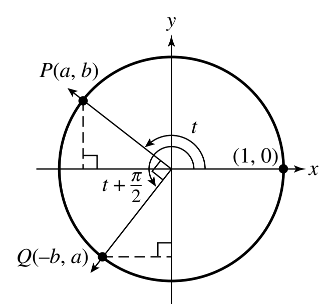
**72.** These coordinates give the lengths of the legs of the triangles from #71, and these triangles are congruent. For example, the length of the horizontal leg of the triangle with vertex $P$ is given by the (absolute value of the) $x$-coordinate of $P$; this must be the same as the (absolute value of the) $y$-coordinate of $Q$.




**73.** One possible answer: Starting from the point $(a, b)$ on the unit circle—at an angle of $t$, so that $\cos t = a$—then measuring a quarter of the way around the circle (which corresponds to adding $\pi/2$ to the angle), we end at $(-b, a)$, so that $\sin(t + \pi/2) = a$. For $(a, b)$ in Quadrant I, this is shown in the figure above; similar illustrations can be drawn for the other quadrants.

**74.** One possible answer: Starting from the point $(a, b)$ on the unit circle—at an angle of $t$, so that $\sin t = b$—then measuring a quarter of the way around the circle (which corresponds to adding $\pi/2$ to the angle), we end at $(-b, a)$, so that $\cos(t + \pi/2) = -b = -\sin t$. For $(a, b)$ in Quadrant I, this is shown in the figure above; similar illustrations can be drawn for the other quadrants.

**75.** Starting from the point $(a, b)$ on the unit circle—at an angle of $t$, so that $\cos t = a$—then measuring a quarter of the way around the circle (which corresponds to adding $\pi/2$ to the angle), we end at $(-b, a)$, so that $\sin(t + \pi/2) = a$. This holds true when $(a, b)$ is in Quadrant II, just as it did for Quadrant I.




**76.** **(a)** Both triangles are right triangles with hypotenuse 1, and the angles at the origin are both $t$ (for the triangle on the left, the angle is the supplement of $\pi - t$). Therefore the vertical legs are also congruent; their lengths correspond to the sines of $t$ and $\pi - t$.

**(b)** The points $P$ and $Q$ are reflections of each other across the $y$-axis, so they are the same distance (but opposite directions) from the $y$-axis. Alternatively, use the congruent triangles argument from part (a).

**77.** Seven decimal places are shown so that the slight differences can be seen. The magnitude of the relative error is less than 1% when $|\theta| < 0.2441$ (approximately). This can be seen by extending the table to larger values of $\theta$, or by graphing $\left|\frac{\sin\theta - \theta}{\sin\theta}\right| - 0.01$.

**78.** Let $(x, y)$ be the coordinates of the point that corresponds to $t$ under the wrapping. Then

$$1 + (\tan t)^2 = 1 + \left(\frac{y}{x}\right)^2 = \frac{x^2 + y^2}{x^2} = \frac{1}{x^2} = (\sec t)^2.$$

(Note that $x^2 + y^2 = 1$ because $(x, y)$ is on the unit circle.)

**79.** This Taylor polynomial is generally a very good approximation for $\sin\theta$—in fact, the relative error (see #77) is less than 1% for $|\theta| < 1$ (approx.). It is better for $\theta$ close to 0; it is slightly larger than $\sin\theta$ when $\theta < 0$ and slightly smaller when $\theta > 0$.

**80.** This Taylor polynomial is generally a very good approximation for $\cos\theta$—in fact, the relative error (see #77) is less than 1% for $|\theta| < 1.2$ (approx.). It is better for $\theta$ close to 0; it is slightly larger than $\cos\theta$ when $\theta \neq 0$.

**77.**

| $\theta$ | $\sin\theta$ | $\sin\theta - \theta$ | $\left\lvert\frac{\sin\theta - \theta}{\sin\theta}\right\rvert$ |
|---|---|---|---|
| −0.03 | −0.0299955 | 0.0000045 | 0.0001500 |
| −0.02 | −0.0199987 | 0.0000013 | 0.0000667 |
| −0.01 | −0.0099998 | 0.0000002 | 0.0000167 |
| 0 | 0 | 0 | — |
| 0.01 | 0.0099998 | −0.0000002 | 0.0000167 |
| 0.02 | 0.0199987 | −0.0000013 | 0.0000667 |
| 0.03 | 0.0299955 | −0.0000045 | 0.0001500 |

**79.**

| $\theta$ | $\sin\theta$ | $\theta - \frac{1}{6}\theta^3$ | $\sin\theta - \left(\theta - \frac{1}{6}\theta^3\right)$ |
|---|---|---|---|
| −0.3 | −0.2955202 | −0.2955000 | −0.0000202 |
| −0.2 | −0.1986693 | −0.1986667 | −0.0000027 |
| −0.1 | −0.0998334 | −0.0998333 | −0.0000001 |
| 0 | 0 | 0 | 0 |
| 0.1 | 0.0998334 | 0.0998333 | 0.0000001 |
| 0.2 | 0.1986693 | 0.1986667 | 0.0000027 |
| 0.3 | 0.2955202 | 0.2955000 | 0.0000202 |

**80.**

| $\theta$ | $\cos\theta$ | $1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4$ | $\cos\theta - (1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4)$ |
|---|---|---|---|
| −0.3 | 0.9553365 | 0.9553375 | −0.0000010 |
| −0.2 | 0.9800666 | 0.9800667 | −0.0000001 |
| −0.1 | 0.9950042 | 0.9950042 | −0.0000000 |
| 0 | 1 | 1 | 0 |
| 0.1 | 0.9950042 | 0.9950042 | −0.0000000 |
| 0.2 | 0.9800666 | 0.9800667 | −0.0000001 |
| 0.3 | 0.9553365 | 0.9553375 | −0.0000010 |

## ■ Section 4.4 Graphs of Sine and Cosine: Sinusoids

### Exploration 1

1. $\pi/2$ (at the point $(0, 1)$)
2. $3\pi/2$ (at the point $(0, -1)$)
3. Both graphs cross the $x$-axis when the $y$-coordinate on the unit circle is 0.
4. (Calculator exploration)
5. The sine function tracks the $y$-coordinate of the point as it moves around the unit circle. After the point has gone completely around the unit circle (a distance of $2\pi$), the same pattern of $y$-coordinates starts over again.
6. Leave all the settings as they are shown at the start of the Exploration, except change $Y_{2T}$ to cos(T).

### Quick Review 4.4

1. In order: +, +, −, −
2. In order: +, −, −, +
3. In order: +, −, +, −
4. $135° \cdot \frac{\pi}{180°} = \frac{3\pi}{4}$
5. $-150° \cdot \frac{\pi}{180°} = -\frac{5\pi}{6}$
6. $450° \cdot \frac{\pi}{180°} = \frac{5\pi}{2}$
7. Starting with the graph of $y_1$, vertically stretch by 3 to obtain the graph of $y_2$.
8. Starting with the graph of $y_1$, reflect across $y$-axis to obtain the graph of $y_2$.
9. Starting with the graph of $y_1$, vertically shrink by 0.5 to obtain the graph of $y_2$.

**10.** Starting with the graph of $y_1$, translate down 2 units to obtain the graph of $y_2$.

### Section 4.4 Exercises

In #1–6, for $y = a \sin x$, the amplitude is $|a|$. If $|a| > 1$, there is a vertical stretch by a factor of $|a|$, and if $|a| < 1$, there is a vertical shrink by a factor of $|a|$. When $a < 0$, there is also a reflection across the $x$-axis.

**1.** Amplitude 2; vertical stretch by a factor of 2.

**2.** Amplitude $2/3$; vertical shrink by a factor of $2/3$.

**3.** Amplitude 4; vertical stretch by a factor of 4, reflection across the $x$-axis.

**4.** Amplitude $7/4$; vertical stretch by a factor of $7/4$, reflection across the $x$-axis.

**5.** Amplitude 0.73; vertical shrink by a factor of 0.73.

**6.** Amplitude 2.34; vertical stretch by a factor of 2.34, reflection across the $x$-axis.

In #7–12, for $y = \cos bx$, the period is $2\pi/|b|$. If $|b| > 1$, there is a horizontal shrink by a factor of $1/|b|$, and if $|b| < 1$, there is a horizontal stretch by a factor of $1/|b|$. When $b < 0$, there is also a reflection across the $y$-axis. For $y = a \cos bx$, $a$ has the same effects as in #1–6.

**7.** Period $2\pi/3$; horizontal shrink by a factor of $1/3$.

**8.** Period $2\pi/(1/5) = 10\pi$; horizontal stretch by a factor of $1/(1/5) = 5$.

**9.** Period $2\pi/7$; horizontal shrink by a factor of $1/7$, reflection across the $y$-axis.

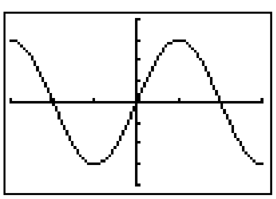
**10.** Period $2\pi/0.4 = 5\pi$; horizontal stretch by a factor of $1/0.4 = 2.5$, reflection across the $y$-axis.

**11.** Period $2\pi/2 = \pi$; horizontal shrink by a factor of $1/2$. Also a vertical stretch by a factor of 3.

**12.** Period $2\pi/(2/3) = 3\pi$; horizontal stretch by a factor of $1/(2/3) = 3/2$. Also a vertical shrink by a factor of $1/4$.

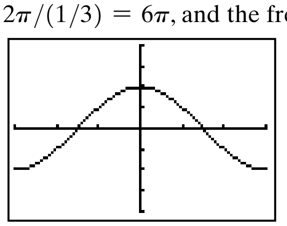
In #13–16, the amplitudes of the graphs for $y = a \sin bx$ and $y = a \cos bx$ are governed by $a$, while the period is governed by $b$, just as in #1–12. The frequency is 1/period.

**13.** For $y = 3 \sin(x/2)$, the amplitude is 3, the period is $2\pi/(1/2) = 4\pi$, and the frequency is $1/(4\pi)$.



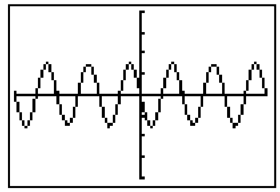
$[-3\pi, 3\pi]$ by $[-4, 4]$

**14.** For $y = 2 \cos(x/3)$, the amplitude is 2, the period is $2\pi/(1/3) = 6\pi$, and the frequency is $1/(6\pi)$.



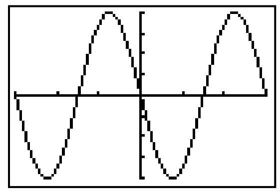
$[-3\pi, 3\pi]$ by $[-4, 4]$

**15.** For $y = -(3/2) \sin 2x$, the amplitude is $3/2$, the period is $2\pi/2 = \pi$, and the frequency is $1/\pi$.



$[-3\pi, 3\pi]$ by $[-4, 4]$

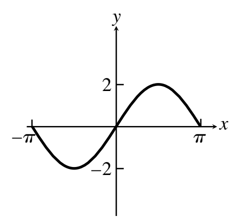
**16.** For $y = -4 \sin(2x/3)$, the amplitude is 4, the period is $2\pi/(2/3) = 3\pi$, and the frequency is $1/(3\pi)$.



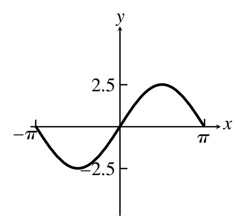
$[-3\pi, 3\pi]$ by $[-4, 4]$

Note: the frequency for each graph in #17–22 is $1/(2\pi)$.

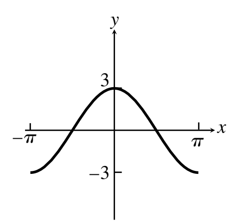
**17.** Period $2\pi$, amplitude $= 2$



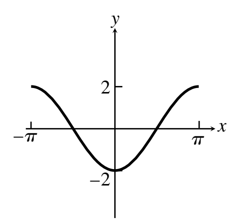

**18.** Period $2\pi$, amplitude $= 2.5$



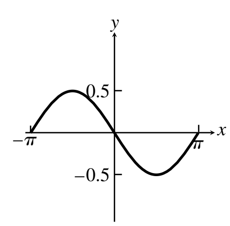

**19.** Period $2\pi$, amplitude $= 3$



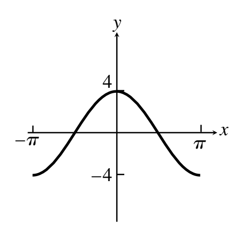

**20.** Period $2\pi$, amplitude $= 2$



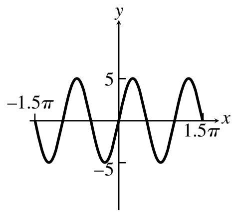

**21.** Period $2\pi$, amplitude $= 0.5$



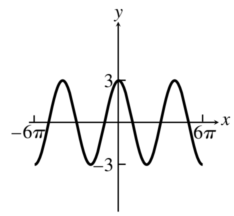

**22.** Period $2\pi$, amplitude $= 4$



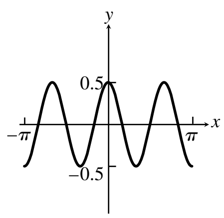
**23.** Period $\pi$, amplitude $= 5$, frequency $= 1/\pi$



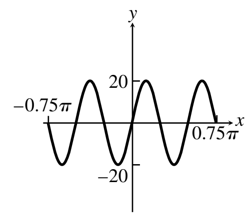

**24.** Period $4\pi$, amplitude $= 3$, frequency $= 1/(4\pi)$



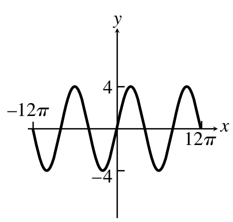

**25.** Period $2\pi/3$, amplitude $= 0.5$, frequency $= 3/2\pi$



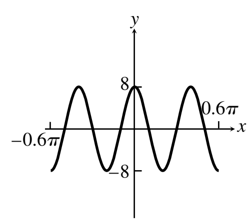

**26.** Period $\pi/2$, amplitude $= 20$, frequency $= 2/\pi$




**27.** Period $8\pi$, amplitude $= 4$, frequency $= 1/(8\pi)$




**28.** Period $2\pi/5$, amplitude $= 8$, frequency $= 5/(2\pi)$




**29.** Period $\pi$; amplitude 1.5; $[-2\pi, 2\pi]$ by $[-2, 2]$

**30.** Period $2\pi/3$; amplitude 2; $\left[-\frac{2\pi}{3}, \frac{2\pi}{3}\right]$ by $[-4, 4]$

**31.** Period $\pi$; amplitude 3; $[-2\pi, 2\pi]$ by $[-4, 4]$

**32.** Period $4\pi$; amplitude 5; $[-4\pi, 4\pi]$ by $[-10, 10]$

**33.** Period 6; amplitude 4; $[-3, 3]$ by $[-5, 5]$

**34.** Period 2; amplitude 3; $[-4, 4]$ by $[-5, 5]$

**35.** Maximum: $2\left(\text{at } -\frac{3\pi}{2} \text{ and } \frac{\pi}{2}\right)$;
minimum: $-2\left(\text{at } -\frac{\pi}{2} \text{ and } \frac{3\pi}{2}\right)$.
Zeros: $0, \pm\pi, \pm 2\pi$.

**36.** Maximum: 3 (at 0); minimum: $-3$ (at $\pm 2\pi$). Zeros: $\pm\pi$.

**37.** Maximum: 1 (at $0, \pm\pi, \pm 2\pi$); minimum: $-1\left(\text{at } \pm\frac{\pi}{2} \text{ and } \pm\frac{3\pi}{2}\right)$. Zeros: $\pm\frac{\pi}{4}, \pm\frac{3\pi}{4}, \pm\frac{5\pi}{4}, \pm\frac{7\pi}{4}$.

**38.** Maximum: $\frac{1}{2}\left(\text{at } -\frac{3\pi}{2} \text{ and } \frac{\pi}{2}\right)$;
minimum: $-\frac{1}{2}\left(\text{at } -\frac{\pi}{2} \text{ and } \frac{3\pi}{2}\right)$. Zeros: $0, \pm\pi, \pm 2\pi$.

**39.** Maximum: $1\left(\text{at } \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}\right)$; minimum: $-1$ (at $0, \pm\pi, \pm 2\pi$).
Zeros: $\pm\frac{\pi}{4}, \pm\frac{3\pi}{4}, \pm\frac{5\pi}{4}, \pm\frac{7\pi}{4}$.

**40.** Maximum: $2\left(\text{at } -\frac{\pi}{2}, \frac{3\pi}{2}\right)$; minimum: $-2\left(\text{at } -\frac{3\pi}{2}, \frac{\pi}{2}\right)$.
Zeros: $0, \pm\pi, \pm 2\pi$.

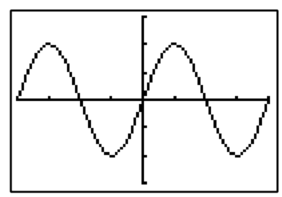
**41.** $y = \sin x$ has to be translated left or right by an odd multiple of $\pi$. One possibility is $y = \sin(x + \pi)$.

**42.** $y = \sin x$ has to be translated right by $\frac{\pi}{2}$ plus an even multiple of $\pi$. One possibility is $y = \sin(x - \pi/2)$.

**43.** Starting from $y = \sin x$, horizontally shrink by $\frac{1}{3}$ and vertically shrink by 0.5. The period is $2\pi/3$.
Possible window: $\left[-\frac{2\pi}{3}, \frac{2\pi}{3}\right]$ by $\left[-\frac{3}{4}, \frac{3}{4}\right]$.

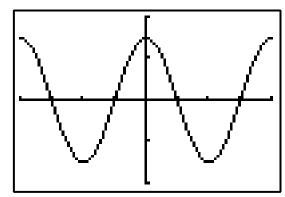
**44.** Starting from $y = \cos x$, horizontally shrink by $\frac{1}{4}$ and vertically stretch by 1.5. The period is $\pi/2$.

Possible window: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ by $[-2, 2]$.



$\left[\frac{-2\pi}{3}, \frac{2\pi}{3}\right]$ by $[-0.75, 0.75]$

**For #43**

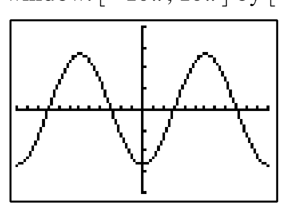


$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ by $[-0.75, 0.75]$

**For #44**

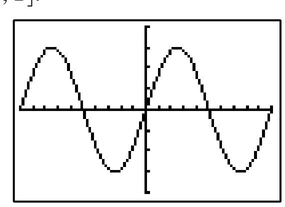
**45.** Starting from $y = \cos x$, horizontally stretch by 3, vertically shrink by $\frac{2}{3}$, reflect across $x$-axis. The period is $6\pi$. Possible window: $[-6\pi, 6\pi]$ by $[-1, 1]$.

**46.** Starting from $y = \sin x$, horizontally stretch by 5 and vertically shrink by $\frac{3}{4}$. The period is $10\pi$. Possible window: $[-10\pi, 10\pi]$ by $[-1, 1]$.



$[-6\pi, 6\pi]$ by $[-1, 1]$

**For #45**

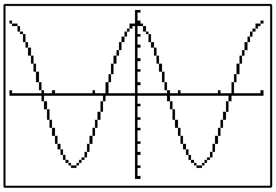


$[-10\pi, 10\pi]$ by $[-1, 1]$

**For #46**

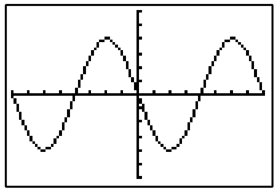
**47.** Starting from $y = \cos x$, horizontally shrink by $\frac{3}{2\pi}$ and vertically stretch by 3. The period is 3. Possible window: $[-3, 3]$ by $[-3.5, 3.5]$.

**48.** Starting from $y = \sin x$, horizontally stretch by $\frac{4}{\pi}$, vertically stretch by 2, and reflect across $x$-axis. The period is 8. Possible window: $[-8, 8]$ by $[-3, 3]$.



