57. (a) $V^2 = (V_{max})^2 \sin^2(120\pi t)$

Using NINT:

$av(V^2) = \frac{1}{1}\int_0^1 (V_{max})^2 \sin^2(120\pi t)\,dt$

$= (V_{max})^2 \int_0^1 \sin^2(120\pi t)\,dt = (V_{max})^2 \frac{1}{2} = \frac{(V_{max})^2}{2}$

$V_{rms} = \sqrt{\frac{(V_{max})^2}{2}} = \frac{V_{max}}{\sqrt{2}}$

(b) $V_{max} = 240\sqrt{2} \approx 339.41$ volts

Chapter 6
Differential Equations and Mathematical Modeling

■ Section 6.1 Antiderivatives and Slope Fields (pp. 303–315)

Exploration 1 Constructing a Slope Field

1. As i and j vary from 1 to 10, 100 ordered pairs are produced. Each ordered pair represents a distinct point in the viewing window.

2. The distance between the points with j fixed and $i = r$ and $i = r + 1$ is the distance between their x-coordinates.

$\left[\text{Xmin} + \left(2(r+1) - 1\right)\frac{h}{2}\right] - \left[\text{Xmin} + (2r - 1)\frac{h}{2}\right]$

$= (\text{Xmin} - \text{Xmin}) + (2r + 2 - 1 - 2r + 1)\frac{h}{2} = h$

3. The distance between the points with i fixed and $j = r$ and $j = r + 1$ is the distance between their y-coordinates.

$\left[\text{Ymin} + (2(r+1) - 1)\frac{k}{2}\right] - \left[\text{Ymin} + (2r - 1)\frac{k}{2}\right]$

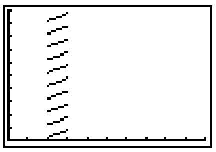
$= (\text{Ymin} - \text{Ymin}) + (2r + 2 - 1 - 2r + 1)\frac{k}{2} = k$

4. Here $h = k = 1$. Each line segment in the third column has slope $\frac{4}{7}$, because the x-coordinate of the midpoint of each line segment is 2.5. The y-coordinates are $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots, \frac{19}{2}$.

The 10 graphs are graphs of the functions

$y = \left(\frac{4}{7}\right)(x - 2.5) + \frac{n}{2}$, $2 \le x \le 3$, for $n = 1, 3, 5, \ldots, 19$.

The length of the line segment can be increased or decreased by adjusting the restriction $2 \le x \le 3$.



[0, 10] by [0, 10]

5. Again $h = k = 1$. The y-coordinate of the midpoint of each line segment is $\frac{7}{2}$. The x-coordinates of the midpoint of each line segment are $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots, \frac{19}{2}$. From left to right the slopes of the line segments are

$\frac{2}{\frac{1}{2}+1}, \frac{2}{\frac{3}{2}+1}, \frac{2}{\frac{5}{2}+1}, \ldots, \frac{2}{\frac{19}{2}+1}$

The 10 graphs are graphs of the functions.

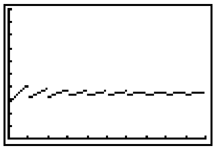
$y_1 = \left(\frac{2}{\frac{1}{2}+1}\right)\left(x - \frac{1}{2}\right) + \frac{7}{2}, \ 0 \le x \le 1,$

$y_2 = \left(\frac{3}{\frac{3}{2}+1}\right)\left(x - \frac{3}{2}\right) + \frac{7}{2}, \ 1 \le x \le 2,$

$y_3 = \left(\frac{2}{\frac{5}{2}+1}\right)\left(x - \frac{5}{2}\right) + \frac{7}{2}, \ 2 \le x \le 3,$

⋮

$y_{10} = \left(\frac{2}{\frac{19}{2}+1}\right)\left(x - \frac{19}{2}\right) + \frac{7}{2}, \ 9 \le x \le 10.$



[0, 10] by [0, 10]

6. For each line segment in part (5), make a column of parallel line segments as in part (4).

7. *WL*

Quick Review 6.1

1. $100(1.06) = \$106.00$

2. $100\left(1 + \frac{0.06}{4}\right)^4 \approx \$106.14$

3. $100\left(1 + \frac{0.06}{12}\right)^{12} \approx \$106.17$

4. $100\left(1 + \frac{0.06}{365}\right)^{365} \approx \$106.18$

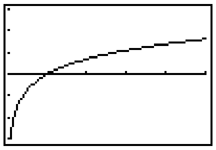
5. $\frac{dy}{dx} = \frac{d}{dx}\sin 3x = (\cos 3x)(3) = 3\cos 3x$

6. $\frac{dy}{dx} = \frac{d}{dx}\tan\frac{5}{2}x = \left(\sec^2\frac{5}{2}x\right)\left(\frac{5}{2}\right) = \frac{5}{2}\sec^2\frac{5}{2}x$

7. $\frac{dy}{dx} = \frac{d}{dx}Ce^{2x} = (Ce^{2x})(2) = 2Ce^{2x}$

8. $\frac{dy}{dx} = \frac{d}{dx}\ln(x+2) = \frac{1}{x+2}$

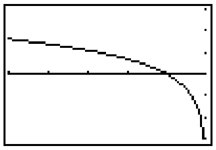
9.



[0.01, 5] by [−3, 3]

By setting the left endpoint at $x = 0.01$ instead of $x = 0$, we avoid an error that occurs when our calculator attempts to calculate $\text{NINT}\left(\frac{1}{x}, x, 1, 0\right)$. The graph appears to be the same as the graph of $y = \ln x$.

10.



[−5, −0.01] by [−3, 3]

By setting the right endpoint at $x = -0.01$ instead of $x = 0$, we avoid an error that occurs when our calculator attempts to calculate $\text{NINT}\left(\frac{1}{x}, x, -1, 0\right)$. The graph appears to be the same as the graph of $y = \ln(-x)$.

Section 6.1 Exercises

1. $\int (x^2 - 2x + 1)\,dx = \frac{x^3}{3} - x^2 + x + C$

Check:

$\frac{d}{dx}\left(\frac{x^3}{3} - x^2 + x + C\right) = x^2 - 2x + 1$

2. $\int (-3x^{-4})\,dx = x^{-3} + C$

Check:

$\frac{d}{dx}(x^{-3} + C) = -3x^{-4}$

3. $\int (x^2 - 4\sqrt{x})\,dx = \int (x^2 - 4x^{1/2})\,dx = \frac{x^3}{3} - \frac{8}{3}x^{3/2} + C$

Check:

$\frac{d}{dx}\left(\frac{x^3}{3} - \frac{8}{3}x^{3/2} + C\right) = x^2 - 4x^{1/2} = x^2 - \sqrt{x}$

4. $\int (8 + \csc x \cot x)\,dx = 8x - \csc x + C$

Check:

$\frac{d}{dx}(8x - \csc x + C) = 8 + \csc x \cot x$

5. $\int e^{4x}\,dx = \frac{1}{4}e^{4x} + C$

Check:

$\frac{d}{dx}\left(\frac{1}{4}e^{4x} + C\right) = e^{4x}$

6. $\int \frac{1}{x+3}\,dx = \ln|x+3| + C$

Check:

$\frac{d}{dx}[\ln|x+3| + C] = \frac{1}{x+3}$

7. $\int (x^5 - 6x + 3)\,dx = \frac{x^6}{6} - 3x^2 + 3x + C$

8. $\int (-x^{-3} + x - 1)\,dx = \frac{x^{-2}}{2} + \frac{x^2}{2} - x + C$

9. $\int \left(e^{t/2} - \frac{5}{t^2}\right)\,dt = \int (e^{t/2} - 5t^{-2})\,dt$

$= 2e^{t/2} + 5t^{-1} + C$

$= 2e^{t/2} + \frac{5}{t} + C$

10. $\int \frac{4}{3}\sqrt[3]{t}\,dt = \int \frac{4}{3}t^{1/3}\,dt = t^{4/3} + C$