$[-3, 3]$ by $[-3.5, 3.5]$

**For #47**



$[-8, 8]$ by $[-3, 3]$

**For #48**

**49.** Starting with $y_1$, vertically stretch by $\frac{5}{3}$.

**50.** Starting with $y_1$, translate right $\frac{\pi}{12}$ units and vertically shrink by $\frac{1}{2}$.

**51.** Starting with $y_1$, horizontally shrink by $\frac{1}{2}$.

**52.** Starting with $y_1$, horizontally stretch by 2 and vertically shrink by $\frac{2}{3}$.

For #53–56, graph the functions or use facts about sine and cosine learned to this point.

**53.** (a) and (b)

**54.** (a) and (b)

**55.** (a) and (b) — both functions equal $\cos x$

**56.** (a) and (c) — $\sin\left(2x + \frac{\pi}{4}\right)$

$$= \sin\left[\left(2x - \frac{\pi}{4}\right) + \frac{\pi}{2}\right] = \cos\left(2x - \frac{\pi}{4}\right)$$

In #57–60, for $y = a \sin(b(x - h))$, the amplitude is $|a|$, the period is $2\pi/|b|$, and the phase shift is $h$.

**57.** One possibility is $y = 3 \sin 2x$.

**58.** One possibility is $y = 2 \sin(2x/3)$.

**59.** One possibility is $y = 1.5 \sin 12(x - 1)$.

**60.** One possibility is $y = 3.2 \sin 14(x - 5)$

**61.** Amplitude 2, period $2\pi$, phase shift $\frac{\pi}{4}$, vertical translation 1 unit up.

**62.** Rewrite as $y = -3.5 \sin\left[2\left(x - \frac{\pi}{4}\right)\right] - 1$.

Amplitude 3.5, period $\pi$, phase shift $\frac{\pi}{4}$, vertical translation 1 unit down.

**63.** Rewrite as $y = 5 \cos\left[3\left(x - \frac{\pi}{18}\right)\right] + 0.5$.

Amplitude 5, period $\frac{2\pi}{3}$, phase shift $\frac{\pi}{18}$, vertical translation $\frac{1}{2}$ units up.

**64.** Amplitude 3, period $2\pi$, phase shift $-3$, vertical translation 2 units down.

**65.** Amplitude 2, period 1, phase shift 0, vertical translation 1 unit up.

**66.** Amplitude 4, period $\frac{2}{3}$, phase shift 0, vertical translation 2 units down.

**67.** Amplitude $\frac{7}{3}$, period $2\pi$, phase shift $-\frac{5}{2}$, vertical translation 1 unit down.

**68.** Amplitude $\frac{2}{3}$, period $8\pi$, phase shift 3, vertical translation 1 unit up.

**69.** $y = 2 \sin 2x$ $(a = 2, b = 2, h = 0, k = 0)$.

**70.** $y = 3 \sin[2(x + 0.5)]$ $(a = 3, b = 2, h = 0.5, k = 0)$.

**71.** **(a)** There are two points of intersection in that interval.

**(b)** The coordinates are $(0, 1)$ and $(2\pi, 1.3^{-2\pi}) \approx (6.28, 0.19)$. In general, two functions intersect where $\cos x = 1$, i.e., $x = 2n\pi$, $n$ an integer.

**72.** $a = 4$ and $b = \frac{2\pi}{3.5} = \frac{4\pi}{7}$.

**73.** The height of the rider is modeled by $h = 30 - 25 \cos\left(\frac{2\pi}{40}t\right)$, where $t = 0$ corresponds to the time when the rider is at the low point. $h = 50$ when $\frac{-4}{5} = \cos\left(\frac{2\pi}{40}t\right)$. Then $\frac{2\pi}{40}t \approx 2.498$, so $t \approx 15.90$ sec.

**74.** The length $L$ must be the distance traveled in 30 min by an object traveling at 540 ft/sec:

$$L = 1800 \text{ sec} \cdot 540 \frac{\text{ft}}{\text{sec}} = 972{,}000 \text{ ft, or about 184 miles.}$$

**75.** **(a)** A model of the depth of the tide is $d = 2 \cos\left[\frac{\pi}{6.2}(t - 7.2)\right] + 9$, where $t$ is hours since midnight. The first low tide is at 1:00 A.M. $(t = 1)$.

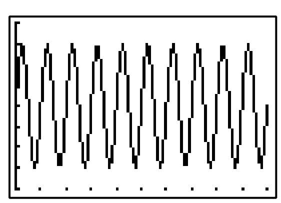
**(b)** At 4:00 A.M. $(t = 4)$: about 8.90 ft. At 9:00 P.M. $(t = 21)$: about 10.52 ft.

**(c)** 4:06 A.M. ($t = 4.1$ — halfway between 1:00 A.M. and 7:12 A.M.).

**76.** **(a)** 1 second.

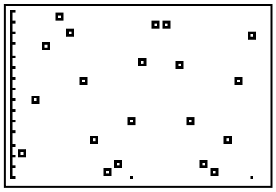
**(b)** Each peak corresponds to a heartbeat —there are 60 per minute.

**(c)**



$[0, 10]$ by $[80, 160]$

**77. (a)** The maximum $d$ is approximately 21.4. The amplitude is $(21.4 - 7.2)/2 = 7.1$.
Scatterplot:

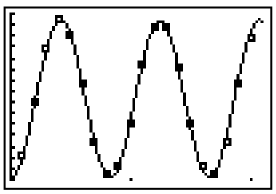


[0, 2.1] by [7, 22]

**(b)** The period appears to be slightly greater than 0.8, say 0.83.

**(c)** Since the function has a minimum at $t = 0$, we use an inverted cosine model:
$d(t) = -7.1 \cos(2\pi t/0.83) + 14.3$.

**(d)**



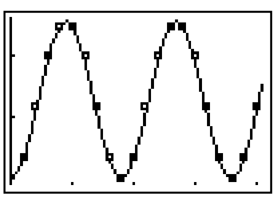
[0, 2.1] by [7, 22]

**78. (a)** The amplitude is 12.7, half the diameter of the turntable.

**(b)** The period is 1.8, as can be seen by measuring from minimum to minimum.

**(c)** Since the function has a minimum at $t = 0$, we use an inverted cosine model:
$d(t) = -12.7 \cos(2\pi t/1.8) + 72.7$.

**(d)**



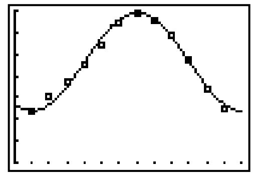
[0, 4.1] by [59, 86]

**79.** One possible answer is $T = 21.5 \cos\left(\frac{\pi}{6}(x - 7)\right) + 57.5$.

Start with the general form sinusoidal function $y = a\cos(b(x - h)) + k$, and find the variables $a, b, h$, and $k$ as follows:

The amplitude is $|a| = \frac{79 - 36}{2} = 21.5$. We can arbitrarily choose to use the positive value, so $a = 21.5$.

The period is 12 months. $12 = \frac{2\pi}{|b|} \Rightarrow |b| = \frac{2\pi}{12} = \frac{\pi}{6}$.
Again, we can arbitrarily choose to use the positive value, so $b = \frac{\pi}{6}$.
The maximum is at month 7, so the phase shift $h = 7$.
The vertical shift $k = \frac{79 + 36}{2} = 57.5$.



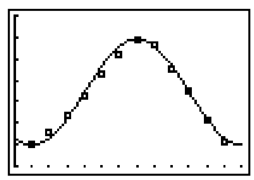
[0, 13] by [10, 80]

**80.** One possible answer is $y = 24 \cos\left(\frac{\pi}{6}(x - 7)\right) + 44$.

Start with the general form sinusoidal function $y = a\cos(b(x - h)) + k$, and find the variables $a, b, h$, and $k$ as follows:

The amplitude is $|a| = \frac{68 - 20}{2} = 24$. We can arbitrarily choose to use the positive value, so $a = 24$.

The period is 12 months. $12 = \frac{2\pi}{|b|} \Rightarrow |b| = \frac{2\pi}{12} = \frac{\pi}{6}$.
Again, we can arbitrarily choose to use the positive value, so $b = \frac{\pi}{6}$.
The maximum is at month 7, so the phase shift $h = 7$.
The vertical shift $k = \frac{68 + 20}{2} = 44$.



[0, 13] by [10, 80]

**81.** False. Since $y = \sin 2x$ is a horizontal *stretch* of $y = \sin 4x$ by a factor of 2, $y = \sin 2x$ has twice the period, not half. Remember, the period of $y = \sin bx$ is $2\pi/|b|$.

**82.** True. Any cosine curve can be converted to a sine curve of the same amplitude and frequency by a phase shift, which can be accomplished by an appropriate choice of $C$ (a multiple of $\pi/2$).

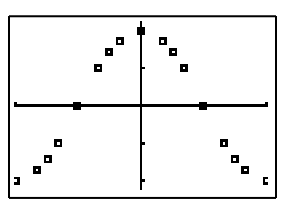
**83.** The minimum and maximum values differ by twice the amplitude. The answer is D.

**84.** Because the graph passes through $(6, 0)$, $f(6) = 0$. And 6 plus exactly two periods equals 96, so $f(96) = 0$ also. But $f(0)$ depends on phase and amplitude, which are unknown. The answer is D.

**85.** For $f(x) = a \sin(bx + c)$, the period is $2\pi/|b|$, which here equals $2\pi/420 = \pi/210$. The answer is C.

**86.** There are 2 solutions per cycle, and 2000 cycles in the interval. The answer is C.

**87. (a)**

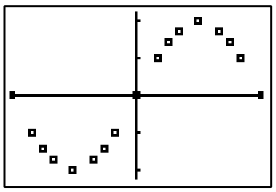


$[-\pi, \pi]$ by $[-1.1, 1.1]$

**(b)** $\cos x \approx 0.0246x^4 + 0x^3 - 0.4410x^2 + 0x + 0.9703$. The coefficients given as "0" here may show up as very small numbers (e.g., $1.44 \times 10^{-14}$) on some calculators. Note that $\cos x$ is an even function, and only the even powers of $x$ have nonzero (or a least "non-small") coefficients.

**(c)** The Taylor polynomial is $1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 = 1 - 0.5000x^2 + 0.04167x^4$; the coefficients are fairly similar.

**88. (a)**



$[-\pi, \pi]$ by $[-1.1, 1.1]$

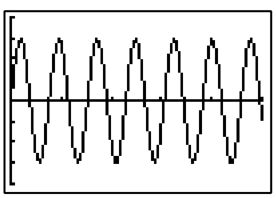
**(b)** $\sin x \approx -0.0872x^3 + 0x^2 + 0.8263x + 0$. The coefficients given as "0" here may show up as very small numbers (e.g., $3.56 \times 10^{-15}$) on some calculators. Note that $\sin x$ is an odd function, and only the odd powers of $x$ have nonzero (or a least "non-small") coefficients.

**(c)** The Taylor polynomial is $x - \frac{1}{6}x^3 = x - 0.16667x^3$; the coefficients are somewhat similar.

**89. (a)** $p = \frac{2\pi}{524\pi} = \frac{1}{262}$ sec

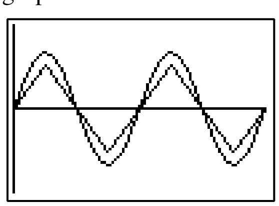
**(b)** $f = 262 \frac{1}{\text{sec}}$ ("cycles per sec"), or 262 Hertz (Hz).

**(c)**



$[0, 0.025]$ by $[-2, 2]$

**90.** Since the cursor moves at a constant rate, its distance from the center must be made up of linear pieces as shown (the slope of the line is the rate of motion). A graph of a sinusoid is included for comparison.



$[0, 2\pi]$ by $[-1.5, 1.5]$

**91. (a)** $a - b$ must equal 1.

**(b)** $a - b$ must equal 2.

**(c)** $a - b$ must equal $k$.

**92. (a)** $a - b$ must equal 1.

**(b)** $a - b$ must equal 2.

**(c)** $a - b$ must equal $k$.

For #93–96, note that $A$ and $C$ are one period apart. Meanwhile, $B$ is located one-fourth of a period to the right of $A$, and the $y$-coordinate of $B$ is the amplitude of the sinusoid.

**93.** The period of this function is $\pi$ and the amplitude is 3. $B$ and $C$ are located (respectively) $\frac{\pi}{4}$ units and $\pi$ units to the right of $A$. Therefore, $B = (0, 3)$ and $C = \left(\frac{3\pi}{4}, 0\right)$.

**94.** The period of this function is $2\pi$ and the amplitude is 4.5. $B$ and $C$ are located (respectively) $\frac{\pi}{2}$ units and $2\pi$ units to the right of $A$. Therefore, $B = \left(\frac{3\pi}{4}, 4.5\right)$ and $C = \left(\frac{9\pi}{4}, 0\right)$

**95.** The period of this function is $\frac{2\pi}{3}$ and the amplitude is 2. $B$ and $C$ are located respectively $\frac{\pi}{6}$ units and $\frac{2\pi}{3}$ units to the right of $A$. Therefore, $B = \left(\frac{\pi}{4}, 2\right)$ and $C = \left(\frac{3\pi}{4}, 0\right)$

**96.** The first coordinate of $A$ is the smallest positive $x$ such that $2x - \pi = n\pi$, $n$ and integer, so $x = \frac{n+1}{2}\pi$ must equal $\frac{\pi}{2}$. The period of this function $\pi$ is and the amplitude is 3. $B$ and $C$ are located (respectively) $\frac{\pi}{4}$ units and $\pi$ units to the right of $A$. Therefore, $A = \left(\frac{\pi}{2}, 0\right)$, $B = \left(\frac{3\pi}{4}, 3\right)$ and $C = \left(\frac{3\pi}{2}, 0\right)$.

**97. (a)** Since $\sin(-\theta) = -\sin\theta$ (because sine is an odd function) $a\sin[-B(x-h)] + k = -a\sin[B(x-h)] + k$. Then any expression with a negative value of $b$ can be rewritten as an expression of the same general form but with a positive coefficient in place of $b$.

**(b)** A sine graph can be translated a quarter of a period to the left to become a cosine graph of the same sinusoid. Thus $y = a\sin\left[b\left((x-h) + \frac{1}{4}\cdot\frac{2\pi}{b}\right)\right] + k$ $= a\sin\left[b\left(x - \left(h - \frac{\pi}{2b}\right)\right)\right] + k$ has the same graph as $y = a\cos[b(x-h)] + k$. We therefore choose $H = h - \frac{\pi}{2b}$.

**(c)** The angles $\theta + \pi$ and $\theta$ determine diametrically opposite points on the unit circle, so they have point symmetry with respect to the origin. The $y$-coordinates are therefore opposites, so $\sin(\theta + \pi) = -\sin\theta$.

**(d)** By the identity in (c). $y = a\sin[b(x-h) + \pi] + k$ $= -a\sin[b(x-h)] + k$. We therefore choose $H = h - \frac{\pi}{b}$.

**(e)** Part (b) shows how to convert $y = a\cos[b(x-h)] + k$ to $y = a\sin[b(x-H)] + k$, and parts (a) and (d) show how to ensure that $a$ and $b$ are positive.

## ■ Section 4.5 Graphs of Tangent, Cotangent, Secant, and Cosecant

### Exploration 1

1. The graphs do not seem to intersect.
2. Set the expressions equal and solve for $x$:
   $-k \cos x = \sec x$
   $-k \cos x = 1/\cos x$
   $-k(\cos x)^2 = 1$
   $(\cos x)^2 = -1/k$
   Since $k > 0$, this requires that the square of cos $x$ be negative, which is impossible. This proves that there is no value of $x$ for which the two functions are equal, so the graphs do not intersect.

### Quick Review 4.5

1. Period $\pi$
2. Period $\frac{2\pi}{3}$
3. Period $6\pi$
4. Period $4\pi$

For #5–8, recall that zeros of rational functions are zeros of the numerator, and vertical asymptotes are found at zeros of the denominator (provided the numerator and denominator have no common zeros).

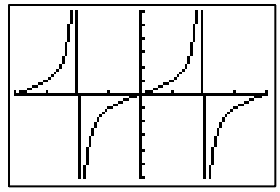
5. Zero: 3. Asymptote: $x = -4$
6. Zero: $-5$. Asymptote: $x = 1$
7. Zero: $-1$. Asymptotes: $x = 2$ and $x = -2$
8. Zero: $-2$. Asymptotes: $x = 0$ and $x = 3$

For #9–10, examine graphs to suggest the answer. Confirm by checking $f(-x) = f(x)$ for even functions and $f(-x) = -f(x)$ for odd functions.

9. Even: $(-x)^2 + 4 = x^2 + 4$
10. Odd: $\frac{1}{(-x)} = -\frac{1}{x}$

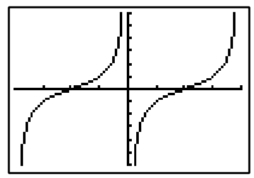
### Section 4.5 Exercises

1. The graph of $y = 2 \csc x$ must be vertically stretched by 2 compared to $y = \csc x$, so $y_1 = 2 \csc x$ and $y_2 = \csc x$.
2. The graph of $y = 5 \tan x$ must be vertically stretched by 10 compared to $y = 0.5 \tan x$, so $y_1 = 5 \tan x$ and $y_2 = 0.5 \tan x$.
3. The graph of $y = 3 \csc 2x$ must be vertically stretched by 3 and horizontally shrunk by $\frac{1}{2}$ compared to $y = \csc x$, so $y_1 = 3 \csc 2x$ and $y_2 = \csc x$.
4. The graph of $y = \cot(x - 0.5) + 3$ must be translated 3 units up and 0.5 units right compared to $y = \cot x$, so $y_1 = \cot(x - 0.5) + 3$ and $y_2 = \cot x$.
5. The graph of $y = \tan 2x$ results from shrinking the graph of $y = \tan x$ horizontally by a factor of $\frac{1}{2}$. There are vertical asymptotes at $x = \ldots, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \ldots$.



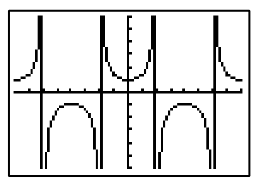
$[-\frac{\pi}{2}, \frac{\pi}{2}]$ by $[-6, 6]$

6. The graph of $y = -\cot 3x$ results from shrinking the graph of $y = \cot x$ horizontally by a factor of $\frac{1}{3}$ and reflecting it across the $x$-axis. There are vertical asymptotes at $x = \ldots, -\frac{2\pi}{3}, -\frac{\pi}{3}, 0, \frac{\pi}{3}, \ldots$.



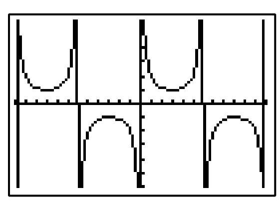
$\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$ by $[-6, 6]$

7. The graph of $y = \sec 3x$ results from shrinking the graph of $y = \sec x$ horizontally by a factor of $\frac{1}{3}$. There are vertical asymptotes at odd multiples of $\frac{\pi}{6}$.



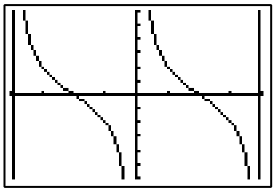
$\left[-\frac{2\pi}{3}, \frac{2\pi}{3}\right]$ by $[-6, 6]$

8. The graph of $y = \csc 2x$ results from shrinking the graph of $y = \csc x$ horizontally by a factor of $\frac{1}{2}$. There are vertical asymptotes at $x = \ldots, -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \ldots$.



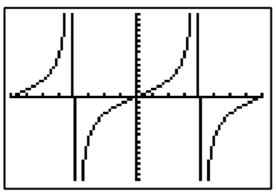
$[-\pi, \pi]$ by $[-6, 6]$

**9.** The graph of $y = 2 \cot 2x$ results from shrinking the graph of $y = \cot x$ horizontally by a factor of $\frac{1}{2}$ and stretching it vertically by a factor of 2. There are vertical asymptotes at $x = \ldots, -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \ldots$.



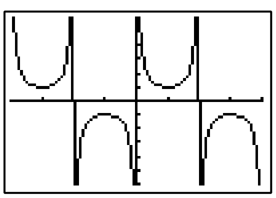
$[-\frac{\pi}{2}, \frac{\pi}{2}]$ by $[-6, 6]$

**10.** The graph of $y = 3 \tan\left(\frac{x}{2}\right)$ results from stretching the graph of $y = \tan x$ horizontally by a factor of 2 and stretching it vertically by a factor of 3. There are vertical asymptotes at $x = \ldots, -\pi, \pi, 3\pi, \ldots$.



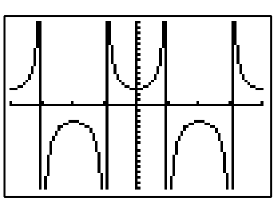
$[-2\pi, 2\pi]$ by $[-15, 15]$

**11.** The graph of $y = \csc\left(\frac{x}{2}\right)$ results from horizontally stretching the graph of $y = \csc x$ by a factor of 2. There are vertical asymptotes at $x = \ldots, -4\pi, -2\pi, 0, 2\pi, \ldots$.



$[-4\pi, 4\pi]$ by $[-6, 6]$

**12.** The graph of $y = 3 \sec 4x$ results from horizontally shrinking the graph of $y = \sec x$ by a factor of $\frac{1}{4}$ and stretching it vertically by a factor of 3. There are vertical asymptotes at odd multiples of $\frac{\pi}{8}$.



$[-\frac{\pi}{2}, \frac{\pi}{2}]$ by $[-15, 15]$

**13.** Graph (a); Xmin $= -\pi$ and Xmax $= \pi$

**14.** Graph (d); Xmin $= -\pi$ and Xmax $= \pi$

**15.** Graph (c); Xmin $= -\pi$ and Xmax $= \pi$

**16.** Graph (b); Xmin $= -\pi$ and Xmax $= \pi$

**17.** Domain: All reals except integer multiples of $\pi$
Range: $(-\infty, \infty)$
Continuous on its domain
Decreasing on each interval in its domain
Symmetric with respect to the origin (odd)
Not bounded above or below
No local extrema
No horizontal asymptotes
Vertical asymptotes $x = k\pi$ for all integers $k$
End behavior: $\lim_{x\to\infty} \cot x$ and $\lim_{x\to-\infty} \cot x$ do not exist.

**18.** Domain: All reals except odd multiples of $\frac{\pi}{2}$
Range: $(-\infty, -1] \cup [1, \infty)$
Continuous on its domain
On each interval centered at an even multiple of $\pi$: decreasing on the left half of the interval and increasing on the right half
On each interval centered at an odd multiple of $\pi$: increasing on the left half of the interval and decreasing on the right half
Symmetric with respect to the $y$-axis (even)
Not bounded above or below
Local minimum 1 at each even multiple of $\pi$, local maximum $-1$ at each odd multiple of $\pi$
No horizontal asymptotes
Vertical asymptotes $x = k\pi/2$ for all odd integers $k$
End behavior: $\lim_{x\to\infty} \sec x$ and $\lim_{x\to-\infty} \sec x$ do not exist.

**19.** Domain: All reals except integer multiples of $\pi$
Range: $(-\infty, -1] \cup [1, \infty)$
Continuous on its domain
On each interval centered at $x = \frac{\pi}{2} + 2k\pi$ ($k$ an integer): decreasing on the left half of the interval and increasing on the right half
On each interval centered at $\frac{3\pi}{2} + 2k\pi$: increasing on the left half of the interval and decreasing on the right half
Symmetric with respect to the origin (odd)
Not bounded above or below
Local minimum 1 at each $x = \frac{\pi}{2} + 2k\pi$, local maximum $-1$ at each $x = \frac{3\pi}{2} + 2k\pi$, where $k$ is an even integer in both cases
No horizontal asymptotes
Vertical asymptotes: $x = k\pi$ for all integers $k$
End behavior: $\lim_{x\to\infty} \csc x$ and $\lim_{x\to-\infty} \csc x$ do not exist.

**20.** Domain: All reals except odd multiples of $\pi$
Range: $(-\infty, \infty)$
Continuous on its domain
Increasing on each interval in its domain
Symmetric with respect to the origin (odd)
Not bounded above or below
No local extrema
No horizontal asymptotes
Vertical asymptotes $x = k\pi$ for all odd integers $k$
End behavior: $\lim_{x\to\infty} \tan(x/2)$ and $\lim_{x\to-\infty} \tan(x/2)$ do not exist.

**21.** Starting with $y = \tan x$, vertically stretch by 3.

**22.** Starting with $y = \tan x$, reflect across $x$-axis.

**23.** Starting with $y = \csc x$, vertically stretch by 3.

**24.** Starting with $y = \tan x$, vertically stretch by 2.

**25.** Starting with $y = \cot x$, horizontally stretch by 2, vertically stretch by 3, and reflect across $x$-axis.

**26.** Starting with $y = \sec x$, horizontally stretch by 2, vertically stretch by 2, and reflect across $x$-axis.

**27.** Starting with $y = \tan x$, horizontally shrink by $\frac{2}{\pi}$ and reflect across $x$-axis and shift up by 2 units.

**28.** Starting with $y = \tan x$, horizontally shrink by $\frac{1}{\pi}$ and vertically stretch by 2 and shift down by 2 units.

**29.** $\sec x = 2$
$\cos x = \frac{1}{2}$
$x = \frac{\pi}{3}$

**30.** $\csc x = 2$
$\sin x = \frac{1}{2}$
$x = \frac{5\pi}{6}$

**31.** $\cot x = -\sqrt{3}$
$\tan x = -\frac{\sqrt{3}}{3}$
$x = \frac{5\pi}{6}$

**32.** $\sec x = -\sqrt{2}$
$\cos x = -\frac{\sqrt{2}}{2}$
$x = \frac{5\pi}{4}$

**33.** $\csc x = 1$
$\sin x = 1$
$x = \frac{5\pi}{2}$

**34.** $\cot x = 1$
$\tan x = 1$
$x = -\frac{3\pi}{4}$

**35.** $\tan x = 1.3$
$x \approx 0.92$

**36.** $\sec x = 2.4$
$\cos x = \frac{1}{2.4}$
$x \approx 1.14$

**37.** $\cot x = -0.6$
$\tan x = -\frac{1}{0.6}$
$x \approx -1.03 + 2\pi$
$\approx 5.25$

**38.** $\csc x = -1.5$
$\sin x = -\frac{1}{1.5}$
$x \approx \pi - (-0.73)$
$\approx 3.87$

**39.** $\csc x = 2$
$\sin x = \frac{1}{2}$
$x \approx 0.52$ or
$x \approx \pi - 0.52$
$\approx 2.62$

**40.** $\tan x = 0.3$
$x \approx 0.29$ or
$x \approx \pi + 0.29$
$\approx 3.43$

**41.** **(a)** One explanation: If $O$ is the origin, the right triangles with hypotenuses $\overline{OP_1}$ and $\overline{OP_2}$, and one leg (each) on the $x$-axis, are congruent, so the legs have the same lengths. These lengths give the magnitudes of the coordinates of $P_1$ and $P_2$; therefore, these coordinates differ only in sign. Another explanation: The reflection of point $(a, b)$ across the origin is $(-a, -b)$.

**(b)** $\tan t = \frac{\sin t}{\cos t} = \frac{b}{a}$.

**(c)** $\tan(t - \pi) = \frac{\sin(t - \pi)}{\cos(t - \pi)} = \frac{-b}{-a} = \frac{b}{a} = \tan t$.

**(d)** Since points on opposite sides of the unit circle determine the same tangent ratio, $\tan(t \pm \pi) = \tan t$ for all numbers $t$ in the domain. Other points on the unit circle yield triangles with different tangent ratios, so no smaller period is possible.

**(e)** The tangent function repeats every $\pi$ units; therefore, so does its reciprocal, the cotangent (see also #43).

**42.** The terminal side passes through $(0, 0)$ and $(\cos x, \sin x)$; the slope is therefore $m = \frac{\sin x - 0}{\cos x - 0} = \frac{\sin x}{\cos x} = \tan x$.

**43.** For any $x$, $\left(\frac{1}{f}\right)(x + p) = \frac{1}{f(x + p)} = \frac{1}{f(x)} = \left(\frac{1}{f}\right)(x)$.
This is not true for any smaller value of $p$, since this is the smallest value that works for $f$.

**44.** **(a), (b)** The angles $t$ and $t + \pi$ determine points $(\cos t, \sin t)$ and $(\cos(t + \pi), \sin(t + \pi))$, respectively. These points are on opposite sides of the unit circle, so they are reflections of each other about the origin. The reflection of any point $(a, b)$ about the origin is $(-a, -b)$, so $\cos(t + \pi) = -\cos t$ and $\sin(t + \pi) = -\sin t$.

**(c)** $\tan(t + \pi) = \frac{\sin(t + \pi)}{\cos(t + \pi)} = \frac{-\sin t}{-\cos t} = \frac{\sin t}{\cos t} = \tan t$.
In order to determine that the period of $\tan t$ is $\pi$, we would need to show that no $p < \pi$ satisfies $\tan(t + p) = \tan t$ for all $t$.

**45.** **(a)** $d = 350 \sec x = \frac{350}{\cos x}$ ft

**(b)** $d \approx 16{,}831$ ft

**46.** **(a)** $x = 800 \cot y = \dfrac{800}{\tan x}$ ft

**(b)** $x \approx 5{,}051$ ft

**(c)** $\dfrac{\pi}{20} \cdot \dfrac{180°}{\pi} = 9°$

For #47–50, the equations can be rewritten (as shown), but generally are easiest to solve graphically.

**47.** $\sin^2 x = \cos x$; $x \approx \pm 0.905$

**48.** $\cos^2 x = \sin x$; $x \approx 0.666$ or $x \approx 2.475$

**49.** $\cos^2 x = \dfrac{1}{5}$; $x \approx \pm 1.107$ or $x \approx \pm 2.034$

**50.** $4\cos^2 x = \sin x$; $x \approx 1.082$ or $x \approx 2.060$

**51.** False. $f(x) = \tan x$ is increasing only over intervals on which it is defined, that is, intervals bounded by consecutive asymptotes.

**52.** True. Asymptotes of the secant function, $\sec x = 1/\cos x$, occur at all odd multiples of $\pi/2$ (where $\cos x = 0$), and these are exactly the zeros of the cotangent function, $\cot x = \cos x/\sin x$.

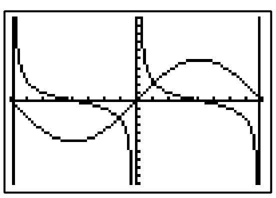
**53.** The cotangent curves are shaped like the tangent curves, but they are mirror images. The reflection of $\tan x$ in the $x$-axis is $-\tan x$. The answer is A.

**54.** $\sec x$ "just barely" intersects its inverse, $\cos x$, and when $\cos x$ is shifted to produce $\sin x$, that curve and the curve of $\sec x$ do not intersect at all. The answer is E.

**55.** $y = k/\sin x$ and the range of $\sin x$ is $[-1, 1]$. The answer is D.

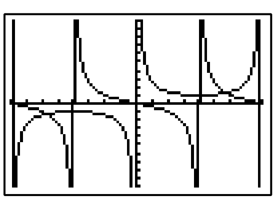
**56.** $y = \csc x = 1/\sin x$ has the same asymptotes as $y = \cos x/\sin x = \cot x$. The answer is C.

**57.** On the interval $[-\pi, \pi]$, $f > g$ on about $(-0.44, 0) \cup (0.44, \pi)$.



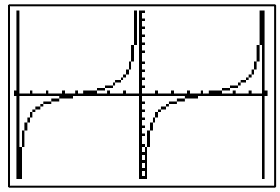
$[-\pi, \pi]$ by $[-10, 10]$

**58.** On the interval $[-\pi, \pi]$, $f > g$ on about $(-\pi, -2.24) \cup \left(-\dfrac{\pi}{2}, 0\right) \cup \left(\dfrac{\pi}{2}, 2.24\right)$.



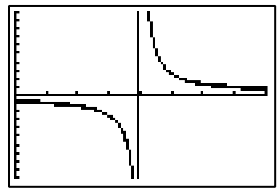
$[-\pi, \pi]$ by $[-10, 10]$

**59.** $\cot x$ is not defined at 0; the definition of "increasing on $(a, b)$" requires that the function be defined everywhere in $(a, b)$. Also, choosing $a = -\pi/4$ and $b = \pi/4$, we have $a < b$ but $f(a) = 1 > f(b) = -1$.



$[-\pi, \pi]$ by $[-10, 10]$

**60.** They look similar on this window, but they are noticeably different at the edges (near 0 and $\pi$). Also, if $f$ were equal to $g$, then it would follow that $\dfrac{1}{f} = -\cos x = \dfrac{1}{g} = x - \dfrac{\pi}{2}$ on this interval, which we know to be false.



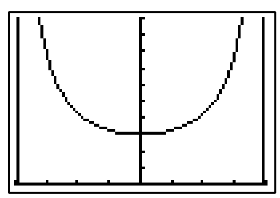
$[0, \pi]$ by $[-10, 10]$

**61.** $\csc x = \sec\left(x - \dfrac{\pi}{2}\right)$ (or $\csc x = \sec\left(x - \left(\dfrac{\pi}{2} + n\pi\right)\right)$ for any integer $n$) This is a translation to the right of $\dfrac{\pi}{2}\left(\text{or } \dfrac{\pi}{2} + n\pi\right)$ units.

**62.** $\cot x = -\tan\left(x - \dfrac{\pi}{2}\right)$ (or $\cot x = -\tan\left(x - \left(\dfrac{\pi}{2} + n\pi\right)\right)$ for any integer $n$).

This is a translation to the right of $\dfrac{\pi}{2}\left(\text{or } \dfrac{\pi}{2} + n\pi\right)$ units, and a reflection in the $x$-axis, in either order.

**63.** $d = 30 \sec x = \dfrac{30}{\cos x}$



$[-0.5\pi, 0.5\pi]$ by $[0, 100]$

**64.** **(a)** For any acute angle $\theta$, $\cos\theta = \sin\left(\dfrac{\pi}{2} - \theta\right)$—the sine of the complement of $\theta$. This can be seen from the right-triangle definition of sine and cosine: if one of the acute angles is $\theta$, then the other acute angle is $\dfrac{\pi}{2} - \theta$, since all three angles in a triangle must add to $\pi$. The side opposite the angle $\theta$ is the side adjacent to the other acute angle.

**(b)** $(\cos t, \sin t)$

**(c)** Using $\triangle ODA \sim \triangle OCB$ (recall "$\sim$" means "similar to"), $\frac{DA}{OD} = \tan t = \frac{BC}{OC} = \frac{BC}{1}$, so $BC = \tan t$.

**(d)** Using $\triangle ODA \sim \triangle OCB$, $\frac{OD}{OA} = \cot t = \frac{OC}{OB} = \frac{1}{OB}$, so $OB = \frac{1}{\cos t} = \sec t$.

**(e)** $\overline{BC}$ is a tangent segment (part of the tangent line); $\overline{OB}$ is a secant segment (part of a secant line, which crosses the circle at two points). The names "cotangent" and "cosecant" arise in the same way as "cosine"—they are the tangent and secant (respectively) of the complement. That is, just as $\overline{BC}$ and $\overline{OB}$ go with $\angle BOC$ (which has measure $t$), they also go with $\angle OBC$ (the complement of $\angle BOC$, with measure $\frac{\pi}{2} - t$).

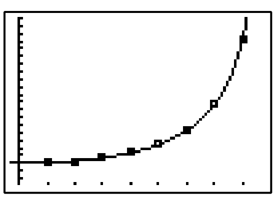
**65.** $0.058 \frac{\text{N}}{\text{m}} = \frac{1}{2}(1.5\text{ m})\left(1050\frac{\text{kg}}{\text{m}^3}\right)\left(9.8\frac{\text{m}}{\text{sec}^2}\right)$ $(4.7 \times 10^{-6}\text{ m})\sec\phi \approx 0.03627 \sec\phi \frac{\text{kg}}{\text{sec}^2}$, so $\sec\phi \approx 1.5990$, and $\phi \approx 0.8952$ radians $\approx 51.29°$.

**66. (a)** $\frac{1}{y} = \frac{1}{a\sec(bx)} = \frac{1}{a}\cdot\frac{1}{\sec(bx)} = \frac{1}{a}\cos(bx) = \frac{1}{a}\sin(bx + \pi/2)$

**(b)** $y = 0.2\sin\left(\frac{1}{6}x + \frac{\pi}{2}\right)$

**(c)** $a = 1/0.2 = 5$ and $b = 1/6$

**(d)** $y = 5\sec\left(\frac{x}{6}\right)$. The scatter plot is shown below, and the fit is very good—so good that you should realize that we made the data up!

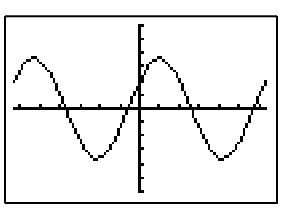


[−0.3, 8.7] by [2.32, 24]

## ■ Section 4.6 Graphs of Composite Trigonometric Functions

### Exploration 1

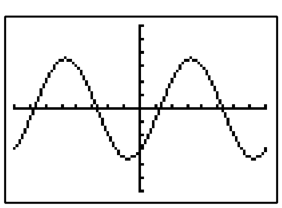
$y = 3\sin x + 2\cos x$



$[-2\pi, 2\pi]$ by $[-6, 6]$

Sinusoid

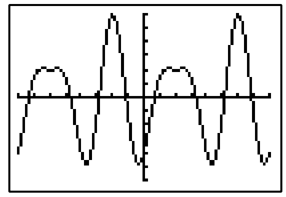
$y = 2\sin x - 3\cos x$



$[-2\pi, 2\pi]$ by $[-6, 6]$

Sinusoid

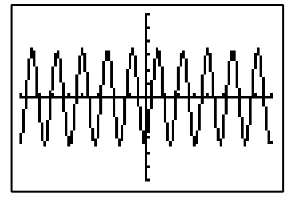
$y = 2\sin 3x - 4\cos 2x$



$[-2\pi, 2\pi]$ by $[-6, 6]$

Not a Sinusoid

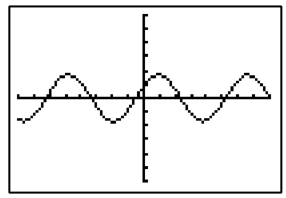
$y = 2\sin(5x + 1) - 5\cos 5x$



$[-2\pi, 2\pi]$ by $[-6, 6]$

Sinusoid

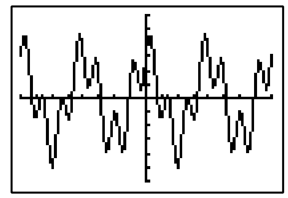
$y = \cos\left(\frac{7x - 2}{5}\right) + \sin\left(\frac{7x}{5}\right)$



$[-2\pi, 2\pi]$ by $[-6, 6]$

Sinusoid

$y = 3\cos 2x + 2\sin 7x$



$[-2\pi, 2\pi]$ by $[-6, 6]$

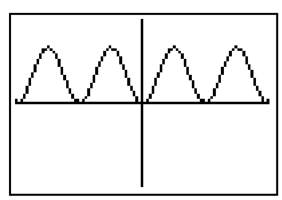
Not a Sinusoid

### Quick Review 4.6

1. Domain: $(-\infty, \infty)$; range: $[-3, 3]$
2. Domain: $(-\infty, \infty)$, range: $[-2, 2]$
3. Domain: $[1, \infty)$; range: $[0, \infty)$
4. Domain: $[0, \infty)$; range: $[0, \infty)$
5. Domain: $(-\infty, \infty)$; range: $[-2, \infty)$
6. Domain: $(-\infty, \infty)$; range: $[1, \infty)$
7. As $x \to -\infty$, $f(x) \to \infty$; as $x \to \infty$, $f(x) \to 0$.
8. As $x \to -\infty$, $f(x) \to -\infty$; as $x \to \infty$, $f(x) \to 0$.
9. $f \circ g(x) = (\sqrt{x})^2 - 4 = x - 4$, domain: $[0, \infty)$. $g \circ f(x) = \sqrt{x^2 - 4}$, domain: $(-\infty, -2] \cup [2, \infty)$.
10. $f \circ g(x) = (\cos x)^2 = \cos^2 x$, domain: $(-\infty, \infty)$. $g \circ f(x) = \cos(x^2)$, domain: $(-\infty, \infty)$.