11. $\int \left(x^3 - \frac{1}{x^3}\right)\,dx = \int (x^3 - x^{-3})\,dx$

$= \frac{x^4}{4} + \frac{x^{-2}}{2} + C$

$= \frac{x^4}{4} + \frac{1}{2x^2} + C$

12. $\int \left(\sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}\right)\,dx = \int (x^{1/3} + x^{-1/3})\,dx$

$= \frac{3}{4}x^{4/3} + \frac{3}{2}x^{2/3} + C$

13. $\int \frac{1}{3}x^{-2/3}\,dx = x^{1/3} + C$

14. $\int (3\sin x - \sin 3x)\,dx = -3\cos x - \frac{\cos 3x}{3} + C$

15. $\int \frac{\pi}{2}\cos\left(\frac{\pi}{2}x\right)\,dx = \sin\left(\frac{\pi}{2}x\right) + C$

16. $\int 2\sec t \tan t\,dt = 2\sec t + C$

17. $\int \left(\frac{2}{x-1} + \frac{1}{x}\right)\,dx = 2\ln|x-1| + \ln|x| + C$

18. $\int \left(\frac{1}{x-2} + \sin 5x - e^{-2x}\right)\,dx$

$= \ln|x-2| - \frac{\cos 5x}{5} + \frac{e^{-2x}}{2} + C$

19. $\int 5\sec^2 5r\,dr = \tan 5r + C$

20. $\int \csc^2 7t\,dt = -\frac{\cot 7t}{7} + C$

21. $\int \cos^2 x\,dx = \int \frac{1 + \cos 2x}{2}\,dx$

$= \int \left(\frac{1}{2} + \frac{\cos 2x}{2}\right)\,dx$

$= \frac{x}{2} + \frac{\sin 2x}{4} + C$

22. $\int \sin^2 x\,dx = \int \frac{1 - \cos 2x}{2}\,dx$

$= \int \left(\frac{1}{2} - \frac{\cos 2x}{2}\right)\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$

23. $\int \tan^2\theta\,d\theta = \int (\sec^2\theta - 1)\,d\theta = \tan\theta - \theta + C$

24. $\int \cot^2 t\, dt = \int (\csc^2 t - 1)\, dt = -\cot t - t + C$

25. **(a)** Graph (b)

(b) The slope is always positive, so (a) and (c) can be ruled out.

26. **(a)** Graph (b)

(b) The solution should have positive slope when x is negative, zero slope when x is zero and negative slope when x is positive since slope $= \frac{dy}{dx} = -x$. Graphs (a) and (c) don't show this slope pattern.

27. $\frac{dy}{dx} = 2x - 1$

$\int \frac{dy}{dx}\, dx = \int (2x - 1)\, dx$

$y = x^2 - x + C$

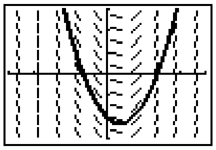
Initial condition: $y(2) = 0$

$0 = 2^2 - 2 + C$

$0 = 2 + C$

$-2 = C$

Solution: $y = x^2 - x - 2$



$[-4, 4]$ by $[-3, 3]$

28. $\frac{dy}{dx} = \frac{1}{x^2} + x$

$\int \frac{dy}{dx}\, dx = \int (x^{-2} + x)\, dx$

$y = -x^{-1} + \frac{x^2}{2} + C$

$y = \frac{x^2}{2} - \frac{1}{x} + C$

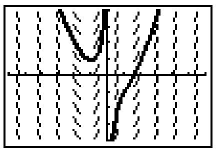
Initial condition: $y(2) = 1$

$1 = \frac{2^2}{2} - \frac{1}{2} + C$

$1 = \frac{3}{2} + C$

$-\frac{1}{2} = C$

Solution: $y = \frac{x^2}{2} - \frac{1}{x} - \frac{1}{2}$



$[-6, 6]$ by $[-4, 4]$

29. $\frac{dy}{dx} = \sec^2 x$

$\int \frac{dy}{dx}\, dx = \int \sec^2 x\, dx$

$y = \tan x + C$

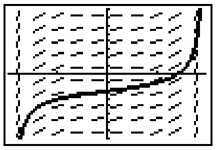
Initial condition: $y\left(\frac{\pi}{4}\right) = -1$

$-1 = \tan \frac{\pi}{4} + C$

$-1 = 1 + C$

$-2 = C$

Solution: $y = \tan x - 2$



$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ by $[-8, 8]$

30. $\frac{dy}{dx} = x^{-2/3}$

$\int \frac{dy}{dx}\, dx = \int x^{-2/3}\, dx$

$y = 3x^{1/3} + C$

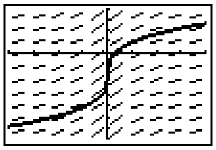
Initial condition: $y(-1) = -5$

$-5 = 3(-1)^{1/3} + C$

$-5 = -3 + C$

$-2 = C$

$y = 3x^{1/3} - 2$



$[-4, 4]$ by $[-8, 4]$

31. $\frac{dy}{dx} = 9x^2 - 4x + 5$

$\int \frac{dy}{dx}\, dx = \int (9x^2 - 4x + 5)\, dx$

$y = 3x^3 - 2x^2 + 5x + C$

Initial condition: $y(-1) = 0$

$0 = 3(-1)^3 - 2(-1)^2 + 5(-1) + C$

$0 = -10 + C$

$10 = C$

Solution: $y = 3x^3 - 2x^2 + 5x + 10$

32. $\dfrac{dy}{dx} = \cos x + \sin x$

$\displaystyle\int \frac{dy}{dx}\,dx = \int (\cos x + \sin x)\,dx$

$y = \sin x - \cos x + C$

Initial condition: $y(\pi) = 1$

$1 = \sin \pi - \cos \pi + C$

$1 = 1 + C$

$0 = C$

Solution: $y = \sin x - \cos x$

33. $\dfrac{dy}{dt} = 2e^{-t}$

$\displaystyle\int \frac{dy}{dt}\,dt = \int 2e^{-t}\,dt$

$y = -2e^{-t} + C$

Initial condition: $y(\ln 2) = 0$

$0 = -2e^{-\ln 2} + C$

$0 = -\dfrac{2}{2} + C$

$1 = C$

Solution: $y = -2e^{-t} + 1$

34. $\dfrac{dy}{dx} = \dfrac{1}{x}$

$\displaystyle\int \frac{dy}{dx}\,dx = \int \frac{1}{x}\,dx$

$y = \ln |x| + C$

Initial condition: $y(e^3) = 0$

$0 = \ln (e^3) + C$

$0 = 3 + C$

$-3 = C$

Solution: $y = \ln |x| - 3$

35. $\dfrac{d^2y}{d\theta^2} = \sin \theta$

$\displaystyle\int \frac{d^2y}{d\theta^2}\,d\theta = \int \sin \theta\,d\theta$

$\dfrac{dy}{d\theta} = -\cos \theta + C_1$

Initial condition: $y'(0) = 0$

$0 = -\cos 0 + C_1$

$0 = -1 + C_1$

$1 = C_1$

First derivative: $\dfrac{dy}{d\theta} = -\cos \theta + 1$

$\displaystyle\int \frac{dy}{d\theta}\,d\theta = \int (-\cos \theta + 1)\,d\theta$