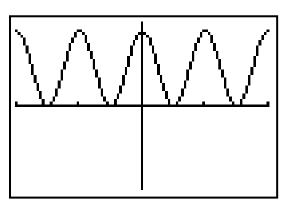
### Section 4.6 Exercises

1. Periodic.



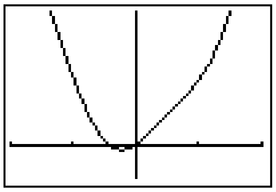
$[-2\pi, 2\pi]$ by $[-1.5, 1.5]$

2. Periodic.



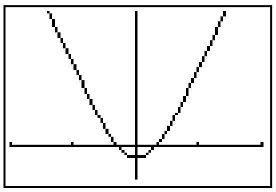
$[-2\pi, 2\pi]$ by $[-2.5, 2.5]$

**3.** Not periodic.



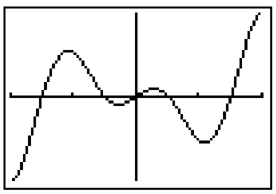
$[-2\pi, 2\pi]$ by $[-5, 20]$

**4.** Not periodic.



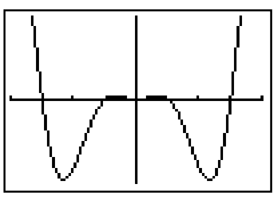
$[-2\pi, 2\pi]$ by $[-5, 20]$

**5.** Not periodic.



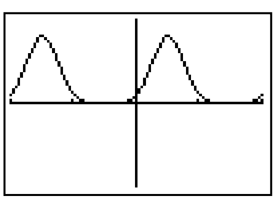
$[-2\pi, 2\pi]$ by $[-6, 6]$

**6.** Not periodic.



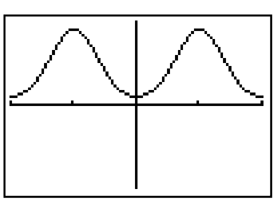
$[-2\pi, 2\pi]$ by $[-12, 12]$

**7.** Periodic.



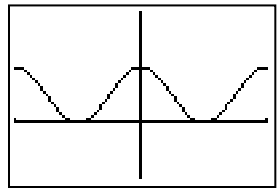
$[-2\pi, 2\pi]$ by $[-10, 10]$

**8.** Periodic.



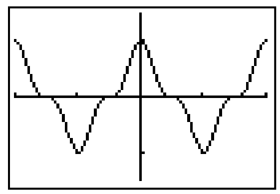
$[-2\pi, 2\pi]$ by $[-40, 40]$

**9.** Since the period of $\cos x$ is $2\pi$, we have $\cos^2(x + 2\pi) = (\cos(x + 2\pi))^2 = (\cos x)^2 = \cos^2 x$. The period is therefore an exact divisor of $2\pi$, and we see graphically that it is $\pi$. A graph for $-\pi \le x \le \pi$ is shown:



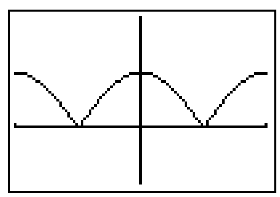
$[-\pi, \pi]$ by $[-1, 2]$

**10.** Since the period of $\cos x$ is $2\pi$, we have $\cos^3(x + 2\pi) = (\cos(x + 2\pi))^3 = (\cos x)^3 = \cos^3 x$. The period is therefore an exact divisor of $2\pi$, and we see graphically that it is $2\pi$. A graph for $-2\pi \le x \le 2\pi$ is shown:



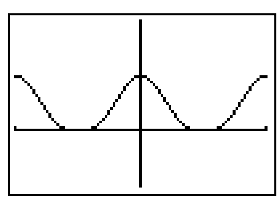
$[-2\pi, 2\pi]$ by $[-1.5, 1.5]$

**11.** Since the period of $\cos x$ is $2\pi$, we have $\sqrt{\cos^2(x + 2\pi)} = \sqrt{(\cos(x + 2\pi))^2} = \sqrt{(\cos x)^2} = \sqrt{\cos^2 x}$. The period is therefore an exact divisor of $2\pi$, and we see graphically that it is $\pi$. A graph for $-\pi \le x \le \pi$ is shown:



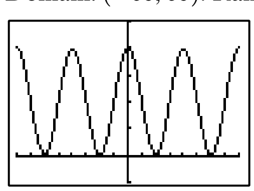
$[-\pi, \pi]$ by $[-1, 2]$

**12.** Since the period of $\cos x$ is $2\pi$, we have $|\cos^3(x + 2\pi)| = |(\cos(x + 2\pi))^3| = |(\cos x)^3| = |\cos^3 x|$. The period is therefore an exact divisor of $2\pi$, and we see graphically that it is $\pi$. A graph for $-\pi \le x \le \pi$ is shown:



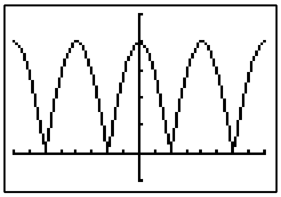
$[-\pi, \pi]$ by $[-1, 2]$

**13.** Domain: $(-\infty, \infty)$. Range: $[0, 1]$.



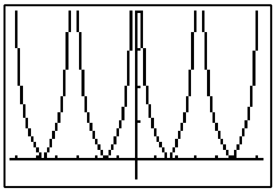
$[-2\pi, 2\pi]$ by $[-0.25, 1.25]$

**14.** Domain: $(-\infty, \infty)$. Range: $[0, 1]$.



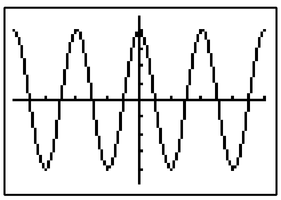
$[-2\pi, 2\pi]$ by $[-0.25, 1.25]$

**15.** Domain: all $x \neq n\pi$, $n$ an integer. Range: $[0, \infty)$.



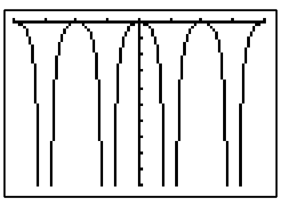
$[-2\pi, 2\pi]$ by $[-0.5, 4]$

**16.** Domain: $(-\infty, \infty)$. Range: $[-1, 1]$.



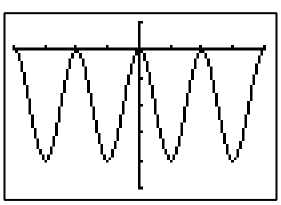
$[-4\pi, 4\pi]$ by $[-1.2, 1.2]$

**17.** Domain: all $x \neq \frac{\pi}{2} + n\pi$, $n$ an integer. Range: $(-\infty, 0]$.



$[-2\pi, 2\pi]$ by $[-10, 0.2]$

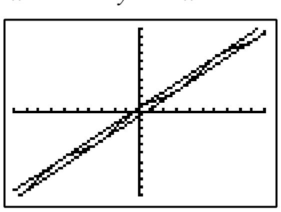
**18.** Domain: $(-\infty, \infty)$. Range: $[-1, 0]$.



$[-2\pi, 2\pi]$ by $[-1.25, .25]$

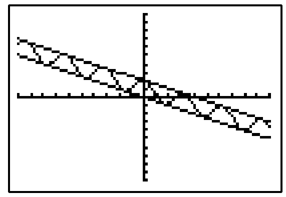
In #19–22, the linear equations are found by setting the cosine term equal to $\pm 1$.

**19.** $2x - 1 \leq y \leq 2x + 1$



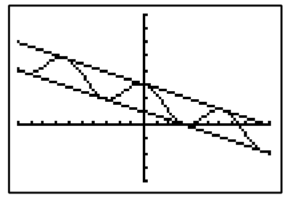
$[-10, 10]$ by $[-20, 20]$

**20.** $-0.5x \leq y \leq 2 - 0.5x$



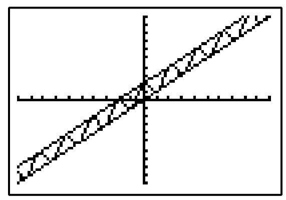
$[-10, 10]$ by $[-10, 10]$

**21.** $1 - 0.3x \leq y \leq 3 - 0.3x$



$[-10, 10]$ by $[-4, 8]$

**22.** $x \leq y \leq x + 2$



$[-10, 10]$ by $[-10, 10]$

For #23–28, the function $y_1 + y_2$ is a sinusoid if both $y_1$ and $y_2$ are sine or cosine functions with the same period.

**23.** Yes (period $2\pi$)

**24.** Yes (period $2\pi$)

**25.** Yes (period 2)

**26.** No

**27.** No

**28.** No

For #29–34, graph the function. Estimate $a$ as the amplitude of the graph (i.e., the height of the maximum). Notice that the value of $b$ is always the coefficient of $x$ in the original functions. Finally, note that $a \sin[b(x - h)] = 0$ when $x = h$, so estimate $h$ using a zero of $f(x)$ where $f(x)$changes from negative to positive.

**29.** $A \approx 3.61$, $b = 2$, and $h \approx 0.49$, so $f(x) \approx 3.61 \sin[2(x - 0.49)]$.

**30.** $A \approx 2.24$, $b = 3$, and $h \approx -0.15$, so $f(x) \approx 2.24 \sin[3(x + 0.15)]$.

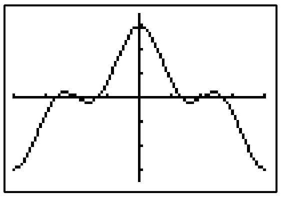
**31.** $A \approx 2.24$, $b = \pi$, and $h \approx 0.35$, so $f(x) \approx 2.24 \sin[\pi(x - 0.35)]$.

**32.** $A \approx 3.16$, $b = 2\pi$, and $h \approx -0.05$, so $f(x) \approx 3.16 \sin[2\pi(x + 0.05)]$.

**33.** $A \approx 2.24$, $b = 1$, and $h \approx -1.11$, so $f(x) \approx 2.24 \sin(x + 1.11)$.

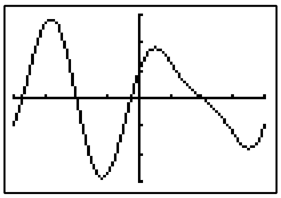
**34.** $A \approx 3.16$, $b = 2$, and $h \approx 0.16$, so $f(x) \approx 3.16 \sin[2(x - 0.16)]$.

**35.** The period is $2\pi$.



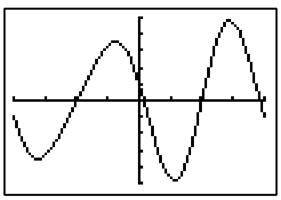
$[-\pi, \pi]$ by $[-3.5, 3.5]$

**36.** The period is $2\pi$.



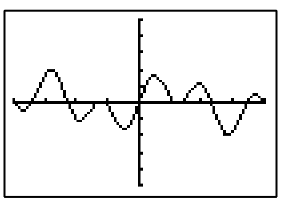
$[-\pi, \pi]$ by $[-3, 3]$

**37.** The period is $2\pi$.



$[-\pi, \pi]$ by $[-5, 5]$

**38.** The period is $2\pi$.



$[-\pi, \pi]$ by $[-5, 5]$

**39.** (a) **40.** (d)

**41.** (c) **42.** (b)

**43.** The damping factor is $e^{-x}$, which goes to zero as $x$ gets large. So damping occurs as $x \to \infty$.

**44.** The damping factor is $x$, which goes to zero as $x$ goes to zero (obviously). So damping occurs as $x \to 0$.

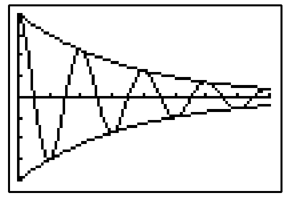
**45.** The amplitude, $\sqrt{5}$, is constant. So there is no damping.

**46.** The amplitude, $\pi^2$, is constant. So there is no damping.

**47.** The damping factor is $x^3$, which goes to zero as $x$ goes to zero. So damping occurs as $x \to 0$.

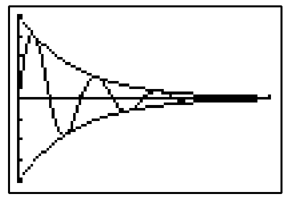
**48.** The damping factor is $(2/3)^x$, which goes to zero as $x$ gets large. So damping occurs as $x \to \infty$.

**49.** $f$ oscillates up and down between $1.2^{-x}$ and $-1.2^{-x}$. As $x \to \infty, f(x) \to 0$.



$[0, 4\pi]$ by $[-1, 1]$

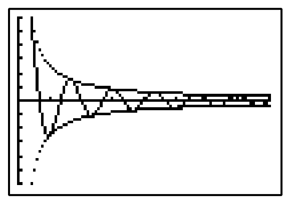
**50.** $f$ oscillates up and down between $2^{-x}$ and $-2^{-x}$. As $x \to \infty, f(x) \to 0$.



$[0, 2\pi]$ by $[-1, 1]$

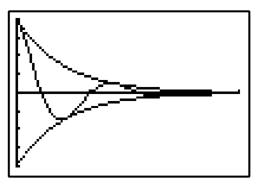
**51.** $f$ oscillates up and down between $\frac{1}{x}$ and $-\frac{1}{x}$.

As $x \to \infty, f(x) \to 0$.



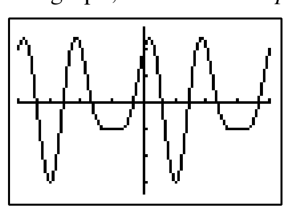
$[0, 4\pi]$ by $[-1.5, 1.5]$

**52.** $f$ oscillates up and down between $e^{-x}$ and $-e^{-x}$. As $x \to \infty, f(x) \to 0$.



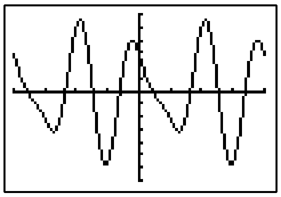
$[0, 1.5\pi]$ by $[-1, 1]$

**53.** Period $2\pi$: $\sin[3(x + 2\pi)] + 2\cos[2(x + 2\pi)] = \sin(3x + 6\pi) + 2\cos(2x + 4\pi) = \sin 3x + 2\cos 2x$. The graph, shows that no $p < 2\pi$ could be the period.



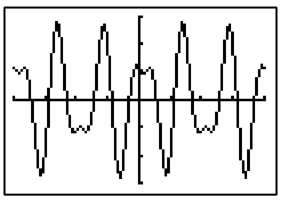
$[-2\pi, 2\pi]$ by $[-3.4, 2.8]$

**54.** Period $2\pi$: $4\cos[2(x + 2\pi)] - 2\cos[3(x + 2\pi) - 1]$
$= 4\cos(2x + 4\pi) - 2\cos(3x - 1 + 6\pi)$
$= 4\cos 2x - 2\cos(3x - 1)$. The graph, shows that no $p < 2\pi$ could be the period.



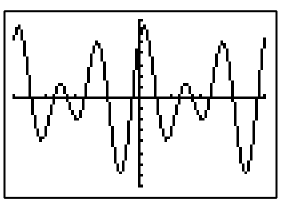
$[-2\pi, 2\pi]$ by $[-7, 6]$

**55.** Period $2\pi$:
$2\sin[3(x + 2\pi) + 1] - \cos[5(x + 2\pi) - 1]$
$= 2\sin(3x + 1 + 6\pi) - \cos(5x - 1 + 10\pi)$
$= 2\sin(3x + 1) - \cos(5x - 1)$. The graph, shows that no $p < 2\pi$ could be the period.



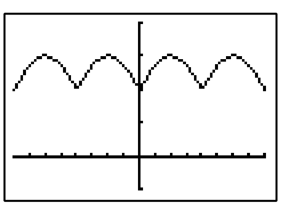
$[-2\pi, 2\pi]$ by $[-3, 3]$

**56.** Period $2\pi$:
$3\cos[2(x + 2\pi) - 1] - 4\sin[3(x + 2\pi) - 2]$
$= 3\cos(2x - 1 + 4\pi) - 4\sin(3x - 2 + 6\pi)$
$= 3\cos(2x - 1) - 4\sin(3x - 2)$. The graph, shows that no $p < 2\pi$ could be the period.



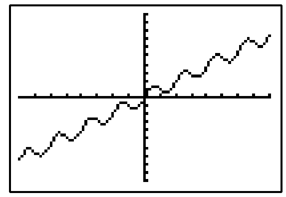
$[-2\pi, 2\pi]$ by $[-8, 7]$

**57.** Period $2\pi$: $\left|\sin\left[\frac{1}{2}(x + 2\pi)\right]\right| + 2 =$
$\left|\sin\left(\frac{1}{2}x + \pi\right)\right| + 2 = \left|-\sin\frac{1}{2}x\right| + 2 = \left|\sin\frac{1}{2}x\right| + 2.$
The graph, shows that no $p < 2\pi$ could be the period.



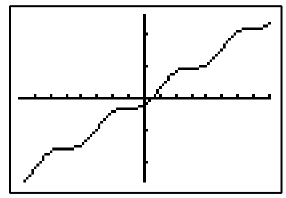
$[-4\pi, 4\pi]$ by $[-1, 4]$

**58.** Not periodic



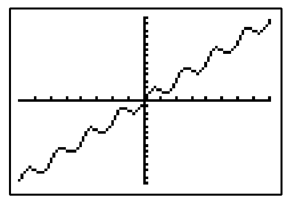
$[-4\pi, 4\pi]$ by $[-50, 50]$

**59.** Not periodic



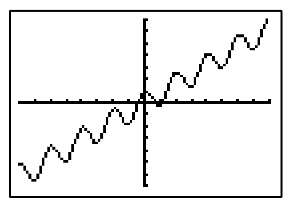
$[-4\pi, 4\pi]$ by $[-13, 13]$

**60.** Not periodic



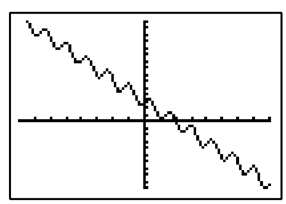
$[-4\pi, 4\pi]$ by $[-13, 13]$

**61.** Not periodic



$[-4\pi, 4\pi]$ by $[-7, 7]$

**62.** Not periodic



$[-4\pi, 4\pi]$ by $[-10, 15]$

For #63–70, graphs may be useful to suggest the domain and range.

**63.** There are no restrictions on the value of $x$, so the domain is $(-\infty, \infty)$. Range: $(-\infty, \infty)$.

**64.** There are no restrictions on the value of $x$, so the domain is $(-\infty, \infty)$. Range: $(-\infty, \infty)$.

**65.** There are no restrictions on the value of $x$, so the domain is $(-\infty, \infty)$. Range: $[1, \infty)$.

**66.** There are no restrictions on the value of $x$, so the domain is $(-\infty, \infty)$. Range: $(-\infty, \infty)$.

**67.** $\sin x$ must be nonnegative, so the domain is $\cdots \cup [-2\pi, -\pi] \cup [0, \pi] \cup [2\pi, 3\pi] \cup \cdots$; that is, all $x$ with $2n\pi \le x \le (2n+1)\pi$, $n$ an integer. Range: $[0, 1]$.

**68.** There are no restrictions on the value of $x$, so the domain is $(-\infty, \infty)$. Range: $[-1, 1]$.

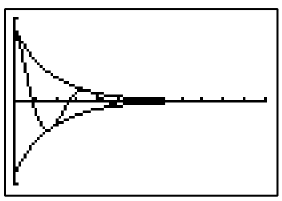
**69.** There are no restrictions on the value of $x$, since $|\sin x| \ge 0$, so the domain is $(-\infty, \infty)$. Range: $[0, 1]$.

**70.** $\cos x$ must be nonnegative, so the domain is

$\cdots \cup \left[-\frac{5\pi}{2}, -\frac{3\pi}{2}\right] \cup \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, \frac{5\pi}{2}\right] \cup \cdots$; that is, all $x$ with $\frac{(4n-1)\pi}{2} \le x \le \frac{(4n+1)\pi}{2}$, $n$ an integer.

Range: $[0, 1]$.

**71. (a)**



$[0, 12]$ by $[-0.5, 0.5]$

**(b)** For $t > 0.51$ (approximately).

**72. (a)** Using $S(t) = 75(1.04)^t + 4\sin\left(\frac{\pi t}{3}\right)$ to estimate sales (in millions of dollars) $t$ years after 2005, we have $S(0) = 75(1.04)^0 + 4\sin\left(\frac{\pi \cdot 0}{3}\right) = 75$ million dollars.

**(b)** The approximate annual growth rate is 4%.

**(c)** In 2013, $t = 8$ so the sales are predicted by $S(8) = 75(1.04)^8 + 4\sin\left(\frac{\pi \cdot 8}{3}\right) \approx 106.1$ million dollars.

**(d)** To find the number of years in each economic cycle, find the period of $\sin\left(\frac{\pi x}{3}\right)$, the trigonometric part of the model. The period of $\sin\left(\frac{\pi x}{3}\right)$ is $\frac{2\pi}{\pi/3} = 6$, so there are 6 years in each economic cycle for the company.

**73.** No. This is suggested by a graph of $y = \sin x^3$; there is no other section of the graph that looks like the section between $-1$ and 1. In particular, there is only one zero of the function in that interval (at $x = 0$); nowhere else can we find an interval this long with only one zero.

**74.** One explanation: The 'v'-shaped section around $x = 0$ is unique — it does not appear anywhere else on the graph.

**75. (a)** — this is obtained by adding $x$ to all parts of the inequality $-1 \le \sin x \le 1$. In the second, after subtracting $x$ from both sides, we are left with $-\sin x \le \sin x$, which is false when $\sin x$ is negative.

**76. (b)** — the first is impossible (even ignoring the middle part) if $x < 0$, since then $-x \not\le x$.

**77.** Graph (d), shown on $[-2\pi, 2\pi]$ by $[-4, 4]$

**78.** Graph (a), shown on $[-2\pi, 2\pi]$ by $[-4, 4]$

**79.** Graph (b), shown on $[-2\pi, 2\pi]$ by $[-4, 4]$

**80.** Graph (c), shown on $[-2\pi, 2\pi]$ by $[-4, 4]$

**81.** False. The behavior near zero, with a relative minimum of 0 at $x = 0$, is not repeated anywhere else.

**82.** False. If two sinusoids have different periods, the sum of the sinusoids is not a sinusoid. Example: $\sin x + \sin 3x$.

**83.** The negative portions of the graph of $y = \sin x$ are reflected in the $x$-axis for $y = |\sin x|$. This halves the period. The answer is B.

**84.** $f(-x) = (-x)\sin(-x) = x\sin x = f(x)$. The answer is C.

**85.** $f(-x) = -x + \sin(-x) = -x - \sin x = -f(x)$. The answer is D.

**86.** The sum of two sinusoids is a sinusoid only when the two sinusoids have the same period. The answer is D.

**87. (a)** Answers will vary — for example,

on a TI-81: $\frac{\pi}{47.5} = 0.0661\ldots \approx 0.07$;

on a TI-82: $\frac{\pi}{47} = 0.0668\ldots \approx 0.07$;

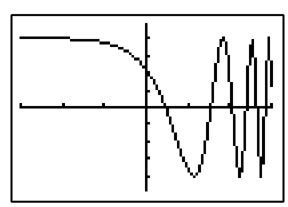
on a TI-85: $\frac{\pi}{63} = 0.0498\ldots \approx 0.05$;

on a TI-92: $\frac{\pi}{119} = 0.0263\ldots \approx 0.03$.

**(b)** Period: $p = \pi/125 = 0.0251\ldots$. For any of the TI graphers, there are from 1 to 3 cycles between each pair of pixels; the graphs produced are therefore inaccurate, since so much detail is lost.

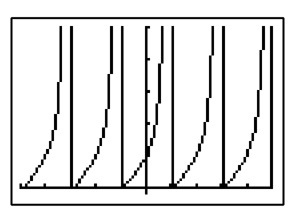
**88.** The amplitude is 3 hr 6 min, or 3.1 hr. The period is 365 days (one could also use 365.26 days), so $b = \frac{2\pi}{365}$. The phase shift $h$ is 80 days, so an answer is $3.1\sin\left[\frac{2\pi}{365}(x - 80)\right] + 12$ hours of daylight, where $x$ is the day of the year.

**89.** Domain: $(-\infty, \infty)$. Range: $[-1, 1]$. Horizontal asymptote: $y = 1$. Zeros at $\ln\left(\frac{\pi}{2} + n\pi\right)$, $n$ a non-negative integer.



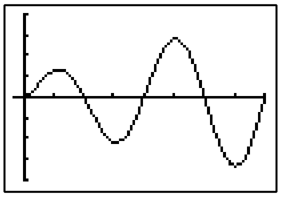
$[-3, 3]$ by $[-1.2, 1.2]$

**90.** Period: $\pi$. Domain: $x \ne n\pi + \frac{\pi}{2}$. Range: $(0, \infty)$. Vertical asympotes at missing points of domain: $x = n\pi + \frac{\pi}{2}$.



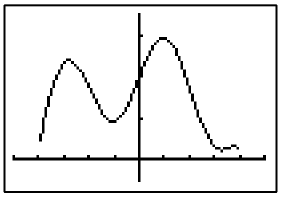
$[-2.5\pi, 2.5\pi]$ by $[-0.2, 5]$

**91.** Domain: $[0, \infty)$. Range: $(-\infty, \infty)$. Zeros at $n\pi$, $n$ a non-negative integer.



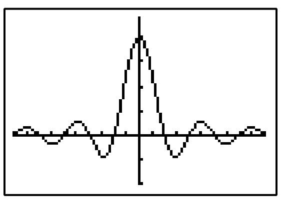
$[-0.5, 4\pi]$ by $[-4, 4]$

**92.** Domain: $[-2, 2]$. Range: $[0, 2.94]$ (approximately). Zeros at $-2$ and 2.



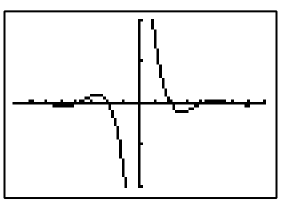
$[-2.5, 2.5]$ by $[-0.5, 3.5]$

**93.** Domain: $(-\infty, 0) \cup (0, \infty)$. Range: approximately $[-0.22, 1)$. Horizontal asymptote: $y = 0$. Zeros at $n\pi$, $n$ a non-zero integer.



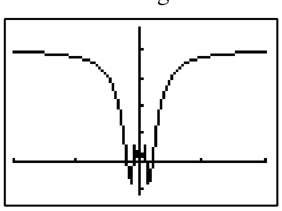
$[-5\pi, 5\pi]$ by $[-0.5, 1.2]$

**94.** Domain: $(-\infty, 0) \cup (0, \infty)$. Range: $(-\infty, \infty)$. Horizontal asymptote: $y = 0$. Vertical asymptote: $x = 0$. Zeros at $n\pi$, $n$ a non-zero integer.



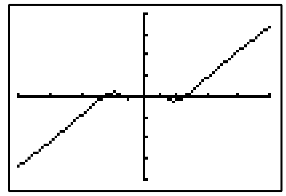
$[-4\pi, 4\pi]$ by $[-0.5, 0.5]$

**95.** Domain: $(-\infty, 0) \cup (0, \infty)$. Range: approximately $[-0.22, 1)$. Horizontal asymptote: $y = 1$. Zeros at $\frac{1}{n\pi}$, $n$ a non-zero integer.



$[-\pi, \pi]$ by $[-0.3, 1.2]$

**96.** Domain: $(-\infty, 0) \cup (0, \infty)$. Range: $(-\infty, \infty)$. Zeros at $\frac{1}{n\pi}$, $n$ a non-zero integer. Note: the graph also suggests the end-behavior asymptote $y = x$.



$[-1, 1]$ by $[-1, 1]$

## ■ Section 4.7 Inverse Trigonometric Functions

### Exploration 1

1. $\tan\theta = \frac{x}{1} = x$
2. $\tan^{-1} x = \tan^{-1}\left(\frac{x}{1}\right) = \theta$
3. $\sqrt{1 + x^2}$ (by the Pythagorean theorem)
4. $\sin(\tan^{-1}(x)) = \sin(\theta) = \frac{x}{\sqrt{1 + x^2}}$
5. $\sec(\tan^{-1}(x)) = \sec(\theta) = \sqrt{1 + x^2}$
6. The hypotenuse is positive in either quadrant. The ratios in the six basic trig functions are the same in every quadrant, so the functions are still valid regardless of the sign of $x$. (Also, the sign of the answer in (4) is negative, as it should be, and the sign of the answer in (5) is negative, as it should be.)

### Quick Review 4.7

1. $\sin x$: positive; $\cos x$: positive; $\tan x$: positive
2. $\sin x$: positive; $\cos x$: negative; $\tan x$: negative
3. $\sin x$: negative; $\cos x$: negative; $\tan x$: positive
4. $\sin x$: negative; $\cos x$: positive; $\tan x$: negative
5. $\sin\frac{\pi}{6} = \frac{1}{2}$
6. $\tan\frac{\pi}{4} = 1$
7. $\cos\frac{2\pi}{3} = -\frac{1}{2}$
8. $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$
9. $\sin\frac{-\pi}{6} = -\frac{1}{2}$
10. $\cos\frac{-\pi}{3} = \frac{1}{2}$

### Section 4.7 Exercises

For #1–12, keep in mind that the inverse sine and inverse tangent functions return values in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and the inverse cosine function gives values in $[0, \pi]$. A calculator may also be useful to suggest the exact answer. (A useful trick is to compute, e.g., $\sin^{-1}(\sqrt{3}/2)\pi$ and observe that this is $\approx 0.333$, suggesting the answer $\pi/3$.)

1. $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$
2. $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$
3. $\tan^{-1}(0) = 0$
4. $\cos^{-1}(1) = 0$

**5.** $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

**6.** $\tan^{-1}(1) = \frac{\pi}{4}$

**7.** $\tan^{-1}(-1) = -\frac{\pi}{4}$

**8.** $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$

**9.** $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$

**10.** $\tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$

**11.** $\cos^{-1}(0) = \frac{\pi}{2}$

**12.** $\sin^{-1}(1) = \frac{\pi}{2}$

**13.** approx. 21.22°

**14.** approx. 42.07°

**15.** approx. −85.43°

**16.** approx. 103.30°

**17.** approx. 1.172

**18.** approx. 1.527

**19.** approx. −0.478

**20.** approx. 2.593

**21.** $y = \tan^{-1}(x^2)$ is equivalent to $\tan y = x^2$, $-\pi/2 < y < \pi/2$. For $x^2$ to get very large, $y$ has to approach $\pi/2$. So $\lim_{x\to\infty} \tan^{-1}(x^2) = \pi/2$ and $\lim_{x\to-\infty} \tan^{-1}(x^2) = \pi/2$.

**22.** $y = (\tan^{-1} x)^2$ is equivalent to $x = \tan(\pm\sqrt{y})$, $0 \le y < \pi^2/4$. For $x$ to get very large, in the positive or negative direction, $y$ has to approach $\pi^2/4$. So $\lim_{x\to\infty} (\tan^{-1}x)^2 = \pi^2/4$ and $\lim_{x\to-\infty} (\tan^{-1}x)^2 = \pi^2/4$.

**23.** $\cos\left(\sin^{-1}\frac{1}{2}\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

**24.** $\sin(\tan^{-1} 1) = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

**25.** $\sin^{-1}\left(\cos\frac{\pi}{4}\right) = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

**26.** $\cos^{-1}\left(\cos\frac{7\pi}{4}\right) = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

**27.** $\cos\left(2\sin^{-1}\frac{1}{2}\right) = \cos\left(2\cdot\frac{\pi}{6}\right) = \frac{1}{2}$

**28.** $\sin[\tan^{-1}(-1)] = \sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

**29.** $\arcsin\left(\cos\frac{\pi}{3}\right) = \arcsin\frac{1}{2} = \frac{\pi}{6}$

**30.** $\arccos\left(\tan\frac{\pi}{4}\right) = \arccos 1 = 0$

**31.** $\cos(\tan^{-1}\sqrt{3}) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

**32.** $\tan^{-1}(\cos\pi) = \tan^{-1}(-1) = -\frac{\pi}{4}$

**33.** Domain: $[-1, 1]$
Range: $[-\pi/2, \pi/2]$
Continuous
Increasing
Symmetric with respect to the origin (odd)
Bounded
Absolute maximum of $\pi/2$, absolute minimum of $-\pi/2$
No asymptotes
No end behavior (bounded domain)

**34.** Domain: $[-1, 1]$
Range: $[0, \pi]$
Continuous
Decreasing
Neither odd nor even (but symmetric with respect to the point $(0, \pi/2)$)
Bounded
Absolute maximum of $\pi$, absolute minimum of 0
No asymptotes
No end behavior (bounded domain)

**35.** Domain: $(-\infty, \infty)$
Range: $(-\pi/2, \pi/2)$
Continuous
Increasing
Symmetric with respect to the origin (odd)
Bounded
No local extrema
Horizontal asymptotes: $y = \pi/2$ and $y = -\pi/2$
End behavior: $\lim_{x\to\infty} \tan^{-1}x = \pi/2$ and $\lim_{x\to-\infty} \tan^{-1}x = -\pi/2$

**36.** Domain: $(-\infty, \infty)$
Range: $(0, \pi)$
Continuous
Decreasing
Neither odd nor even (but symmetric with respect to the point $(0, \pi/2)$)
Bounded
No local extrema
Horizontal asymptotes: $y = \pi$ and $y = 0$
End behavior: $\lim_{x\to\infty} \cot^{-1}x = 0$ and $\lim_{x\to-\infty} \cot^{-1}x = \pi$

**37.** Domain: $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Starting from $y = \sin^{-1} x$, horizontally shrink by $\frac{1}{2}$.

**38.** Domain: $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Range: $[0, 3\pi]$. Starting from $y = \cos^{-1} x$, horizontally shrink by $\frac{1}{2}$ and vertically stretch by 3 (either order).

**39.** Domain: $(-\infty, \infty)$. Range: $\left(-\frac{5\pi}{2}, \frac{5\pi}{2}\right)$. Starting from $y = \tan^{-1} x$, horizontally stretch by 2 and vertically stretch by 5 (either order).

**40.** Domain: $[-2, 2]$. Range: $[0, 3\pi]$. Starting from $y = \arccos x = \cos^{-1} x$, horizontally stretch by 2 and vertically stretch by 3 (either order).

**41.** First set $\theta = \sin^{-1} x$ and solve $\sin\theta = 1$, yielding $\theta = \frac{\pi}{2} + 2n\pi$ for integers $n$. Since $\theta = \sin^{-1} x$ must be in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we have $\sin^{-1} x = \frac{\pi}{2}$, so $x = 1$.

**42.** First set $y = \cos x$ and solve $\cos^{-1} y = 1$, yielding $y = \cos 1$. Then solve $\cos x = \cos 1$, which gives $x = 1 + 2n\pi$ or $x = -1 + 2n\pi$, for all integers $n$.

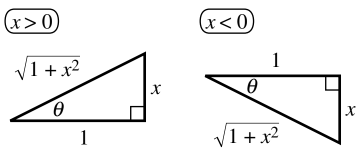
**43.** Divide both sides of the equation by 2, leaving $\sin^{-1} x = \frac{1}{2}$, so $x = \sin\frac{1}{2} \approx 0.479$.

**44.** If $\tan^{-1} x = -1$, then $x = \tan(-1) \approx -1.557$.

**45.** For any $x$ in $[0, \pi]$, $\cos(\cos^{-1}x), = x$. Hence, $x = \frac{1}{3}$.

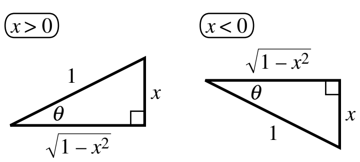
**46.** For any $x$ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\sin^{-1}(\sin x) = x$. Since $\frac{\pi}{10}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $x = \frac{\pi}{10}$.

**47.** Draw a right triangle with horizontal leg 1, vertical leg $x$ (if $x > 0$, draw the vertical leg "up"; if $x < 0$, draw it down), and hypotenuse $\sqrt{1 + x^2}$. The acute angle adjacent to the leg of length 1 has measure $\theta = \tan^{-1} x$ (take $\theta < 0$ if $x < 0$), so $\sin\theta = \sin(\tan^{-1} x) = \frac{x}{\sqrt{1 + x^2}}$.

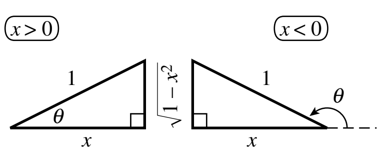



**48.** Use the same triangles as in #47: draw a triangle with horizontal leg 1, vertical leg $x$ (up or down as $x > 0$ or $x < 0$), and hypotenuse $\sqrt{1 + x^2}$. The acute angle adjacent to the leg of length 1 has measure $\theta = \tan^{-1} x$ (take $\theta < 0$ if $x < 0$), so $\cos\theta = \cos(\tan^{-1} x) = \frac{1}{\sqrt{1 + x^2}}$.

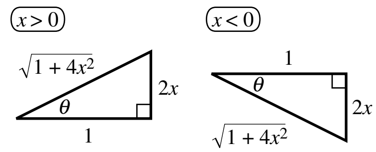
**49.** Draw a right triangle with horizontal leg $\sqrt{1 - x^2}$, vertical leg $x$ (if $x > 0$, draw the vertical leg "up"; if $x < 0$, draw it down), and hypotenuse 1. The acute angle adjacent to the horizontal leg has measure $\theta = \arcsin x$ (take $\theta < 0$ if $x < 0$), so $\tan\theta = \tan(\arcsin x) = \frac{x}{\sqrt{1 - x^2}}$.



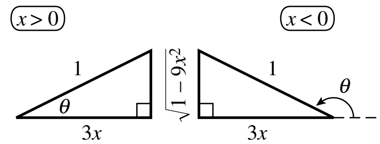

**50.** Draw a right triangle with horizontal leg $x$ (if $x > 0$, draw the horizontal leg right; if $x < 0$, draw it left), vertical leg $\sqrt{1 - x^2}$, and hypotenuse 1. If $x > 0$, let $\theta$ be the acute angle adjacent to the horizontal leg; if $x < 0$, let $\theta$ be the supplement of this angle. Then $\theta = \arccos x$, so $\cot\theta = \cot(\arccos x) = \frac{x}{\sqrt{1 - x^2}}$.




**51.** Draw a right triangle with horizontal leg 1, vertical leg $2x$ (up or down as $x > 0$ or $x < 0$), and hypotenuse $\sqrt{1 + 4x^2}$. The acute angle adjacent to the leg of length 1 has measure $\theta = \arctan 2x$ (take $\theta < 0$ if $x < 0$), so $\cos\theta = \cos(\arctan 2x) = \frac{1}{\sqrt{1 + 4x^2}}$.

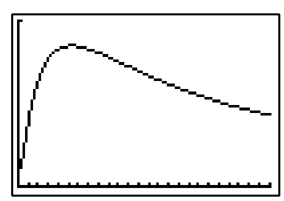



**52.** Draw a right triangle with horizontal leg $3x$ (if $x > 0$, draw the horizontal leg right; if $x < 0$, draw it left), vertical leg $\sqrt{1 - 9x^2}$, and hypotenuse 1. If $x > 0$, let $\theta$ be the acute angle adjacent to the horizontal leg; if $x < 0$, let $\theta$ be the supplement of this angle. Then $\theta = \arccos 3x$, so $\sin\theta = \sin(\arccos 3x) = \sqrt{1 - 9x^2}$.




**53. (a)** Call the smaller (unlabeled) angle in the lower left $\alpha$; then $\tan\alpha = \frac{2}{x}$, or $\alpha = \tan^{-1}\frac{2}{x}$ (since $\alpha$ is acute). Also, $\theta + \alpha$ is the measure of one acute angle in the right triangle formed by a line parallel to the floor and the wall; for this triangle $\tan(\theta + \alpha) = \frac{14}{x}$. Then $\theta + \alpha = \tan^{-1}\frac{14}{x}$ (since $\theta + \alpha$ is acute), so $\theta = \tan^{-1}\frac{14}{x} - \alpha = \tan^{-1}\frac{14}{x} - \tan^{-1}\frac{2}{x}$.

**(b)** Graph is shown. The actual maximum occurs at $x \approx 5.29$ ft, where $\theta \approx 48.59°$.

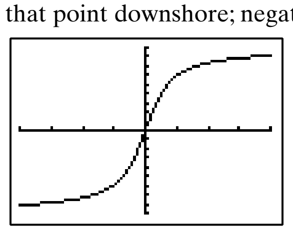


$[0, 25]$ by $[0, 55]$

**(c)** Either $x \approx 1.83$ or $x \approx 15.31$—these round to 2 ft or 15 ft.

**54. (a)** $\theta$ is one acute angle in the right triangle with leg lengths $x$ (opposite) and 3 (adjacent); thus $\tan\theta = \frac{x}{3}$, and $\theta = \tan^{-1}\frac{x}{3}$ (since $\theta$ is acute).