$y = -\sin \theta + \theta + C_2$

Initial condition: $y(0) = -3$

$-3 = -\sin 0 + 0 + C_2$

$-3 = C_2$

Solution: $y = -\sin \theta + \theta - 3$

36. $\dfrac{d^2y}{dx^2}\,dx = 2 - 6x$

$\displaystyle\int \frac{d^2y}{dx^2}\,dx = \int (2 - 6x)\,dx$

$\dfrac{dy}{dx} = 2x - 3x^2 + C_1$

Initial condition: $y'(0) = 4$

$4 = 2(0) - 3(0)^2 + C_1$

$4 = C_1$

First derivative: $\dfrac{dy}{dx} = 2x - 3x^2 + 4$

$\displaystyle\int \frac{dy}{dx}\,dx = \int (2x - 3x^2 + 4)\,dx$

$y = x^2 - x^3 + 4x + C_2$

Initial condition: $y(0) = 1$

$1 = 0^2 - 0^3 + 4(0) + C_2$

$1 = C_2$

Solution: $y = x^2 - x^3 + 4x + 1$

or $y = -x^3 + x^2 + 4x + 1$

37. $\dfrac{d^3y}{dt^3} = \dfrac{1}{t^3}$

$\displaystyle\int \frac{d^3y}{dt^3}\,dt = \int t^{-3}\,dt$

$\dfrac{d^2y}{dt^2} = -\dfrac{1}{2}t^{-2} + C_1$

Initial condition: $y''(1) = 2$

$2 = -\dfrac{1}{2}(1)^{-2} + C_1$

$2 = -\dfrac{1}{2} + C_1$

$\dfrac{5}{2} = C_1$

Second derivative: $\dfrac{d^2y}{dt^2} = -\dfrac{1}{2}t^{-2} + \dfrac{5}{2}$

$\displaystyle\int \frac{d^2y}{dt^2}\,dt = \int\left(-\frac{1}{2}t^{-2} + \frac{5}{2}\right)\,dt$

$\dfrac{dy}{dt} = \dfrac{1}{2}t^{-1} + \dfrac{5}{2}t + C_2$

Initial condition: $y'(1) = 3$

$3 = \dfrac{1}{2}(1)^{-1} + \dfrac{5}{2}(1) + C_2$

$3 = 3 + C_2$

$0 = C_2$

First derivative: $\dfrac{dy}{dt} = \dfrac{1}{2}t^{-1} + \dfrac{5}{2}t$

$\displaystyle\int \frac{dy}{dt}\,dt = \int\left(\frac{1}{2}t^{-1} + \frac{5}{2}t\right)\,dt$

$y = \dfrac{1}{2}\ln|t| + \dfrac{5}{4}t^2 + C_3$

Initial condition: $y(1) = 1$

$1 = \dfrac{1}{2}\ln 1 + \dfrac{5}{4}(1)^2 + C_3$

$1 = \dfrac{5}{4} + C_3$

$-\dfrac{1}{4} = C_3$

Solution: $y = \dfrac{1}{2}\ln|t| + \dfrac{5}{4}t^2 - \dfrac{1}{4}$

38. $\dfrac{d^4y}{d\theta^4} = \sin\theta + \cos\theta$

$\displaystyle\int \frac{d^4y}{d\theta^4}\,d\theta = \int(\sin\theta + \cos\theta)\,d\theta$

$\dfrac{d^3y}{d\theta^3} = -\cos\theta + \sin\theta + C_1$

Initial condition: $y^{(3)}(0) = -3$

$-3 = -\cos 0 + \sin 0 + C_1$

$-3 = -1 + C_1$

$-2 = C_1$

Third derivative: $\dfrac{d^3y}{d\theta^3} = -\cos\theta + \sin\theta - 2$

$\displaystyle\int \frac{d^3y}{d\theta^3}\,d\theta = \int(-\cos\theta + \sin\theta - 2)\,d\theta$

$\dfrac{d^2y}{d\theta^2} = -\sin\theta - \cos\theta - 2\theta + C_2$

Initial condition: $y''(0) = -1$

$-1 = -\sin 0 - \cos 0 - 2(0) + C_2$

$-1 = -1 + C_2$

$0 = C_2$

Second derivative: $\dfrac{d^2y}{d\theta^2} = -\sin\theta - \cos\theta - 2\theta$

$\displaystyle\int \frac{d^2y}{d\theta^2}\,d\theta = \int(-\sin\theta - \cos\theta - 2\theta)\,d\theta$

$\dfrac{dy}{d\theta} = \cos\theta - \sin\theta - \theta^2 + C_3$

Initial condition: $y'(0) = -1$

$-1 = \cos 0 - \sin 0 - 0^2 + C_3$

$-1 = 1 + C_3$

$-2 = C_3$

First derivative: $\dfrac{dy}{d\theta} = \cos\theta - \sin\theta - \theta^2 - 2$

$\displaystyle\int \frac{dy}{d\theta}\,d\theta = \int(\cos\theta - \sin\theta - \theta^2 - 2)\,d\theta$

$y = \sin\theta + \cos\theta - \dfrac{\theta^3}{3} - 2\theta + C_4$

Initial condition: $y(0) = -3$

$-3 = \sin 0 + \cos 0 - \dfrac{0^3}{3} - 2(0) + C_4$

$-3 = 1 + C_4$

$-4 = C_4$

Solution: $y = \sin\theta + \cos\theta - \dfrac{\theta^3}{3} - 2\theta - 4$

39. $\dfrac{ds}{dt} = v = 9.8t + 5$

$\displaystyle\int \frac{ds}{dt}\,dt = \int(9.8t + 5)\,dt$

$s = 4.9t^2 + 5t + C$

Initial condition: $s(0) = 10$
$10 = 4.9(0)^2 + 5(0) + C$
$10 = C$
Solution: $s = 4.9t^2 + 5t + 10$

40. $\frac{ds}{dt} = v = \sin \pi t$

$\int \frac{ds}{dt}\,dt = \int \sin \pi t\,dt$

$s = -\frac{1}{\pi}\cos \pi t + C$

Initial condition: $s(1) = 0$

$0 = -\frac{1}{\pi}\cos \pi + C$

$0 = \frac{1}{\pi} + C$

$-\frac{1}{\pi} = C$

Solution: $s = -\frac{1}{\pi}\cos \pi t - \frac{1}{\pi}$

or $s = -\frac{1}{\pi}(1 + \cos \pi t)$

41. $\frac{dv}{dt} = a = 32$

$\int \frac{dv}{dt}\,dt = \int 32\,dt$

$v = 32t + C_1$

Initial condition: $v(0) = 20$

$20 = 32(0) + C_1$

$20 = C_1$

Velocity: $\frac{ds}{dt} = v = 32t + 20$

$\int \frac{ds}{dt}\,dt = \int (32t + 20)\,dt$

$s = 16t^2 + 20t + C_2$

Initial condition: $s(0) = 0$

$0 = 16(0)^2 + 20(0) + C_2$

$0 = 0$

Solution: $s = 16t^2 + 20t$

42. $\frac{dv}{dt} = a = \cos t$

$\int \frac{dv}{dt}\,dt = \int \cos t\,dt$

$v = \sin t + C_1$

Initial condition: $v(0) = -1$

$-1 = \sin 0 + C_1$

$-1 = C_1$

Velocity: $\frac{ds}{dt} = v = \sin t - 1$

$\int \frac{ds}{dt}\,dt = \int (\sin t - 1)\,dt$

$s = -\cos t - t + C_2$

Initial condition: $s(0) = 1$

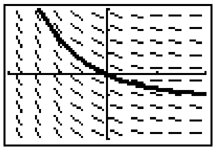
$1 = -\cos 0 - 0 + C_2$

$1 = -1 + C_2$

$2 = C_2$

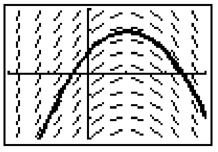
Solution: $s = -\cos t - t + 2$

43.



[−2, 2] by [−3, 3]

44.