**(b)** Graph is shown (using DEGREE mode). Negative values of $x$ correspond to the point $Q$ being "upshore" from $P$ ("into" the picture) instead of downshore (as shown in the illustration). Positive angles are angles that point downshore; negative angles point upshore.



$[-20, 20]$ $[-90, 90]$

**(c)** $\theta = \tan^{-1} 5 \approx 78.69°$

**55.** **(a)** $\theta = \tan^{-1}\dfrac{s}{500}$.

**(b)** As $s$ changes from 10 to 20 ft, $\theta$ changes from about 1.1458° to 2.2906°—it almost exactly doubles (a 99.92% increase). As $s$ changes from 200 to 210 ft, $\theta$ changes from about 21.80° to 22.78°—an increase of less than 1°, and a very small relative change (only about 4.25%).

**(c)** The $x$-axis represents the height and the $y$-axis represents the angle: the angle cannot grow past 90° (in fact, it *approaches* but never exactly equals 90°).

**56.** **(a)** $\text{Sin}(x)$ exists for all $x$, but $\sin^{-1}(x)$ is restricted to $[-1, 1]$. The domain of $f(x)$ is $[-1, 1]$. The range is $[-1, 1]$.

**(b)** Since the domains of $\sin^{-1}(x)$ and $\cos^{-1}(x)$ are $[-1, 1]$, the domain of $g(x)$ is $[-1, 1]$. The range is $\left\{\dfrac{\pi}{2}\right\}$.

**(c)** Since $|\sin(x)| \le 1$ for all $x$, $h(x)$ exists for all $x$ and its domain is $(-\infty, \infty)$. The range is $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

**(d)** $\text{Sin}(x)$ exists for all $x$, but $\cos^{-1}(x)$ is restricted to $[-1, 1]$. The domain of $k(x)$ is $[-1, 1]$. The range is $[0, 1]$.

**(e)** Since $|\sin(x)| \le 1$ for all $x$, $q(x)$ exists for all $x$ and its domain is $(-\infty, \infty)$. The range is $[0, \pi]$.

**57.** False. This is only true for $-1 \le x \le 1$, the domain of the $\sin^{-1}$ function. For $x < -1$ and for $x > 1$, $\sin(\sin^{-1} x)$ is undefined.

**58.** True. The end behavior of $y = \arctan x$ determines two horizontal asymptotes, since $\lim_{x\to-\infty} \arctan x = -\pi/2$ and $\lim_{x\to\infty} \arctan x = \pi/2$.

**59.** $\cos(5\pi/6) = -\sqrt{3}/2$, so $\cos^{-1}(-\sqrt{3}/2) = 5\pi/6$. The answer is E.

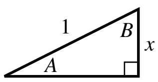
**60.** $\sin^{-1}(\sin \pi) = \sin^{-1} 0 = 0$. The answer is C.

**61.** $\sec(\tan^{-1} x) = \sqrt{1 + \tan^2(\tan^{-1} x)} = \sqrt{1 + x^2}$. The answer is C.

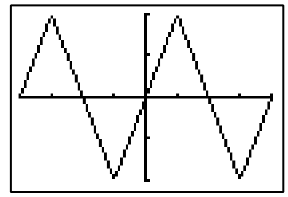
**62.** The range of $f(x) = \arcsin x = \sin^{-1} x$ is, by definition, $[-\pi/2, \pi/2]$. The answer is E.

**63.** The cotangent function restricted to the interval $(0, \pi)$ is one-to-one and has an inverse. The unique angle $y$ between 0 and $\pi$ (non-inclusive) such that $\cot y = x$ is called the inverse cotangent (or arccotangent) of $x$, denoted $\cot^{-1} x$ or $\text{arccot}\, x$. The domain of $y = \cot^{-1} x$ is $(-\infty, \infty)$ and the range is $(0, \pi)$.

**64.** In the triangle below, $A = \sin^{-1} x$ and $B = \cos^{-1} x$. Since $A$ and $B$ are complementary angles, $A + B = \pi/2$. The left-hand side of the equation is only defined for $-1 \le x \le 1$.

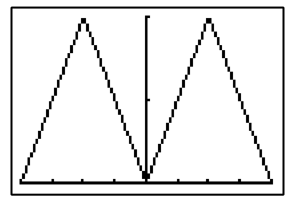



**65.** **(a)** Domain all reals, range $[-\pi/2, \pi/2]$, period $2\pi$.



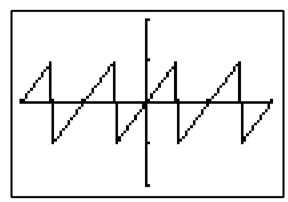
$[-2\pi, 2\pi]$ by $[-0.5\pi, 0.5\pi]$

**(b)** Domain all reals, range $[0, \pi]$, period $2\pi$.



$[-2\pi, 2\pi]$ by $[-0, \pi]$

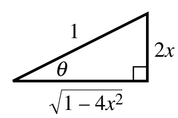
**(c)** Domain all reals except $\pi/2 + n\pi$ ($n$ an integer), range $(-\pi/2, \pi/2)$, period $\pi$. Discontinuity is not removable.



$[-2\pi, 2\pi]$ by $[-\pi, \pi]$

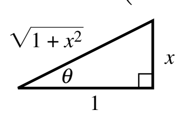
**66.** **(a)** Let $\theta = \sin^{-1}(2x)$. Then the adjacent side of the right triangle is $\sqrt{1^2 - (2x)^2} = \sqrt{1 - 4x^2}$.

$$\cos(\theta) = \frac{\sqrt{1 - 4x^2}}{1} = \sqrt{1 - 4x^2}$$



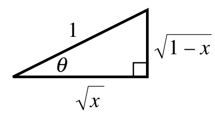

**(b)** Let $\theta = \tan^{-1}(x)$. Then the hypotenuse is $\sqrt{1 + x^2}$.

$$\sec^2(\theta) = \left(\frac{\sqrt{1 + x^2}}{1}\right)^2 = 1 + x^2$$

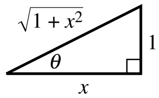



**(c)** Let $\theta = \cos^{-1}(\sqrt{x})$. Then the opposite side of the right triangle is $\sqrt{1^2 - (\sqrt{x})^2} = \sqrt{1 - x}$.

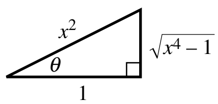
$$\sin(\theta) = \frac{\sqrt{1 - x}}{1} = \sqrt{1 - x}$$



**(d)** Let $\theta = \cot^{-1}(x)$. Then the hypotenuse is $\sqrt{1^2 + x^2} = \sqrt{1 + x^2}$. $-\csc^2(\theta) = -\left(\frac{\sqrt{1 + x^2}}{1}\right)^2$
$= -(1 + x^2) = -x^2 - 1$.

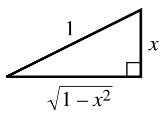



**(e)** Let $\theta = \sec^{-1}(x^2)$. Then the opposite leg of the right triangle is $\sqrt{(x^2)^2 - 1^2} = \sqrt{x^4 - 1}$.
$\tan(\theta) = \frac{\sqrt{x^4 - 1}}{1} = \sqrt{x^4 - 1}$.

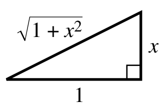



**67.** $y = \frac{\pi}{2} - \tan^{-1} x$.
(Note that $y = \tan^{-1}\left(\frac{1}{x}\right)$ does not have the correct range for negative values of $x$.)

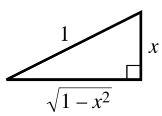
**68. (a)** $\cos(\sin^{-1} x)$ or $\sin(\cos^{-1} x)$




**(b)** $\sin(\tan^{-1} x)$ or $\cos(\cot^{-1} x)$




**(c)** $\tan(\sin^{-1} x)$ or $\cot(\cos^{-1} x)$




**69.** In order to transform the arctangent function to a function that has horizontal asymptotes at $y = 24$ and $y = 42$, we need to find $a$ and $d$ that will satisfy the equation $y = a \tan^{-1} x + d$. In other words, we are shifting the horizontal asymptotes of $y = \tan^{-1} x$ from $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$ to the new asymptotes $y = 24$ and $y = 42$.

Solving $y = a \tan^{-1} x + d$ and $y = 24$ for $\tan^{-1} x$ in terms of $a$ and $d$ yields $24 = a \tan^{-1} x + d$; so, $\frac{24 - d}{a} = \tan^{-1}x$. We know that $y = 24$ is the lower horizontal asymptote and thus it corresponds to $y = -\frac{\pi}{2}$.

So, $\frac{24 - d}{a} = \tan^{-1}x = -\frac{\pi}{2} \Rightarrow \frac{24 - d}{a} = -\frac{\pi}{2}$. Solving this for $d$ in terms of $a$ yields $d = 24 + \left(\frac{\pi}{2}\right)a$.

Solving $y = a \tan^{-1} x + d$ and $y = 42$ for $\tan^{-1} x$ in terms of $a$ and $d$ yields $42 = a \tan^{-1} x + d$; so, $\frac{42 - d}{a} = \tan^{-1}x$. We know that $y = 42$ is the upper horizontal asymptote and thus it corresponds to $y = \frac{\pi}{2}$.

So, $\frac{42 - d}{a} = \tan^{-1}x = \frac{\pi}{2} \Rightarrow \frac{42 - d}{a} = \frac{\pi}{2}$. Solving this for $d$ in terms of $a$ yields $d = 42 - \left(\frac{\pi}{2}\right)a$.

If $d = 24 + \left(\frac{\pi}{2}\right)a$ and $d = 42 - \left(\frac{\pi}{2}\right)a$, then $24 + \left(\frac{\pi}{2}\right)a = 42 - \left(\frac{\pi}{2}\right)a$. So, $18 = \pi a$, and $a = \frac{18}{\pi}$.

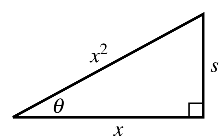
Substitute this value for $a$ into either of the two equations for $d$ to get: $d = 24 + \left(\frac{\pi}{2}\right)\left(\frac{18}{\pi}\right) = 24 + 9 = 33$ or $d = 42 - \left(\frac{\pi}{2}\right)\left(\frac{18}{\pi}\right) = 42 - 9 = 33$.

The arctangent function with horizontal asymptotes at $y = 24$ and $y = 42$ will be $y = \frac{18}{\pi}\tan^{-1}x + 33$.

**70. (a)** As in Example 5, $\theta$ can only be in Quadrant I or Quadrant IV, so the horizontal side of the triangle can only be positive.

**(b)** $\tan\left(\sin^{-1}\left(\frac{1}{x}\right)\right) = \frac{x}{s} = \frac{x}{\sqrt{x^4 - x^2}}$

**(c)** $\sin\left(\cos^{-1}\left(\frac{1}{x}\right)\right) = \frac{s}{x^2} = \frac{\sqrt{x^4 - x^2}}{x^2}$
(See figure below.)

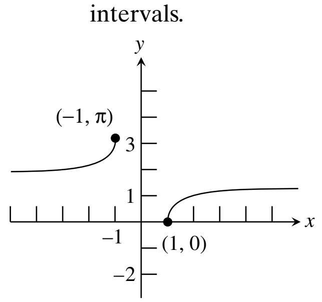


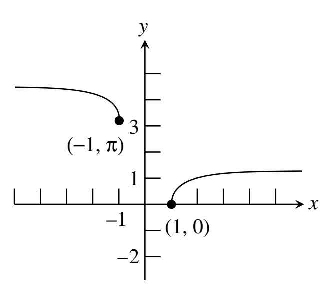

**71. (a)** The horizontal asymptote of the graph on the left is $y = \frac{\pi}{2}$.

**(b)** The two horizontal asymptotes of the graph on the right are $y = \frac{\pi}{2}$ and $y = \frac{3\pi}{2}$.

**(c)** The graph of $y = \cos^{-1}\left(\frac{1}{x}\right)$ will look like the graph on the left.

**(d)** The graph on the left is increasing on both connected intervals.



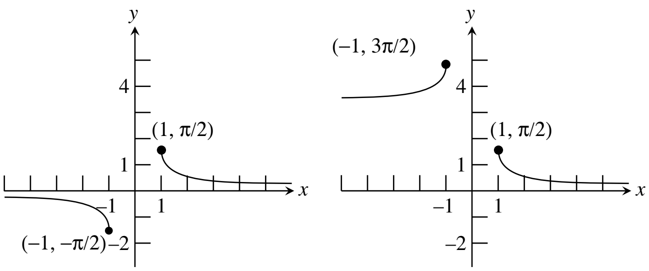



**72. (a)** The horizontal asymptote of the graph on the left is $y = 0$.

**(b)** The two horizontal asymptotes of the graph on the right are $y = 0$ and $y = \pi$.

**(c)** The graph of $y = \sin^{-1}\left(\frac{1}{x}\right)$ will look like the graph on the left.

**(d)** The graph on the left is decreasing on both connected intervals.




## ■ Section 4.8 Solving Problems with Trigonometry

### Exploration 1

1. The parametrization should produce the unit circle.
2. The grapher is actually graphing the unit circle, but the $y$-window is so large that the point never seems to get above or below the $x$-axis. It is flattened vertically.
3. Since the grapher is plotting points along the unit circle, it covers the circle at a constant speed. Toward the extremes its motion is mostly vertical, so not much horizontal progress (which is all that we see) occurs. Toward the middle, the motion is mostly horizontal, so it moves faster.
4. The directed distance of the point from the origin at any $T$ is exactly $\cos T$, and $d = \cos t$ models simple harmonic motion.

### Quick Review 4.8

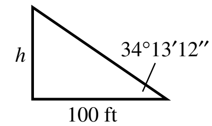
1. $b = 15 \cot 31° \approx 24.964$, $c = 15 \csc 31° \approx 29.124$
2. $a = 25 \cos 68° \approx 9.365$, $b = 25 \sin 68° \approx 23.180$
3. $b = 28 \cot 28° - 28 \cot 44° \approx 23.665$, $c = 28 \csc 28° \approx 59.642$, $a = 28 \csc 44° \approx 40.308$
4. $b = 21 \cot 31° - 21 \cot 48° \approx 16.041$, $c = 21 \csc 31° \approx 40.774$, $a = 21 \csc 48° \approx 28.258$
5. complement: 58°, supplement: 148°
6. complement: 17°, supplement: 107°
7. 45°
8. 202.5°
9. Amplitude: 3; period: $\pi$
10. Amplitude: 4; period: $\pi/2$

### Section 4.8 Exercises

All triangles in the supplied figures are right triangles.

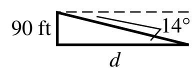
1. $\tan 60° = \frac{h}{300 \text{ ft}}$, so $h = 300 \tan 60° = 300\sqrt{3} \approx 519.62$ ft.

2. $\tan 34°13'12'' = \frac{h}{100 \text{ ft}}$, so $h = 100 \tan 34°13'12'' \approx 68.01$




3. Let $d$ be the length of the horizontal leg. Then $\tan 10° = \frac{120 \text{ ft}}{d}$, so $d = \frac{120}{\tan 10°} = 120 \cot 10° \approx 680.55$ ft.

4. $\tan 14° = \frac{90 \text{ ft}}{d}$, so $d = \frac{90}{\tan 14°} = 90 \cot 14° \approx 361$ ft.

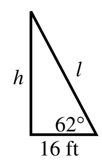



5. Let $\ell$ be the wire length (the hypotenuse); then $\cos 80° = \frac{5 \text{ ft}}{\ell}$, so $\ell = \frac{5}{\cos 80°} = 5 \sec 80° \approx 28.79$ ft.

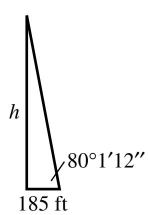
   Let $h$ be the tower height (the vertical leg); then $\tan 80° = \frac{h}{5 \text{ ft}}$, so $h = 5 \tan 80° \approx 28.36$ ft.

6. $\cos 62° = \frac{16 \text{ ft}}{\ell}$, so $\ell = \frac{16}{\cos 62°} = 16 \sec 62° \approx 34.08$ ft.

   $\tan 62° = \frac{h}{16 \text{ ft}}$, so $h = 16 \tan 62° \approx 30.09$ ft.

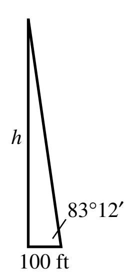



7. $\tan 80°1'12'' = \frac{h}{185 \text{ ft}}$, so $h = 185 \tan 80°1'12'' \approx 1051$ ft.

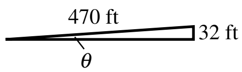



8. Let $h$ be the height of the smokestack; then $\tan 38° = \frac{h}{1580 \text{ ft}}$, so $h = 1580 \tan 38° \approx 1234.43$ ft.

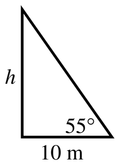
9. $\tan 83°12' = \frac{h}{100 \text{ ft}}$, so $h = 100 \tan 83°12' \approx 839$ ft.



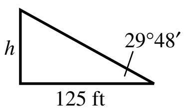
**10.** $\theta = \sin^{-1}\dfrac{32\text{ ft}}{470\text{ ft}} \approx 3.9°$




**11.** $\tan 55° = \dfrac{h}{10\text{ m}}$, so $h = 10\tan 55° \approx 14.3$ m

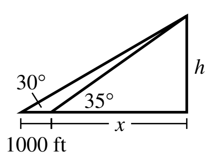



**12.** $\tan 29°48' = \dfrac{h}{125\text{ ft}}$, so $h = 125\tan 29°48' \approx 71.6$ ft




**13.** $\tan 35° = \dfrac{LP}{4.25\text{ mi}}$, so $LP = 4.25\tan 35° \approx 2.98$ mi.

**14.** $\tan 35° = \dfrac{h}{x}$ and $\tan 30° = \dfrac{h}{x + 1000}$, so $x = h\cot 35°$ and $x + 1000 = h\cot 30°$. Then $h\cot 35° = h\cot 30° - 1000$, so $h = \dfrac{1000}{\cot 30° - \cot 35°} \approx 3290.5$ ft.

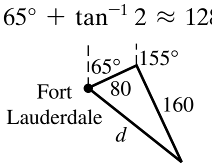



**15.** Let $x$ be the elevation of the bottom of the deck, and $h$ be the height of the deck. Then $\tan 30° = \dfrac{x}{200\text{ ft}}$ and $\tan 40° = \dfrac{x + h}{200\text{ ft}}$, so $x = 200\tan 30°$ ft and $x + h = 200\tan 40°$ ft. Therefore $h = 200(\tan 40° - \tan 30°) \approx 52.35$ ft.

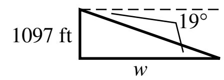
**16.** Let $d$ be the distance traveled, and let $x$ be car's ending distance from the base of the building. Then $\tan 15° = \dfrac{100\text{ ft}}{d + x}$ and $\tan 33° = \dfrac{100\text{ ft}}{x}$, so $d + x = 100\cot 15°$ ft and $x = 100\cot 33°$ ft. Therefore $d = 100(\cot 15° - \cot 33°) \approx 219$ ft.

**17.** The two legs of the right triangle are the same length (30 knots · 2 hr = 60 naut mi), so both acute angles are 45°. The length of the hypotenuse is the distance: $60\sqrt{2} \approx 84.85$ naut mi. The bearing is $95° + 45° = 140°$.

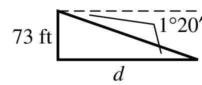
**18.** The two legs of the right triangle are 40 knots · 2 hr = 80 naut mi and 40 knots · 4 hr = 160 naut mi. The distance can be found with the Pythagorean Theorem: $d = 80\sqrt{5} \approx 178.885$ naut mi. The acute angle at Fort Lauderdale has measure $\tan^{-1} 2$, so the bearing is $65° + \tan^{-1} 2 \approx 128.435°$ (see figure below)




**19.** The difference in elevations is 1097 ft. If the width of the canyon is $w$, then $\tan 19° = \dfrac{1097\text{ ft}}{w}$, so $w = 1097\cot 19° \approx 3186$ ft.




**20.** The distance from the base of the tower $d$ satisfies $\tan 1°20' = \dfrac{73\text{ ft}}{d}$, so $d = 73\cot 1°20' \approx 3136$ ft.

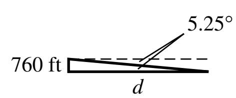



**21.** The acute angle in the triangle has measure $180° - 117° = 63°$, so $\tan 63° = \dfrac{\ell}{325\text{ ft}}$. Then $\ell = 325\tan 63° \approx 638$ ft.

**22.** $\tan 17° = \dfrac{h}{12\text{ mi}}$, so $h = 12\tan 17° \approx 3.67$ mi.

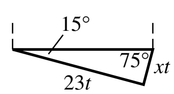
**23.** If $h$ is the height of the vertical span, $\tan 15° = \dfrac{h}{36.5\text{ ft}}$, so $h = 36.5\tan 15° \approx 9.8$ ft.

**24.** The distance $d$ satisfies $\tan 5.25° = \dfrac{760\text{ ft}}{d}$, so $d = 760\cot 5.25° \approx 8271$ ft.




**25.** Let $d$ be the distance from the boat to the shore, and let $x$ be the short leg of the smaller triangle. For the two triangles, the larger acute angles are 70° and 80°. Then $\tan 80° = \dfrac{d}{x}$ and $\tan 70° = \dfrac{d}{x + 550}$, or $x = d\cot 80°$ and $x + 550 = d\cot 70°$. Therefore $d = \dfrac{550}{\cot 70° - \cot 80°} \approx 2931$ ft.

**26.** If $t$ is the time until the boats collide, the law enforcement boat travels $23t$ naut mi. During that same time, the smugglers' craft travels $xt$ naut mi, where $x$ is that craft's speed. These two distances are the legs of a right triangle (shown); then $\tan 15° = \dfrac{xt}{23t} = \dfrac{x}{23}$, so $x = 23\tan 15° \approx 6.2$ knots.




**27.** **(a)** Frequency: $\dfrac{\omega}{2\pi} = \dfrac{16\pi}{2\pi} = 8$ cycles/sec.

**(b)** $d = 6\cos 16\pi t$ inches.

**(c)** When $t = 2.85$, $d \approx 1.854$; this is about 4.1 in. left of the starting position (when $t = 0$, $d = 6$).

**28.** **(a)** Frequency: $\dfrac{\omega}{2\pi} = \dfrac{\pi}{2\pi} = \dfrac{1}{2}$ cycle/sec.

**(b)** $d = 18\cos \pi t$ cm.

**(c)** Since 1 cycle takes 2 sec, there are 30 cycles/min.

**29.** The frequency is 2 cycles/sec, so $\omega = 2 \cdot 2\pi = 4\pi$ radians/sec. Assuming the initial position is $d = 3$ cm: $d = 3 \cos 4\pi t$.

**30.** $\frac{\omega}{2\pi} = 528$, so $\omega = 1056\pi$ radians/sec.

**31.** **(a)** The amplitude is $a = 25$ ft, the radius of the wheel.

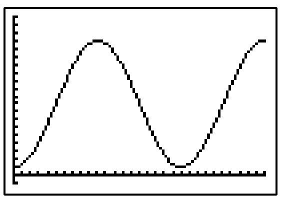
**(b)** $k = 33$ ft, the height of the center of the wheel.

**(c)** $\frac{\omega}{2\pi} = \frac{1}{20}$ rotations/sec, so $\omega = \pi/10$ radians/sec.

**32.** **(a)** $\frac{\omega}{2\pi} = 3 \text{ rpm} = \frac{1}{20}$ rotations/sec, so $\omega = \pi/10$.

One possibility is $h = -8 \cos\frac{\pi t}{10} + 9$ m.

**(b)**



[0, 30] by [−1, 20]

**(c)** $h(4) \approx 6.5$ m; $h(10) = 17$ m.

**33.** **(a)** Given a period of 12, we have $12 = \frac{2\pi}{|b|}$.

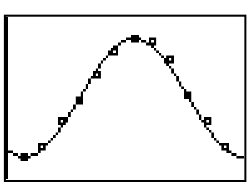
$12|b| = 2\pi$ so $|b| = \frac{2\pi}{12} = \frac{\pi}{6}$. We select the positive value so $b = \frac{\pi}{6}$.

**(b)** Using the high temperature of 82 and a low temperature of 48, we find $|a| = \frac{82 - 48}{2} = \frac{34}{2}$ so $|a| = 17$ and we will select the positive value.

$k = \frac{82 + 48}{2} = 65$

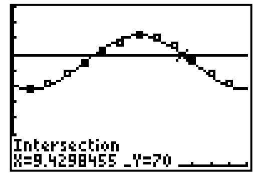
**(c)** $h$ is halfway between the times of the minimum and maximum. Using the maximum at time $t = 7$ and the minimum at time $t = 1$, we have $\frac{7 - 1}{2} = 3$. So, $h = 1 + 3 = 4$.

**(d)** The fit is very good for $y = 17 \sin\left(\frac{\pi}{6}(t - 4)\right) + 65$.



[0, 13] by [42, 88]

**(e)** There are several ways to find when the mean temperature will be 70°. *Graphical solution*: Graph the line $t = 70$ with the curve shown above, and find the intersection of the two curves. The two intersections are at $t \approx 4.57$ and $t \approx 9.43$.




[0, 13] by [42, 88]

*Algebraic solution*: Solve $17 \sin\left(\frac{\pi}{6}(t - 4)\right) + 65 = 70$ for $t$.

$$17 \sin\left(\frac{\pi}{6}(t - 4)\right) + 65 = 70$$
$$\sin\left(\frac{\pi}{6}(t - 4)\right) = \frac{5}{17}$$
$$\frac{\pi}{6}(t - 4) = \sin^{-1}\left(\frac{5}{17}\right)$$
$$\frac{\pi}{6}(t - 4) \approx 0.299, 2.842$$

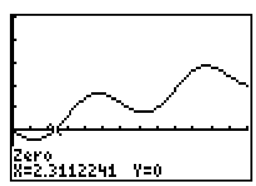
Note: $\sin \theta = \sin (\pi - \theta)$

$$t \approx 4.57, 9.43$$

Using either method to find $t$, find the day of the year as follows: $\frac{4.57}{12} \cdot 365 \approx 139$ and $\frac{9.43}{12} \cdot 365 \approx 287$. These represent May 19 and October 14.

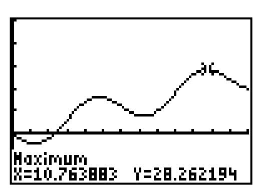
**34.** **(d)** All have the correct period, but the others are incorrect in various ways. Equation (a) oscillates between −25 and +25, while equation (b) oscillates between −17 and +33. Equation (c) is the closest among the incorrect formulas: it has the right maximum and minimum values, but it does not have the property that $h(0) = 8$. This is accomplished by the horizontal shift in (d).

**35.** **(a)** Solve this graphically by finding the zero of the function $P = 2t - 7 \sin\left(\frac{\pi t}{3}\right)$. The zero occurs at approximately 2.3. The function is positive to the right of the zero. So, the shop began to make a profit in March.




[0, 13] by [–20, 50]

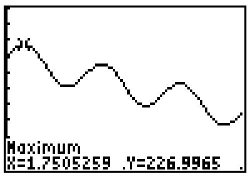
**(b)** Solve this graphically by finding the maximum of the function $P = 2t - 7 \sin\left(\frac{\pi t}{3}\right)$. The maximum occurs at approximately 10.76, so the shop enjoyed its greatest profit in November.




[0, 13] by [–20, 50]

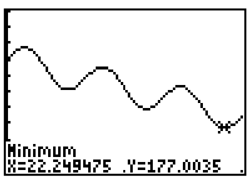
**36. (a)** Using the function $W = 220 - 1.5t + 9.81 \sin\left(\frac{\pi t}{4}\right)$, where $t$ is measured in months after January 1 of the first year and $W$ is measured in pounds, we have $t = 0$ at the beginning. This gives $W = 220 - 1.5(0) + 9.81 \sin\left(\frac{\pi \cdot 0}{4}\right) = 220$ pounds. At the end of two years, $t = 24$, which gives $W = 220 - 1.5(24) + 9.81 \sin\left(\frac{\pi \cdot 24}{4}\right) = 184$ pounds.

**(b)** Solve this graphically by finding the maximum of the function $W = 220 - 1.5t + 9.81 \sin\left(\frac{\pi t}{4}\right)$. The maximum occurs at $t = 1.75$, where $W \approx 227$.




[0, 24] by [150, 250]

**(c)** Solve this graphically by finding the minimum of the function $W = 220 - 1.5t + 9.81 \sin\left(\frac{\pi t}{4}\right)$. The minimum occurs at $t \approx 22.2$, where $W \approx 177$.




[0, 24] by [150, 250]

**37.** True. The frequency and the period are reciprocals: $f = 1/T$. So the higher the frequency, the shorter the period.

**38.** False. One nautical mile equals about 1.15 statute miles, and one knot is one nautical mile per hour. So, in the time that the car travels 1 statute mile, the ship travels about 1.15 statute miles. Therefore the ship is traveling faster.

**39.** If the building height in feet is $x$, then $\tan 58° = x/50$. So $x = 50 \tan 58° \approx 80$. The answer is D.

**40.** By the Law of Cosines, the distance is

$c = \sqrt{a^2 + b^2 - 2ab\cos\theta}$

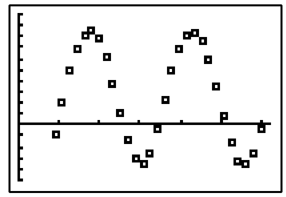
$= \sqrt{40^2 + 20^2 + 2(40)(20)\cos 60°} \approx 53$ naut. mi.

The answer is B.

**41.** Model the tide level as a sinusoidal function of time, $t$. 6 hr, 12 min = 372 min is a half-period, and the amplitude is half of $13 - 9 = 4$. So use the model $f(t) = 2\cos(\pi t/372) - 11$ with $t = 0$ at 8:15 PM. This takes on a value of $-10$ at $t = 124$. The answer is D.

**42.** The answer is A.

**43. (a)**



[0, 0.0062] by [−0.5, 1]

**(b)** The first is the best. This can be confirmed by graphing all three equations.

**(c)** About $\frac{2464}{2\pi} = \frac{1232}{\pi} \approx 392$ oscillations/sec.

**44. (a)** Newborn: about 6 hours. Four-year-old: about 24 hours. Adult: about 24 hours.

**(b)** The adult sleep cycle is perhaps most like a sinusoid, though one might also pick the newborn cycle. At least one can perhaps say that the four-year-old sleep cycles is *least* like a sinusoid.

**45.** The 7-gon can be split into 14 congruent right triangles with a common vertex at the center. The legs of these triangles measure $a$ and 2.5. The angle at the center is $\frac{2\pi}{14} = \frac{\pi}{7}$, so $a = 2.5 \cot\frac{\pi}{7} \approx 5.2$ cm

**46.** The 7-gon can be split into 14 congruent right triangles with a common vertex at the center. The legs of these triangles measure $a$ and 2.5, while the hypotenuse has length $r$. The angle at the center is $\frac{2\pi}{14} = \frac{\pi}{7}$, so

$r = 2.5 \csc\frac{\pi}{7} \approx 5.8$ cm

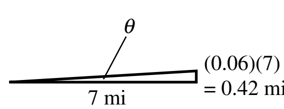
**47.** Choosing point $E$ in the center of the rhombus, we have $\Delta AEB$ with right angle at $E$, and $m\angle EAB = 21°$. Then $AE = 18\cos 21°$ in., $BE = 18 \sin 21°$ in., so that $AC = 2AE \approx 33.6$ in. and $BD = 2\,BE \approx 12.9$ in.

**48. (a)** $BE = 20\tan 50° \approx 23.8$ ft.

**(b)** $CD = BE + 45\tan 20° \approx 40.2$ ft.

**(c)** $AE + ED = 20\sec 50° + 45\sec 20° \approx 79$ ft, so the total distance across the top of the roof is about 158 ft.

**49.** $\theta = \tan^{-1} 0.06 \approx 3.4$

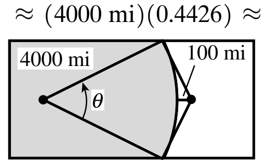



**50.** Observe that there are two (congruent) right triangles with hypotenuse 4100 mi (see figure below). The acute angle adjacent to the 4000 mi leg has measure $\cos^{-1}\frac{4000}{4100}$

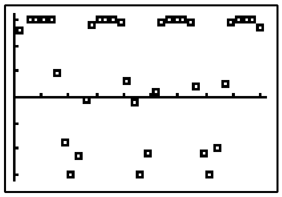
$= \cos^{-1}\frac{40}{41}$, so $\theta = 2\cos^{-1}\frac{40}{41} \approx 25.361°$

$\approx 0.4426$ radians. The arc length is $s = r\theta$

$\approx (4000 \text{ mi})(0.4426) \approx 1771$ mi.



**51. (a)**



$[0, 0.0092]$ by $[-1.6, 1.6]$

**(b)** One pretty good match is $y = 1.51971\sin[2467(t - 0.0002)]$ (that is, $a = 1.51971$, $b = 2467$, $h = 0.0002$). Answers will vary but should be close to these values. A good estimate of $a$ can be found by noting the highest and lowest values of "Pressure" from the data. For the value of $b$, note the time between maxima (approx. $0.0033728 - 0.0008256 = 0.0025472$ sec); this is the period, so $b \approx \dfrac{2\pi}{0.0025472} \approx 2467$. Finally, since 0.0008256 is the location of the first peak after $t = 0$, choose $h$ so that $2467(0.0008256 - h) \approx \dfrac{\pi}{2}$. This gives $h \approx 0.0002$.

**(c)** Frequency: about $\dfrac{2467}{2\pi} \approx \dfrac{1}{0.0025472} \approx 393$ Hz. It appears to be a G.

**(d)** Exercise 41 had $b \approx 2464$, so the frequency is again about 392 Hz; it also appears to be a G.

## ■ Chapter 4 Review

**1.** On the positive $y$-axis (between quadrants I and II); $\dfrac{5\pi}{2} \cdot \dfrac{180°}{\pi} = 450°$.

**2.** Quadrant II; $\dfrac{3\pi}{4} \cdot \dfrac{180°}{\pi} = 135°$.

**3.** Quadrant III; $-135° \cdot \dfrac{\pi}{180°} = -\dfrac{3\pi}{4}$.

**4.** Quadrant IV; $-45° \cdot \dfrac{\pi}{180°} = -\dfrac{\pi}{4}$.

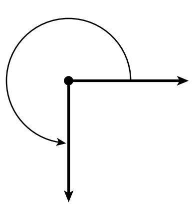
**5.** Quadrant I; $78° \cdot \dfrac{\pi}{180°} = \dfrac{13\pi}{30}$.

**6.** Quadrant II; $112° \cdot \dfrac{\pi}{180°} = \dfrac{28\pi}{45}$.

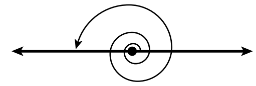
**7.** Quadrant I; $\dfrac{\pi}{12} \cdot \dfrac{180°}{\pi} = 15°$.

**8.** Quadrant II; $\dfrac{7\pi}{10} \cdot \dfrac{180°}{\pi} = 126°$.

**9.** $270°$ or $\dfrac{3\pi}{2}$ radians



**10.** $900°$ or $5\pi$ radians



For #11–16, it may be useful to plot the given points and draw the terminal side to determine the angle. Be sure to make your sketch on a "square viewing window."

**11.** $\theta = \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = 30° = \dfrac{\pi}{6}$ radians

**12.** $\theta = 135° = \dfrac{3\pi}{4}$ radians

**13.** $\theta = 120° = \dfrac{2\pi}{3}$ radians

**14.** $\theta = 225° = \dfrac{5\pi}{4}$ radians

**15.** $\theta = 360° + \tan^{-1}(-2) \approx 296.565 \approx 5.176$ radians

**16.** $\theta = \tan^{-1} 2 \approx 63.435° \approx 1.107$ radians

**17.** $\sin 30° = \dfrac{1}{2}$

**18.** $\cos 330° = \dfrac{\sqrt{3}}{2}$

**19.** $\tan(-135°) = 1$

**20.** $\sec(-135°) = -\sqrt{2}$

**21.** $\sin\dfrac{5\pi}{6} = \dfrac{1}{2}$

**22.** $\csc\dfrac{2\pi}{3} = \dfrac{2}{\sqrt{3}}$

**23.** $\sec\left(-\dfrac{\pi}{3}\right) = 2$

**24.** $\tan\left(-\dfrac{2\pi}{3}\right) = \sqrt{3}$

**25.** $\csc 270° = -1$

**26.** $\sec 180° = -1$

**27.** $\cot(-90°) = 0$

**28.** $\tan 360° = 0$

**29.** Reference angle: $\dfrac{\pi}{6} = 30°$; use a 30–60 right triangle with side lengths $\sqrt{3}$, $(-)1$, and 2 (hypotenuse).
$\sin\left(-\dfrac{\pi}{6}\right) = -\dfrac{1}{2}$, $\cos\left(-\dfrac{\pi}{6}\right) = \dfrac{\sqrt{3}}{2}$, $\tan\left(-\dfrac{\pi}{6}\right) = -\dfrac{1}{\sqrt{3}}$;
$\csc\left(-\dfrac{\pi}{6}\right) = -2$, $\sec\left(-\dfrac{\pi}{6}\right) = \dfrac{2}{\sqrt{3}}$; $\cot\left(-\dfrac{\pi}{6}\right) = -\sqrt{3}$.

**30.** Reference angle: $\dfrac{\pi}{4} = 45°$; use a 45–45 right triangle with side lengths $(-)1$, 1, and $\sqrt{2}$ (hypotenuse).
$\sin\dfrac{19\pi}{4} = \dfrac{1}{\sqrt{2}}$, $\cos\dfrac{19\pi}{4} = -\dfrac{1}{\sqrt{2}}$, $\tan\dfrac{19\pi}{4} = -1$;
$\csc\dfrac{19\pi}{4} = \sqrt{2}$, $\sec\dfrac{19\pi}{4} = -\sqrt{2}$; $\cot\dfrac{19\pi}{4} = -1$.

**31.** Reference angle: 45°; use a 45–45 right triangle with side lengths $(-)1$, $(-)1$, and $\sqrt{2}$ (hypotenuse).
$\sin(-135°) = -\dfrac{1}{\sqrt{2}}$, $\cos(-135°) = -\dfrac{1}{\sqrt{2}}$,
$\tan(-135°) = 1$; $\csc(-135°) = -\sqrt{2}$,
$\sec(-135°) = -\sqrt{2}$, $\cot(-135°) = 1$.

**32.** Reference angle: 60°; use a 30–60 right triangle with side lengths 1, $\sqrt{3}$, and 2 (hypotenuse).
$\sin 420° = \dfrac{\sqrt{3}}{2}$, $\cos 420° = \dfrac{1}{2}$, $\tan 420° = \sqrt{3}$;
$\csc 420° = \dfrac{2}{\sqrt{3}}$, $\sec 420° = 2$, $\cot 420° = \dfrac{1}{\sqrt{3}}$.

**33.** The hypotenuse length is 13 cm, so $\sin\alpha = \frac{5}{13}$, $\cos\alpha = \frac{12}{13}$, $\tan\alpha = \frac{5}{12}$, $\csc\alpha = \frac{13}{5}$, $\sec\alpha = \frac{13}{12}$, $\cot\alpha = \frac{12}{5}$.

For #34–35, since we are using a right triangle, we assume that $\theta$ is acute.

**34.** Draw a right triangle with legs 5 (adjacent) and $\sqrt{7^2 - 5^2} = \sqrt{24} = 2\sqrt{6}$, and hypotenuse 7. $\sin\theta = \frac{2\sqrt{6}}{7}$, $\cos\theta = \frac{5}{7}$, $\tan\theta = \frac{2\sqrt{6}}{5}$; $\csc\theta = \frac{7}{2\sqrt{6}}$, $\sec\theta = \frac{7}{5}$, $\cot\theta = \frac{5}{2\sqrt{6}}$.

**35.** Draw a right triangle with legs 8 (adjacent) and 15, and hypotenuse $\sqrt{8^2 + 15^2} = \sqrt{289} = 17$. $\sin\theta = \frac{15}{17}$, $\cos\theta = \frac{8}{17}$, $\tan\theta = \frac{15}{8}$; $\csc\theta = \frac{17}{15}$, $\sec\theta = \frac{17}{8}$, $\cot\theta = \frac{8}{15}$.