[−2, 3] by [−3, 3]

45. $\frac{d}{dx}(\tan^{-1} x + C) = \frac{1}{1 + x^2}$

46. $\frac{d}{dx}(\sin^{-1} x + C) = \frac{1}{\sqrt{1 - x^2}}$

47. $\frac{d}{dx}(\sec^{-1} x + C) = \frac{1}{|x|\sqrt{x^2 - 1}}$

48. $\frac{d}{dx}(-\cos^{-1} x + C) = \frac{1}{\sqrt{1 - x^2}}$

49. **(a)** $\frac{dy}{dx} = x - \frac{1}{x^2}$

$\int \frac{dy}{dx}\,dx = \int (x - x^{-2})\,dx$

$y = \frac{x^2}{2} + x^{-1} + C = \frac{x^2}{2} + \frac{1}{x} + C$

Initial condition: $y(1) = 2$

$2 = \frac{1^2}{2} + \frac{1}{1} + C$

$2 = \frac{3}{2} + C$

$\frac{1}{2} = C$

Solution: $y = \frac{x^2}{2} + \frac{1}{x} + \frac{1}{2},\ x > 0$

49. continued

(b) Again, $y = \frac{x^2}{2} + \frac{1}{x} + C.$

Initial condition: $y(-1) = 1$

$1 = \frac{(-1)^2}{2} + \frac{1}{(-1)} + C$

$1 = -\frac{1}{2} + C$

$\frac{3}{2} = C$

Solution: $y = \frac{x^2}{2} + \frac{1}{x} + \frac{3}{2}, x < 0$

(c) For $x < 0$, $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{x} + \frac{x^2}{2} + C_1\right)$

$= -\frac{1}{x^2} + x$

$= x - \frac{1}{x^2}.$

For $x > 0$, $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{x} + \frac{x^2}{2} + C_2\right)$

$= -\frac{1}{x^2} + x$

$= x - \frac{1}{x^2}.$

And for $x = 0$, $\frac{dy}{dx}$ is undefined.

(d) Let C_1 be the value from part (b), and let C_2 be the value from part (a). Thus, $C_1 = \frac{3}{2}$ and $C_2 = \frac{1}{2}$.

(e) $y(2) = -1$ $\qquad$ $y(-2) = 2$

$-1 = \frac{1}{2} + \frac{2^2}{2} + C_2$ $\qquad$ $2 = \frac{1}{(-2)} + \frac{(-2)^2}{2} + C_1$

$-1 = \frac{5}{2} + C_2$ $\qquad$ $2 = \frac{3}{2} + C_1$

$-\frac{7}{2} = C_2$ $\qquad$ $\frac{1}{2} = C_1$

Thus, $C_1 = \frac{1}{2}$ and $C_2 = -\frac{7}{2}$.

50. $\int \frac{dr}{dx}\, dx = \int (3x^2 - 6x + 12)\, dx$

$r = x^3 - 3x^2 + 12x + C$

Initial condition: $r(0) = 0$

$0 = 0^3 - 3(0)^2 + 12(0) + C$

$0 = C$

Solution: $r(x) = x^3 - 3x^2 + 12x$

51. $\int \frac{dc}{dx}\, dx = \int (3x^2 - 12x + 15)\, dx$

$c = x^3 - 6x^2 + 15x + C$

Initial condition $c(0) = 400$

$400 = 0^3 - 6(0)^2 + 15(0) + C$

$400 = C$

Solution: $c(x) = x^3 - 6x^2 + 15x + 400$

52. (a) $\int f(x)\, dx = \int \frac{d}{dx}(x^2e^x)\, dx = x^2e^x + C$

(b) $\int g(x)\, dx = \int \frac{d}{dx}(x \sin x)\, dx = x \sin x + C$

(c) $\int [-f(x)]\, dx = -\int f(x)\, dx = -x^2e^x + C$

(d) $\int [-g(x)]\, dx = -\int g(x)\, dx = -x \sin x + C$

(e) $\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx$

$= x^2e^x + x \sin x + C$

(f) $\int [f(x) - g(x)]\, dx = \int f(x)\, dx - \int g(x)\, dx$

$= x^2e^x - x \sin x + C$

(g) $\int [x + f(x)]\, dx = \int x\, dx + \int f(x)\, dx = \frac{x^2}{2} + x^2e^x + C$

(h) $\int [g(x) - 4]\, dx = \int g(x)\, dx - \int 4\, dx = x \sin x - 4x + C$

53. (a) $\int \frac{d^2s}{dt^2}\, dt = \int -k\, dt$

$\frac{ds}{dt} = -kt + C_1$

Initial condition: $\frac{ds}{dt} = 88$ when $t = 0$

$88 = (-k)(0) + C_1$

$88 = C_1$

Velocity: $\frac{ds}{dt} = -kt + 88$

$\int \frac{ds}{dt}\, dt = \int (-kt + 88)\, dt$

$s = -\frac{k}{2}t^2 + 88t + C_2$

Initial condition: $s = 0$ when $t = 0$

$0 = -\frac{k}{2}(0)^2 + 88(0) + C_2$

$0 = C_2$

Solution: $s = -\frac{kt^2}{2} + 88t$

(b)

$$\frac{ds}{dt} = 0$$

$$-kt + 88 = 0$$

$$t = \frac{88}{k}$$

(c)

$$s\left(\frac{88}{k}\right) = 242$$

$$-\frac{k}{2}\left(\frac{88}{k}\right)^2 + 88\left(\frac{88}{k}\right) = 242$$

$$\frac{3872}{k} = 242$$

$$k = 16 \text{ ft/sec}^2$$

54. We first solve $\frac{d^2s}{dt^2} = -k$ with the initial conditions $s'(0) = 44$ and $s(0) = 0$.

$\int \frac{d^2s}{dt^2}\,dt = -k$

$\frac{ds}{dt} = -kt + C_1$

Initial condition: $s'(0) = 44$

$44 = (-k)(0) + C_1$

$44 = C_1$

Velocity: $\frac{ds}{dt} = -kt + 44$

$\int \frac{ds}{dt}\,dt = \int(-kt + 44)\,dt$

$s = -\frac{k}{2}t^2 + 44t + C_2$

Initial condition: $s(0) = 0$

$0 = -\frac{k}{2}(0)^2 + 44(0) + C_2$

$0 = C_2$

Position: $s = -\frac{k}{2}t^2 + 44t$

Now, $\frac{ds}{dt} = -kt + 44 = 0$ when $t = \frac{44}{k}$, so it takes $\frac{44}{k}$ seconds to stop, and we require:

$$\begin{aligned} s\left(\frac{44}{k}\right) &= 45 \\ -\frac{k}{2}\left(\frac{44}{k}\right)^2 + 44\left(\frac{44}{k}\right) &= 45 \\ \frac{968}{k} &= 45 \\ k = \frac{968}{45} &\approx 21.5 \end{aligned}$$

It requires a constant deceleration of approximately 21.5 ft/sec^2.

55. $\int \frac{d^2s}{dt^2}\,dt = \int -5.2\,dt$

$\frac{ds}{dt} = -5.2t + C_1$

Initial condition: $\frac{ds}{dt} = 0$ when $t = 0$

$0 = -5.2(0) + C_1$

$0 = C_1$

Velocity: $\frac{ds}{dt} = -5.2t$

$\int \frac{ds}{dt}\,dt = \int -5.2\,dt$

$s = -2.6t^2 + C_2$

Initial condition: $s = 4$ when $t = 0$

$4 = -2.6(0)^2 + C_2$

$4 = C_2$

Position: $s(t) = -2.6t^2 + 4$

Solving $s(t) = 0$, we have $t^2 = \frac{4}{2.6}$, so the positive solution is $t \approx 1.240$ sec. They took about 1.240 sec to fall.

56. Solving $\frac{d^2s}{dt^2} = a$, $s(0) = s_0$, and $v(0) = v_0$:

$\int \frac{d^2s}{dt^2}\,dt = \int a\,dt$

$\frac{ds}{dt} = at + C_1$

Initial condition: $s'(0) = v_0$

$v_0 = (a)(0) + C_1$

$v_0 = C_1$

Velocity: $\frac{ds}{dt} = at + v_0$

$\int \frac{ds}{dt}\,dt = \int (at + v_0)\,dt$

$s = \frac{a}{2}t^2 + v_0 t + C_2$

Initial condition: $s(0) = s_0$

$s_0 = \frac{a}{2}(0)^2 + (v_0)(0) + C_2$

$s_0 = C_2$

Position: $s = \frac{a}{2}t^2 + v_0 t + s_0$

57. We use the method of Example 7.

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{2}{5}h\right)^2 h = \frac{4\pi}{75}h^3$

$$\begin{aligned} \frac{dV}{dt} &= \frac{d}{dt}\left(\frac{4\pi}{75}h^3\right) \\ -\frac{1}{6}\sqrt{h} &= \frac{4\pi}{25}h^2\frac{dh}{dt} \\ -\frac{25}{24\pi} &= h^{3/2}\,\frac{dh}{dt} \\ -\int \frac{25}{24\pi}\,dt &= \int h^{3/2}\frac{dh}{dt}\,dt \\ -\frac{25}{24\pi}t &= \frac{2}{5}h^{5/2} + C \end{aligned}$$