**36.** $\theta \approx 64.623°$

**37.** $x \approx 4.075$ radians

**38.** $x \approx 0.220$ or $x \approx 2.922$ radians

For #39–44, choose whichever of the following formulas is appropriate:

$$a = \sqrt{c^2 - b^2} = c\sin\alpha = c\cos\beta = b\tan\alpha = \frac{b}{\tan\beta}$$

$$b = \sqrt{c^2 - a^2} = c\cos\alpha = c\sin\beta = a\tan\beta = \frac{a}{\tan\alpha}$$

$$c = \sqrt{a^2 + b^2} = \frac{a}{\cos\beta} = \frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{b}{\cos\alpha}$$

If one angle is given, subtract from 90° to find the other angle. If neither $\alpha$ nor $\beta$ is given, find the value of one of the trigonometric functions, then use a calculator to approximate the value of one angle, then subtract from 90° to find the other.

**39.** $a = c\sin\alpha = 15\sin 35° \approx 8.604$, $b = c\cos\alpha = 15\cos 35° \approx 12.287$, $\beta = 90° - \alpha = 55°$

**40.** $a = \sqrt{c^2 - b^2} = \sqrt{10^2 - 8^2} = 6$. For the angles, we know $\cos\alpha = \frac{8}{10} = \frac{4}{5}$; using a calculator, we find $\alpha \approx 36.87°$, so that $\beta = 90° - \alpha \approx 53.13°$.

**41.** $b = a\tan\beta = 7\tan 48° \approx 7.774$, $c = \frac{a}{\cos\beta} = \frac{7}{\cos 48°} \approx 10.461$, $\alpha = 90° - \beta = 42°$

**42.** $a = c\sin\alpha = 8\sin 28° \approx 3.756$, $b = c\cos\alpha \approx 8\cos 28° = 7.064$, $\beta = 90° - \alpha = 62°$

**43.** $a = \sqrt{c^2 - b^2} = \sqrt{7^2 - 5^2} = \sqrt{24} = 2\sqrt{6} \approx 4.90$. For the angles, we know $\cos\alpha = \frac{5}{7}$; using a calculator, we find $\alpha \approx 44.42°$, so that $\beta = 90° - \alpha \approx 45.58°$.

**44.** $c = \sqrt{a^2 + b^2} = \sqrt{2.5^2 + 7.3^2} = \sqrt{59.54} \approx 7.716$. For the angles, we know $\tan\alpha = \frac{2.5}{7.3}$; using a calculator, we find $\alpha \approx 18.90°$, so that $\beta = 90° - \alpha \approx 71.10°$.

**45.** $\sin x < 0$ and $\cos x < 0$: Quadrant III

**46.** $\cos x < 0$ and $\frac{1}{\sin x} > 0$: Quadrant II

**47.** $\sin x > 0$ and $\cos x < 0$: Quadrant II

**48.** $\frac{1}{\cos x} < 0$ and $\frac{1}{\sin x} > 0$: Quadrant II

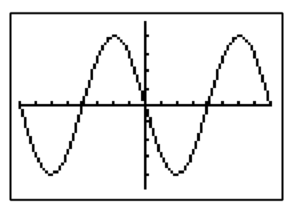
**49.** The distance $OP = 3\sqrt{5}$, so $\sin\theta = \frac{2}{\sqrt{5}}$, $\cos\theta = -\frac{1}{\sqrt{5}}$, $\tan\theta = -2$; $\csc\theta = \frac{\sqrt{5}}{2}$, $\sec\theta = -\sqrt{5}$, $\cot\theta = -\frac{1}{2}$.

**50.** $OP = \sqrt{193}$, so $\sin\theta = \frac{7}{\sqrt{193}}$, $\cos\theta = \frac{12}{\sqrt{193}}$, $\tan\theta = \frac{7}{12}$; $\csc\theta = \frac{\sqrt{193}}{7}$, $\sec\theta = \frac{\sqrt{193}}{12}$, $\cot\theta = \frac{12}{7}$.

**51.** $OP = \sqrt{34}$, so $\sin\theta = -\frac{3}{\sqrt{34}}$, $\cos\theta = -\frac{5}{\sqrt{34}}$, $\tan\theta = \frac{3}{5}$; $\csc\theta = -\frac{\sqrt{34}}{3}$, $\sec\theta = -\frac{\sqrt{34}}{5}$, $\cot\theta = \frac{5}{3}$.

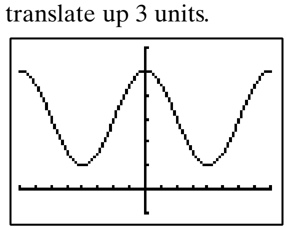
**52.** $OP = \sqrt{97}$, so $\sin\theta = \frac{9}{\sqrt{97}}$, $\cos\theta = \frac{4}{\sqrt{97}}$, $\tan\theta = \frac{9}{4}$; $\csc\theta = \frac{\sqrt{97}}{9}$, $\sec\theta = \frac{\sqrt{97}}{4}$, $\cot\theta = \frac{4}{9}$.

**53.** Starting from $y = \sin x$, translate left $\pi$ units.



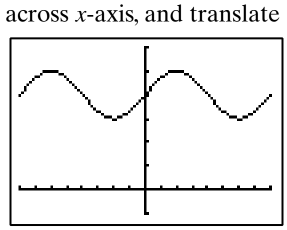
$[-2\pi, 2\pi]$ by $[-1.2, 1.2]$

**54.** Starting from $y = \cos x$, vertically stretch by 2 then translate up 3 units.



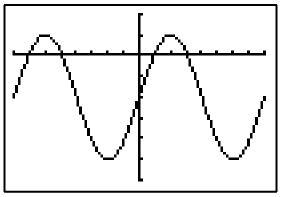
$[-2\pi, 2\pi]$ by $[-1, 6]$

**55.** Starting from $y = \cos x$, translate left $\frac{\pi}{2}$ units, reflect across $x$-axis, and translate up 4 units.



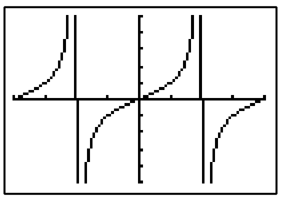
$[-2\pi, 2\pi]$ by $[-1, 6]$

**56.** Starting from $y = \sin x$, translate right $\pi$ units, vertically stretch by 3, reflect across $x$-axis, and translate down 2 units.



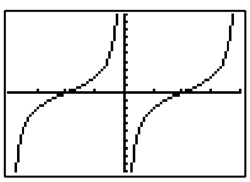
$[-2\pi, 2\pi]$ by $[-6, 2]$

**57.** Starting from $y = \tan x$, horizontally shrink by $\frac{1}{2}$.



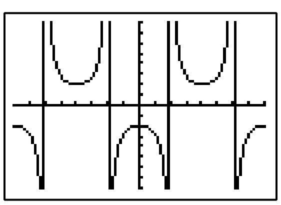
$[-0.5\pi, 0.5\pi]$ by $[-5, 5]$

**58.** Starting from $y = \cot x$, horizontally shrink by $\frac{1}{3}$, vertically stretch by 2, and reflect across $x$-axis (in any order).



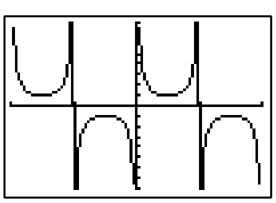
$\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$ by $[-10, 10]$

**59.** Starting from $y = \sec x$, horizontally stretch by 2, vertically stretch by 2, and reflect across $x$-axis (in any order).



$[-4\pi, 4\pi]$ by $[-8, 8]$

**60.** Starting from $y = \csc \pi x$, horizontally shrink by $\frac{1}{\pi}$.



$[-2, 2]$ by $[-5, 5]$

For #61–66, recall that for $y = a \sin[b(x - h)]$ or $y = a \cos[b(x - h)]$, the amplitude is $|a|$, the period is $\frac{2\pi}{|b|}$, and the phase shift is $h$. The domain is always $(-\infty, \infty)$, and the range is $[-|a|, |a|]$.

**61.** $f(x) = 2 \sin 3x$. Amplitude: 2; period: $\frac{2\pi}{3}$; phase shift: 0; domain: $(-\infty, \infty)$; range: $[-2, 2]$.

**62.** $g(x) = 3 \cos 4x$. Amplitude: 3; period: $\frac{\pi}{2}$; phase shift: 0; domain: $(-\infty, \infty)$; range: $[-3, 3]$.

**63.** $f(x) = 1.5 \sin\left[2\left(x - \frac{\pi}{8}\right)\right]$. Amplitude: 1.5; period: $\pi$; phase shift: $\frac{\pi}{8}$; domain: $(-\infty, \infty)$; range: $[-1.5, 1.5]$.

**64.** $g(x) = -2 \sin\left[3\left(x - \frac{\pi}{9}\right)\right]$. Amplitude: 2; period: $\frac{2\pi}{3}$; phase shift: $\frac{\pi}{9}$; domain: $(-\infty, \infty)$; range: $[-2, 2]$.

**65.** $y = 4 \cos\left[2\left(x - \frac{1}{2}\right)\right]$. Amplitude: 4; period: $\pi$; phase shift: $\frac{1}{2}$; domain: $(-\infty, \infty)$; range: $[-4, 4]$.

**66.** $g(x) = -2 \cos\left[3\left(x + \frac{1}{3}\right)\right]$. Amplitude: 2; period: $\frac{2\pi}{3}$; phase shift: $-\frac{1}{3}$; domain: $(-\infty, \infty)$; range: $[-2, 2]$.

For #67–68, graph the function. Estimate $a$ as the amplitude of the graph (i.e., the height of the maximum). Notice that the value of $b$ is always the coefficient of $x$ in the original functions. Finally, note that $a \sin[b(x - h)] = 0$ when $x = h$, so estimate $h$ using a zero of $f(x)$ where $f(x)$ changes from negative to positive.

**67.** $a \approx 4.47$, $b = 1$, and $h \approx 1.11$, so $f(x) \approx 4.47 \sin(x - 1.11)$.

**68.** $a \approx 3.61$, $b = 2$, and $h \approx -1.08$, so $f(x) \approx 3.61 \sin[2(x + 1.08)]$.

**69.** $\approx 49.996° \approx 0.873$ radians

**70.** $\approx 61.380° \approx 1.071$ radians

**71.** $45° = \frac{\pi}{4}$ radians

**72.** $60° = \frac{\pi}{3}$ radians

**73.** Starting from $y = \sin^{-1} x$, horizontally shrink by $\frac{1}{3}$. Domain: $\left[-\frac{1}{3}, \frac{1}{3}\right]$. Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

**74.** Starting from $y = \tan^{-1} x$, horizontally shrink by $\frac{1}{2}$. Domain: $(-\infty, \infty)$. Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

**75.** Starting from $y = \sin^{-1} x$, translate right 1 unit, horizontally shrink by $\frac{1}{3}$, translate up 2 units. Domain: $\left[0, \frac{2}{3}\right]$. Range: $\left[2 - \frac{\pi}{2}, 2 + \frac{\pi}{2}\right]$.

**76.** Starting from $y = \cos^{-1} x$, translate left 1 unit, horizontally shrink by $\frac{1}{2}$, translate down 3 units. Domain: $[-1, 0]$. Range: $[-3, \pi - 3]$.

**77.** $x = \frac{5\pi}{6}$

**78.** $x = \frac{\pi}{6}$

**79.** $x = \frac{3\pi}{4}$

**80.** $\frac{5\pi}{3}$

**81.** $\frac{3\pi}{2}$

**82.** $\frac{5\pi}{6}$

**83.** As $|x| \to \infty$, $\frac{\sin x}{x^2} \to 0$.

**84.** As $x \to \infty$, $\frac{3}{5}e^{-x/12}\sin(2x - 3) \to 0$; as $x \to -\infty$, the function oscillates from positive to negative, and tends to $\infty$ in absolute value.

**85.** $\tan(\tan^{-1} 1) = \tan \frac{\pi}{4} = 1$

**86.** $\cos^{-1}\left(\cos \frac{\pi}{3}\right) = \cos^{-1} \frac{1}{2} = \frac{\pi}{3}$

**87.** $\tan(\sin^{-1} \frac{3}{5}) = \frac{\sin \theta}{\cos \theta}$, where $\theta$ is an angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with $\sin \theta = \frac{3}{5}$. Then $\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{0.64}$ $= 0.80$ and $\tan \theta = 0.75$

**88.** $\cos^{-1} \cos\left(-\frac{\pi}{7}\right) = -\frac{\pi}{7}$

**89.** Periodic; period $\pi$. Domain $x \neq \frac{\pi}{2} + n\pi$, $n$ an integer. Range: $[1, \infty)$.

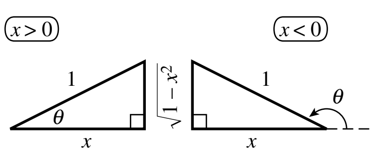
**90.** Not periodic. Domain: $(-\infty, \infty)$. Range: $[-1, 1]$.

**91.** Not periodic. Domain: $x \neq \frac{\pi}{2} + n\pi$, $n$ an integer. Range: $[-\infty, \infty)$.

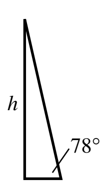
**92.** Periodic; period $2\pi$. Domain: $(-\infty, \infty)$. Range: approximately $[-5, 4.65]$.

**93.** $s = r\theta = (2)\left(\frac{2\pi}{3}\right) = \frac{4\pi}{3}$

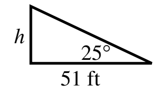
**94.** Draw a right triangle with horizontal leg $x$ (if $x > 0$, draw the horizontal leg right; if $x < 0$, draw it left), vertically leg $\sqrt{1 - x^2}$, and hypotenuse 1. If $x > 0$, let $\theta$ be the acute angle adjacent to the horizontal leg; if $x < 0$, let $\theta$ be the supplement of this angle. Then $\theta = \cos^{-1} x$, so $\tan \theta = \tan(\cos^{-1} x) = \frac{\sqrt{1 - x^2}}{x}$.



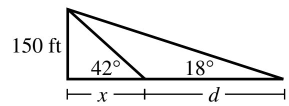

**95.** $\tan 78° = \frac{h}{100 \text{ m}}$, so $h = 100 \tan 78° \approx 470$ m.




**96.** $\tan 25° = \frac{h}{51 \text{ ft}}$, so $h = 51 \tan 25° \approx 23.8$ ft.

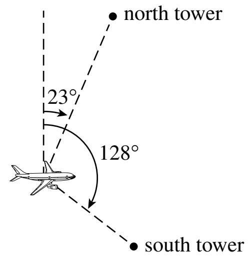



**97.** $\tan 42° = \frac{150 \text{ ft}}{x}$ and $\tan 18° = \frac{150 \text{ ft}}{d + x}$, so $x = 150 \cot 42°$ and $d + x = 150 \cot 18°$. Then $d = 150 \cot 18° - 150 \cot 42° \approx 295$ ft.

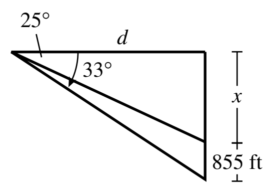



**98.** $\tan 22° = \frac{PQ}{4}$, so $PQ = 4 \tan 22° \approx 1.62$ mi.

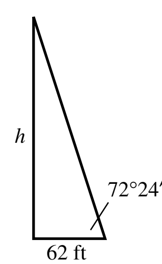
**99.** See figure below.




**100.** $\tan 25° = \frac{x}{d}$ and $\tan 33° = \frac{x + 855}{d}$, so $x = d \tan 25°$ and $x + 855 = d \tan 33°$. Then $d \tan 25° + 855 =$ $d \tan 33°$, so $d = \frac{855}{\tan 33° - \tan 25°} \approx 4670$ ft.



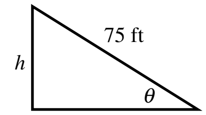

**101.** $\tan 72°24' = \frac{h}{62 \text{ ft}}$, so $h = 62 \tan 72°24' \approx 195.4$ ft




**102.** Let $\theta$ be the angle of elevation. Note that $\sin \theta = \frac{h}{75 \text{ ft}}$, so $h = 75 \sin \theta$.

**(a)** If $\theta = 22°$, then $h = 75 \sin 22° \approx 28$ ft.

**(b)** If $\theta = 27°$, then $h = 75 \sin 27° \approx 34$ ft.



**103.** $s = r\theta = (44 \text{ in.})\left(6° \cdot \frac{\pi}{180°}\right) = \frac{22\pi}{15} \approx 4.6$ in.

**104.** The blade sweeps out $\frac{110}{360} = \frac{11}{36}$ of a circle; take this fraction of (the area of a 20 in.-radius circle minus the area of a 4 in.-radius circle):

$$A = \frac{11}{36}[\pi(20)^2 - \pi(4)^2] = \frac{352\pi}{3} \approx 368.6 \text{ in}^2$$

**105.** *Solve algebraically*: Set $T(x) = 32$ and solve for $x$.

$$37.3 \sin\left[\frac{2\pi}{365}(x - 114)\right] + 26 = 32$$

$$\sin\left[\frac{2\pi}{365}(x - 114)\right] = \frac{6}{37.3}$$

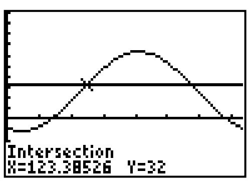
$$\frac{2\pi}{365}(x - 114) = \sin^{-1}\left(\frac{6}{37.3}\right)$$

$$\frac{2\pi}{365}(x - 114) \approx 0.162, 2.98$$

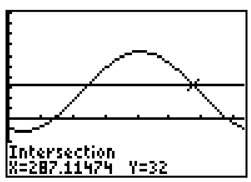
Note: $\sin\theta = \sin(\pi - \theta)$

$$x \approx 123, 287$$

*Solve graphically*: Graph $T(x) = 32$ and $T(x) = 37.3 \sin\left[\frac{2\pi}{365}(x - 114)\right] + 25$ on the same set of axes, and then determine the intersections.




[0, 365] by [–50, 100]




[0, 365] by [–50, 100]

Using either method, we would expect the average temperature to be 32°F on day 123 (May 3) and day 287 (October 14).

**106.** Set $h(x) = 0$ and solve for $x$.

$$0 = 35 \cos\left(\frac{x}{35}\right) + 17$$

$$-\frac{17}{35} = \cos\frac{x}{35}$$

$$\frac{x}{35} = \cos^{-1}\left(-\frac{17}{35}\right) \approx 2.078 \text{ rad}$$

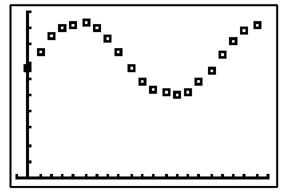
$$x = (35)(2.078)$$

$$x \approx 72.7 \text{ ft.}$$

## Chapter 4 Project

Solutions are based on the sample data shown in the table.

**1.**



$[-0.1, 2.1]$ by $[0, 1]$

**2.** The peak value seems to occur between $x = 0.4$ and $x = 0.5$, so let $h = 0.45$. The difference of the two extreme values is $0.931 - 0.495 = 0.436$, so let $a \approx 0.436/2 \approx 0.22$. The average of the two extreme values is $(0.931 + 0.495)/2 = 0.713$, so let $k = 0.71$. The time interval from $x = 0.5$ to $x = 1.3$, which equals 0.8, is right around a half-period, so let $b = \pi/0.8 \approx 3.93$. Then the equation is $y \approx 0.22 \cos(3.93(x - 0.45)) + 0.71$.

**3.** The constant $a$ represents half the distance the pendulum bob swings as it moves from its highest point to its lowest point. And $k$ represents the distance from the detector to the pendulum bob when it is in mid-swing.

**4.** Since the sine and cosine functions differ only by a phase shift, only $h$ would change.

**5.** The regression yields $y \approx 0.22 \sin(3.87x - 0.16) + 0.71$. Most calculator/computer regression models are expressed in the form $y = a \sin(bx + f) + k$, where $-f/b = h$ in the equation $y = a \sin(b(x - h)) + k$. Here, the regression equation can be rewritten as $y \approx 0.22 \sin(3.87(x - 0.04)) + 0.71$. The difference in the two values of $h$ for the cosine and sine models is 0.41, which is right around a quarter-period, as it should be.