Initial condition: $h = 10$ when $t = 0$.

$$\begin{aligned} -\frac{25}{24\pi}(0) &= \frac{2}{5}(10)^{5/2} + C \\ C &= -\frac{2}{5}(10)^{5/2} \\ -\frac{25}{24\pi}t &= \frac{2}{5}h^{5/2} - \frac{2}{5}(10)^{5/2} \\ -\frac{125t}{48\pi} &= h^{5/2} - 10^{5/2} \\ h^{5/2} &= -\frac{125t}{48\pi} + 10^{5/2} \\ h &= \left(-\frac{125t}{48\pi} + 10^{5/2}\right)^{2/5} \end{aligned}$$

The height is given by $h = \left(-\frac{125t}{48\pi} + 10^{5/2}\right)^{2/5}$ and the volume is given by

$V = \frac{4\pi}{75}h^3 = \frac{4\pi}{75}\left(-\frac{125t}{48\pi} + 10^{5/2}\right)^{6/5}$.

58. (a) $y = 500e^{0.0475t}$

(b) $1000 = 500e^{0.0475t}$

$2 = e^{0.0475t}$

$\ln 2 = 0.0475t$

$t = \frac{\ln 2}{0.0475} \approx 14.6$

It will take approximately 14.6 years.

59. (a) $y = 1200e^{0.0625t}$

(b) $3600 = 1200e^{0.0625t}$

$3 = e^{0.0625t}$

$\ln 3 = 0.0625t$

$t = \frac{\ln 3}{0.0625} \approx 17.6$

It will take approximately 17.6 years.

60. (a) $\int x^2 \cos x\, dx = \int_0^x t^2 \cos t\, dt + C$

(b) We require $\int_0^0 t^2 \cos t\, dt + C = 1$, so $C = 1$.

The required antiderivative is $\int_0^x t^2 \cos t\, dt + 1$.

61. (a) $\int xe^x\, dx = \int_0^x te^t\, dt + C$

(b) We require $\int_0^0 te^t\, dt + C = 1$, so $C = 1$.

The required antiderivative is $\int_0^x te^t\, dt + 1$.

62. (a) $\int \frac{d^2y}{dx^2}\, dx = \int 6x\, dx$

$\frac{dy}{dx} = 3x^2 + C_1$

Initial condition (horizontal tangent): $y'(0) = 0$

$0 = 3(0)^2 + C_1$

$0 = C_1$

First derivative: $\frac{dy}{dx} = 3x^2$

$\int \frac{dy}{dx}\, dx = \int 3x^2\, dx$

$y = x^3 + C_2$

Initial condition (contains (0, 1)): $y(0) = 1$

$1 = (0)^3 + C_2$

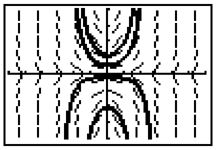
$1 = C_2$

Solution: $y = x^3 + 1$

(b) Only one function satisfies the differential equation on

$(-\infty, \infty)$ and the initial conditions.

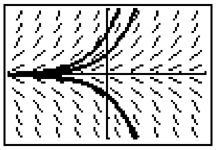
63. Use differential equation graphing mode.
For reference, the equations of the solution curves are as follows.
(1, 1): $y = e^{(x^2-1)/2}$
(−1, 2): $y = 2e^{(x^2-1)/2}$
(0, −2): $y = -2e^{x^2/2}$
(−2, −1): $y = -e^{(x^2-4)/2}$



[−6, 6] by [−4, 4]

The concavity of each solution curve indicates the sign of y''.

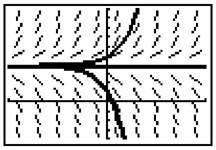
64. Use differential equation graphing mode.
For reference, the equations of the solution curves are as follows.
(0, 1): $y = e^x$
(0, 2): $y = 2e^x$
(0, −1): $y = -e^x$



[−4, 4] by [−3, 3]

The concavity of each solution curve indicates the sign of y''.

65. Use differential equation graphing mode.
For reference, the equations of the solution curves are as follows.
(0, 1): $y = -3e^{2x} + 4$
(0, 4): $y = 4$
(0, 5): $y = e^{2x} + 4$



[−3, 3] by [−4, 10]

The concavity of each solution curve indicates the sign of y''.

66. Use differential equation graphing mode.

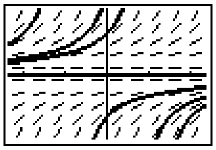
For reference, the equations of the solution curves are as follows.

(0, 1): $y = -\frac{1}{x-1}$

(0, 2): $y = -\frac{2}{2x-1}$

(0, −1): $y = -\frac{1}{x+1}$

(0, 0): $y = 0$



[−2.35, 2.35] by [−1.55, 1.55]

The concavity of each solution curve indicates the sign of y''.

67. (a) $\frac{d}{dx}(\ln x + C) = \frac{1}{x}$ for $x > 0$

(b) $\frac{d}{dx}[\ln(-x) + C] = \frac{1}{-x}\frac{d}{dx}(-x) = \left(\frac{1}{-x}\right)(-1) = \frac{1}{x}$ for $x < 0$

(c) For $x > 0$, $\ln|x| + C = \ln x + C$, which is a solution to the differential equation, as we showed in part (a). For $x < 0$, $\ln|x| + C = \ln(-x) + C$, which is a solution to the differential equation, as we showed in part (b). Thus, $\frac{d}{dx}\ln|x| = \frac{1}{x}$ for all x except 0.

(d) For $x < 0$, we have $y = \ln(-x) + C_2$, which is a solution to the differential equation, as we showed in part (a). For $x > 0$, we have $y = \ln x + C_1$, which is a solution to the differential equation, as we showed part (b). Thus, $\frac{dy}{dx} = \frac{1}{x}$ for all x except 0.

■ Section 6.2 Integration by Substitution (pp. 315–323)

Exploration 1 Supporting Indefinite Integrals Graphically

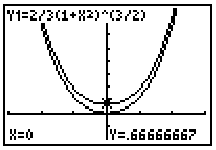
1. $\int \sqrt{1+x^2} \cdot 2x\,dx = \int \sqrt{u}\,du$

$= \frac{2}{3}u^{3/2} + C$

$= \frac{2}{3}(1+x^2)^{3/2} + C$

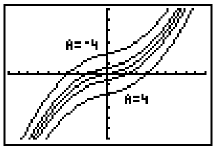
2. Their derivatives are equal: $\frac{dy_1}{dx} = \frac{dy_2}{dx} = \sqrt{1+x^2} \cdot 2x$.

3. $y_1 = y_2 + \frac{2}{3}$. By Corollary 3 to the Mean Value Theorem of Section 4.2, y_1 and y_2 must differ by a constant. We find that constant by evaluating the two functions at $x = 0$.



[−3, 3] by [−2, 8]

4.



[−10, 10] by [−30, 30]

5. The derivative with respect to x of each function graphed in part (4) is equal to $\sqrt{1+x^2}$.

Exploration 2 Two Routes to the Integral

1. $\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \int_0^2 \sqrt{u}\,du = \frac{2}{3}u^{3/2}\Big]_0^2 = \frac{4\sqrt{2}}{3}$

2. $\int 3x^2\sqrt{x^3+1}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} = \frac{2}{3}(x^3+1)^{3/2}$ so $\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \frac{2}{3}(x^3+1)^{3/2}\Big]_{-1}^{1} = \frac{4\sqrt{2}}{3}$.

Quick Review 6.2

1. $\int_0^2 x^4\,dx = \frac{1}{5}x^5\Big]_0^2 = \frac{1}{5}(2)^5 - \frac{1}{5}(0)^5 = \frac{32}{5}$

2. $\int_1^5 \sqrt{x-1}\,dx = \int_1^5 (x-1)^{1/2}\,dx = \frac{2}{3}(x-1)^{3/2}\Big]_1^5$

$= \frac{2}{3}(4)^{3/2} - \frac{2}{3}(0)^{3/2}$

$= \frac{2}{3}(8) = \frac{16}{3}$

3. $\frac{dy}{dx} = 3^x$

4. $\frac{dy}{dx} = 3^x$

5. $\frac{dy}{dx} = 4(x^3 - 2x^2 + 3)^3(3x^2 - 4x)$

6. $\frac{dy}{dx} = 2\sin(4x-5)\cos(4x-5) \cdot 4$

$= 8\sin(4x-5)\cos(4x-5)$

7. $\frac{dy}{dx} = \frac{1}{\cos x} \cdot -\sin x = -\tan x$

8. $\frac{dy}{dx} = \frac{1}{\sin x} \cdot \cos x = \cot x$

9. $\frac{dy}{dx} = \frac{1}{\sec x + \tan x} \cdot (\sec x \tan x + \sec^2 x)$

$= \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$

$= \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x}$

$= \sec x$

10. $\frac{dy}{dx} = \frac{1}{\csc x + \cot x}(-\csc x \cot x - \csc^2 x)$

$= -\frac{\csc x \cot x + \csc^2 x}{\csc x + \cot x}$

$= -\frac{\csc x(\cot x + \csc x)}{\csc x + \cot x}$

$= -\csc x$

Section 6.2 Exercises

1. $u = 3x$

$du = 3\,dx$

$\frac{1}{3}du = dx$

$\int \sin 3x\,dx = \frac{1}{3}\int \sin u\,du$

$= -\frac{1}{3}\cos u + C$

$= -\frac{1}{3}\cos 3x + C$

Check: $\frac{d}{dx}\left(-\frac{1}{3}\cos 3x + C\right) = -\frac{1}{3}(-\sin 3x)(3) = \sin 3x$

2. $u = 2x^2$

$du = 4x\,dx$

$x\,dx = \frac{1}{4}\,du$

$\int x\cos(2x^2)\,dx = \frac{1}{4}\int \cos u\,du$

$= \frac{1}{4}\sin u + C$

$= \frac{1}{4}\sin(2x^2) + C$

Check: $\frac{d}{dx}\left(\frac{1}{4}\sin(2x^2) + C\right) = \frac{1}{4}\cos(2x^2)(4x) = x\cos(2x^2)$

3. $u = 2x$

$du = 2\,dx$

$\frac{1}{2}\,du = dx$

$\int \sec 2x \tan 2x\,dx = \frac{1}{2}\int \sec u \tan u\,du$

$= \frac{1}{2}\sec u + C$

$= \frac{1}{2}\sec 2x + C$

Check: $\frac{d}{dx}\left(\frac{1}{2}\sec 2x + C\right) = \frac{1}{2}\sec 2x \tan 2x \cdot 2 = \sec 2x \tan$

$2x$

4. $u = 7x - 2$

$du = 7\,dx$

$\frac{1}{7}\,du = dx$

$\int 28(7x-2)^3\,dx = \frac{1}{7}\int 28u^3\,du = u^4 + C = (7x-2)^4 + C$

Check: $\frac{d}{dx}[(7x-2)^4 + C] = 4(7x-2)^3(7) = 28(7x-2)^3$

5. $u = \frac{x}{3}$

$du = \frac{1}{3}\,dx$

$3\,du = dx$

$\int \frac{dx}{x^2+9} = \int \frac{3u}{9u^2+9}$

$= \frac{3}{9}\int \frac{du}{u^2+1}$

$= \frac{1}{3}\int \frac{du}{u^2+1}$

$= \frac{1}{3}\tan^{-1} u + C$

$= \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$

Check: $\frac{d}{dx}\left(\frac{1}{3}\tan^{-1}\frac{x}{3} + C\right) = \frac{1}{3}\,\frac{1}{1+\left(\frac{x}{3}\right)^2} \cdot \frac{1}{3} = \frac{1}{9+x^2}$

6. $u = 1 - r^3$

$du = -3r^2\,dr$

$-\frac{1}{3}\,du = r^2\,dr$

$\int \frac{9r^2\,dr}{\sqrt{1-r^3}} = 9\left(-\frac{1}{3}\right)\int \frac{du}{\sqrt{u}}$

$= -3\int u^{-1/2}\,du$

$= -3(2)u^{1/2} + C$

$= -6\sqrt{1-r^3} + C$

Check: $\frac{d}{dx}(-6\sqrt{1-r^3} + C) \quad = -6\left(\frac{1}{2\sqrt{1-r^3}}\right)(-3r^2)$

$= \frac{9r^2}{\sqrt{1-r^3}}$

7. $u = 1 - \cos\frac{t}{2}$

$du = \frac{1}{2}\sin\frac{t}{2}\,dt$

$2\,du = \sin\frac{t}{2}\,dt$

$\int \left(1 - \cos\frac{t}{2}\right)^2 \sin\frac{t}{2}\,dt = 2\int u^2\,du$

$= \frac{2}{3}u^3 + C$

$= \frac{2}{3}\left(1 - \cos\frac{t}{2}\right)^3 + C$

Check: $\frac{d}{dx}\left[\frac{2}{3}\left(1 - \cos\frac{t}{2}\right)^3 + C\right]$

$= 2\left(1 - \cos\frac{t}{2}\right)^2\left(\sin\frac{t}{2}\right)\left(\frac{1}{2}\right)$

$= \left(1 - \cos\frac{t}{2}\right)^2 \sin\frac{t}{2}$

8. $u = y^4 + 4y^2 + 1$

$du = (4y^3 + 8y)\,dy$

$du = 4(y^3 + 2y)\,dy$

$\frac{1}{4}\,du = (y^3 + 2y)\,dy$

$$\int 8(y^4 + 4y^2 + 1)^2(y^3 + 2y)\,dy = 8\left(\frac{1}{4}\right)\int u^2\,du = \frac{2}{3}u^3 + C = \frac{2}{3}(y^4 + 4y^2 + 1)^3 + C$$

Check: $\frac{d}{dx}\left[\frac{2}{3}(y^4 + 4y^2 + 1)^3 + C\right]$

$= 2(y^4 + 4y^2 + 1)^2(4y^3 + 8y)$

$= 8(y^4 + 4y^2 + 1)^2(y^3 + 2y)$

9. Let $u = 1 - x$

$du = -dx$

$$\int \frac{dx}{(1-x)^2} = -\int \frac{du}{u^2} = u^{-1} + C = \frac{1}{1-x} + C$$

10. Let $u = x + 2$

$du = dx$

$$\int \sec^2 (x+2)\,dx = \int \sec^2 u\,du = \tan u + C = \tan (x+2) + C$$

11. Let $u = \tan x$

$du = \sec^2 x\,dx$

$$\int \sqrt{\tan x}\sec^2 x\,dx = \int u^{1/2}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(\tan x)^{3/2} + C$$

12. Let $u = \theta + \frac{\pi}{2}$

$du = d\theta$

$$\int \sec\left(\theta + \frac{\pi}{2}\right)\tan\left(\theta + \frac{\pi}{2}\right)d\theta = \int \sec u \tan u\,du = \sec u + C = \sec\left(\theta + \frac{\pi}{2}\right) + C$$

13. Let $u = \ln x$

$du = \frac{1}{x}\,dx$

$$\int_e^6 \frac{dx}{x \ln x} = \int_1^{\ln 6} \frac{du}{u} = \ln |u|\Big]_1^{\ln 6} = \ln (\ln 6)$$

14. Let $u = \tan x$

$du = \sec^2 x\,dx$

$$\int_{-\pi/4}^{\pi/4} \tan^2 x \sec^2 x\,dx = \int_{-1}^{1} u^2\,du = \frac{1}{3}u^3\Big]_{-1}^{1} = \frac{1}{3}(1) - \frac{1}{3}(-1)^3 = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

15. Let $u = 3z + 4$

$du = 3\,dz$

$\frac{1}{3}\,du = dz$

$$\int \cos (3z+4)\,dz = \frac{1}{3}\int \cos u\,du = \frac{1}{3}\sin u + C = \frac{1}{3}\sin (3z+4) + C$$

16. Let $u = \cot x$

$du = -\csc^2 x\,dx$

$$\int \sqrt{\cot x}\csc^2 x\,dx = -\int u^{1/2}\,du = -\frac{2}{3}u^{3/2} + C = -\frac{2}{3}(\cot x)^{3/2} + C$$

17. Let $u = \ln x$

$du = \frac{1}{x}\,dx$

$$\int \frac{\ln^6 x}{x}\,dx = \int u^6\,du = \frac{1}{7}u^7 + C = \frac{1}{7}(\ln^7 x) + C$$

18. Let $u = \tan \frac{x}{2}$

$du = \frac{1}{2}\sec^2 \frac{x}{2}\,dx$

$$\int \tan^7 \frac{x}{2}\sec^2 \frac{x}{2}\,dx = 2\int u^7\,du = 2 \cdot \frac{1}{8}u^8 + C = \frac{1}{4}\tan^8 \frac{x}{2} + C$$

19. Let $u = s^{4/3} - 8$

$du = \frac{4}{3}s^{1/3}\,ds$

$\frac{3}{4}du = s^{1/3}\,ds$

$$\int s^{1/3}\cos(s^{4/3} - 8)\,ds = \frac{3}{4}\int \cos u\,du$$
$$= \frac{3}{4}\sin u + C$$
$$= \frac{3}{4}\sin(s^{4/3} - 8) + C$$

20. $\int \frac{dx}{\sin^2 3x} = \int \csc^2 3x\,dx$

Let $u = 3x$

$du = 3\,dx$

$\frac{1}{3}\,du = dx$

$$\int \csc^2 3x\,dx = \frac{1}{3}\int \csc^2 u\,du$$
$$= -\frac{1}{3}\cot u + C$$
$$= -\frac{1}{3}\cot(3x) + C$$

21. Let $u = \cos(2t + 1)$

$du = -\sin(2t + 1)(2)\,dt$

$-\frac{1}{2}du = \sin(2t + 1)\,dt$

$$\int \frac{\sin(2t + 1)}{\cos^2(2t + 1)}\,dt = -\frac{1}{2}\int u^{-2}\,du$$
$$= \frac{1}{2}u^{-1} + C$$
$$= \frac{1}{2\cos(2t + 1)} + C$$
$$= \frac{1}{2}\sec(2t + 1) + C$$

22. Let $u = 2 + \sin t$

$du = \cos t\,dt$

$$\int \frac{6\cos t}{(2 + \sin t)^2}\,dt = 6\int u^{-2}\,du$$
$$= -6u^{-1} + C$$
$$= -\frac{6}{2 + \sin t} + C$$

23. $\int_{\pi/4}^{3\pi/4} \cot x\,dx = \int_{\pi/4}^{3\pi/4} \frac{\cos x}{\sin x}\,dx$

Let $u = \sin x$

$du = \cos x\,dx$

$$\int_{\pi/4}^{3\pi/4} \frac{\cos x}{\sin x}\,dx = \int_{x=\pi/4}^{x=3\pi/4} \frac{1}{u}\,du$$
$$= \ln|u|\Big]_{x=\pi/4}^{x=3\pi/4}$$
$$= \ln|\sin x|\Big]_{\pi/4}^{3\pi/4}$$
$$= \ln\left|\frac{\sqrt{2}}{2}\right| - \ln\left|\frac{\sqrt{2}}{2}\right| = 0$$

24. Let $u = x + 2$

$du = dx$

$$\int_0^7 \frac{dx}{x + 2} = \int_2^9 \frac{1}{u}\,du$$
$$= \ln u\Big]_2^9$$
$$= \ln 9 - \ln 2 \approx 1.504$$

25. Let $u = x^2 + 1$

$du = 2x\,dx$

$x\,dx = \frac{1}{2}\,du$

$$\int_{-1}^{3} \frac{x\,dx}{x^2 + 1} = \frac{1}{2}\int_2^{10} \frac{1}{u}\,du$$
$$= \frac{1}{2}\ln|u|\Big]_2^{10}$$
$$= \frac{1}{2}(\ln 10 - \ln 2) = \frac{1}{2}\ln 5 \approx 0.805$$

26. $\int_0^5 \frac{40\,dx}{x^2+25} = \int_0^5 \frac{\frac{40}{25}}{\left(\frac{x}{5}\right)^2 + \left(\frac{25}{25}\right)^2}\,dx$

$= \frac{40}{25}\int_0^5 \frac{1}{\left(\frac{x}{5}\right)^2 + 1}\,dx$

Let $u = \frac{x}{5}$

$du = \frac{1}{5}\,dx$

$5\,du = dx$

$\int_0^5 \frac{40\,dx}{x^2+25} = \frac{8}{5}(5)\int_0^1 \frac{1}{u^2+1}\,du$

$= 8 \arctan u\Big]_0^1$

$= 8(\arctan 1)$

$= 8\left(\frac{\pi}{4}\right) = 2\pi$

27. $\int \frac{dx}{\cot 3x} = \int \frac{\sin 3x}{\cos 3x}\,dx$

Let $u = \cos 3x$

$du = -3 \sin 3x\,dx$

$-\frac{1}{3}\,du = \sin 3x\,dx$

$\int \frac{dx}{\cot 3x} = -\frac{1}{3}\int \frac{1}{u}\,du$

$= -\frac{1}{3}\ln|u| + C$

$= -\frac{1}{3}\ln|\cos 3x| + C$

(An equivalent expression is $\frac{1}{3}\ln|\sec 3x| + C$.)

28. Let $u = 5x + 8$

$du = 5\,dx$

$\frac{1}{5}\,du = dx$

$\int \frac{dx}{\sqrt{5x+8}} = \frac{1}{5}\int u^{-1/2}\,du$

$= \frac{1}{5} \cdot 2u^{1/2} + C$

$= \frac{2}{5}\sqrt{5x+8} + C$

29. $\int \sec x\,dx = \int \sec x \cdot \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right)dx$

$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx$

Let $u = \sec x + \tan x$

$du = \sec x \tan x + \sec^2 x\,dx$

$\int \sec x\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|\sec x + \tan x| + C$

30. $\int \csc x\,dx = \int \csc x\left(\frac{\csc x + \cot x}{\csc x + \cot x}\right)dx$

$= \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x}\,dx$

Let $u = \csc x + \cot x$

$du = -\csc x \cot x - \csc^2 x\,dx$

$\int \csc x\,dx = -\int \frac{1}{u}\,du$

$= -\ln|u| + C$

$= -\ln|\csc x + \cot x| + C$

31. Let $u = y + 1$

$du = dy$

$\int_0^3 \sqrt{y+1}\,dy = \int_1^4 u^{1/2}\,du$

$= \frac{2}{3}u^{3/2}\Big]_1^4$

$= \frac{2}{3}(4)^{3/2} - \frac{2}{3}(1)^{3/2}$

$= \frac{2}{3}(8) - \frac{2}{3} = \frac{14}{3}$

32. Let $u = 1 - r^2$

$du = -2r\,dr$

$-\frac{1}{2}\,du = r\,dr$

$\int_0^1 r\sqrt{1-r^2}\,dr = -\frac{1}{2}\int_1^0 u^{1/2}\,du$

$= -\frac{1}{2} \cdot \frac{2}{3}u^{3/2}\Big]_1^0$

$= -\frac{1}{3}(0) + \frac{1}{3}(1) = \frac{1}{3}$

33. Let $u = \tan x$

$du = \sec^2 x\,dx$

$\int_{-\pi/4}^0 \tan x \sec^2 x\,dx = \int_{-1}^0 u\,du$

$= \frac{1}{2}u^2\Big]_{-1}^0$

$= \frac{1}{2}(0) - \frac{1}{2}(-1)^2$

$= -\frac{1}{2}$

34. Let $u = 4 + r^2$

$du = 2r\,dr$

$\frac{1}{2}\,du = r\,dr$

$\int_{-1}^1 \frac{5r}{(4+r^2)^2}\,dr = \frac{5}{2}\int_5^5 u^{-2}\,du = 0$

35. Let $u = 1 + \theta^{3/2}$

$du = \frac{3}{2}\theta^{1/2}\, d\theta$

$\frac{2}{3}\, du = \theta^{1/2}\, d\theta$

$$\int_0^1 \frac{10\sqrt{\theta}}{(1+\theta^{3/2})^2}\, d\theta = \frac{2}{3}(10)\int_1^2 u^{-2}\, du = -\frac{20}{3}u^{-1}\Big]_1^2 = -\frac{20}{3}\left(\frac{1}{2} - 1\right) = -\frac{20}{3}\left(-\frac{1}{2}\right) = \frac{10}{3}$$

36. Let $u = 4 + 3 \sin x$

$du = 3 \cos x\, dx$

$\frac{1}{3}\, du = \cos x\, dx$

$$\int_{-\pi}^{\pi} \frac{\cos x}{\sqrt{4 + 3\sin x}}\, dx = \frac{1}{3}\int_4^4 u^{-1/2}\, du = 0$$

37. Let $u = t^5 + 2t$

$du = (5t^4 + 2)\, dt$

$$\int_0^1 \sqrt{t^5 + 2t}\,(5t^4 + 2)\, dt = \int_0^3 u^{1/2}\, du = \frac{2}{3}u^{3/2}\Big]_0^3 = \frac{2}{3}(3)^{3/2} = \frac{2}{3}\sqrt{27} = 2\sqrt{3}$$

38. Let $u = \cos 2\theta$

$du = -2 \sin 2\theta\, d\theta$

$-\frac{1}{2}\, du = \sin 2\theta\, d\theta$

$$\int_0^{\pi/6} \cos^{-3} 2\theta \sin 2\theta\, d\theta = -\frac{1}{2}\int_1^{1/2} u^{-3}\, du = -\frac{1}{2}\cdot\left(-\frac{1}{2}\right)u^{-2}\Big]_1^{1/2} = \frac{1}{4}\left(\left(\frac{1}{2}\right)^{-2} - 1\right) = \frac{1}{4}(3) = \frac{3}{4}$$

39. $\frac{dy}{dx} = (y + 5)(x + 2)$

$\frac{dy}{y+5} = (x + 2)dx$

Integrate both sides.

$\int \frac{dy}{y+5} = \int (x + 2)\, dx$

On the left,

let $u = y + 5$

$du = dy$

$\int \frac{1}{u}\, du = \frac{1}{2}x^2 + 2x + C$

$\ln |u| = \frac{1}{2}x^2 + 2x + C$

$\ln |y + 5| = \frac{1}{2}x^2 + 2x + C$

$|y + 5| = e^{(1/2)x^2 + 2x + C}$

$|y + 5| = e^C e^{(1/2)x^2 + 2x}$

We now let $C' = e^C$ or $C' = -e^C$, depending on whether $(y + 5)$ is positive or negative. Then

$y + 5 = C' e^{(1/2)x^2 + 2x}$

$y = C' e^{(1/2)x^2 + 2x} - 5$

Since C' represents an arbitrary constant (note that even the value $C' = 0$ gives a solution to the original differential equation), we may write the solution as

$y = Ce^{(1/2)x^2 + 2x} - 5.$

40. $\frac{dy}{dx} = x\sqrt{y} \cos^2 \sqrt{y}$

$\frac{dy}{\sqrt{y}\cos^2\sqrt{y}} = x\, dx$

Integrate both sides.

$\int \frac{dy}{\sqrt{y}\cos^2\sqrt{y}} = \int x\, dx$

On the left, let $u = \sqrt{y}$

$du = \frac{1}{2}y^{-1/2}\, dy$

$2\, du = y^{-1/2}\, dy$

$2\int \frac{du}{\cos^2 u} = \frac{1}{2}x^2 + C$

$2\int \sec^2 u\, du = \frac{1}{2}x^2 + C$

$2 \tan u = \frac{1}{2}x^2 + C$

$2 \tan \sqrt{y} = \frac{1}{2}x^2 + C$

$\tan \sqrt{y} = \frac{1}{4}x^2 + C$

(Note: technically, C is now $C' = \frac{C}{2}$. But C's are generic.)

$\sqrt{y} = \tan^{-1}\left(\frac{x^2}{4} + C\right)$

$y = \left[\tan^{-1}\left(\frac{x^2}{4} + C\right)\right]^2$

41. $\frac{dy}{dx} = (\cos x)e^{y+\sin x}$

$\frac{dy}{dx} = (\cos x)(e^y e^{\sin x})$

$\frac{dy}{e^y} = \cos x\, e^{\sin x}\, dx$

Integrate both sides.

$\int \frac{dy}{e^y} = \int \cos x\, e^{\sin x}\, dx$

On the right, let $u = \sin x$

$du = \cos x\, dx$

$-e^{-y} = \int e^u\, du$

$-e^{-y} = e^u + C$

$-e^{-y} = e^{\sin x} + C$

$e^{-y} = -e^{\sin x} + C$

(Note: technically C is now $C' = -C$.)

$-y = \ln(C - e^{\sin x})$

$y = -\ln(C - e^{\sin x})$

42. $\frac{dy}{dx} = e^{x-y}$

$\frac{dy}{dx} = e^x e^{-y}$

$\frac{dy}{e^{-y}} = e^x\, dx$

Integrate both sides.

$\int \frac{dy}{e^{-y}} = \int e^x\, dx$

$\int e^y\, dy = \int e^x\, dx$

$e^y = e^x + C$

$y = \ln(e^x + C)$

43. $\frac{dy}{dx} = -2xy^2$

$-\frac{dy}{y^2} = 2x\, dx$

$-\int \frac{dy}{y^2} = \int 2x\, dx$

$y^{-1} = x^2 + C$

$y = \frac{1}{x^2 + C}$

$y(1) = \frac{1}{1 + C} = 0.25$

$1 + C = 4$

$C = 3$

$y = \frac{1}{x^2 + 3}$

44. $\frac{dy}{dx} = \frac{4\sqrt{y}\ln x}{x}$

$\frac{dy}{\sqrt{y}} = 4\frac{\ln x}{x}\, dx$

Integrate both sides.

$\int \frac{dy}{\sqrt{y}} = 4\int \frac{\ln x}{x}\, dx$

On the right, let $u = \ln x$

$du = \frac{1}{x}\, dx$

$2y^{1/2} = 4\int u\, du$

$2y^{1/2} = 4\left(\frac{1}{2}u^2\right) + C$

$2y^{1/2} = 2(\ln x)^2 + C$

$y^{1/2} = (\ln x)^2 + C$

$y = [(\ln x)^2 + C]^2$

$y(e) = [(\ln e)^2 + C]^2 = 1$

$(1 + C)^2 = 1$

$C = 0$

$y = (\ln x)^4$

Note: Absolute value signs are not needed because the original problem involved $\ln x$, so we know that $x > 0$.

45. (a) Let $u = x + 1$

$du = dx$

$\int \sqrt{x+1}\, dx = \int u^{1/2}\, du$

$= \frac{2}{3}u^{3/2} + C$

$= \frac{2}{3}(x+1)^{3/2} + C$

Alternatively, $\frac{d}{dx}\left(\frac{2}{3}(x+1)^{3/2} + C\right) = \sqrt{x+1}$.

(b) By Part 1 of the Fundamental Theorem of Calculus, $\frac{dy_1}{dx} = \sqrt{x+1}$ and $\frac{dy_2}{dx} = \sqrt{x+1}$, so both are antiderivatives of $\sqrt{x+1}$.

(c) Using NINT to find the values of y_1 and y_2, we have:

x	0	1	2	3	4
y_1	0	1.219	2.797	4.667	6.787
y_2	−4.667	−3.448	−1.869	0	2.120
$y_1 - y_2$	4.667	4.667	4.667	4.667	4.667

$C = 4\frac{2}{3}$

(d) $C = y_1 - y_2$

$= \int_0^x \sqrt{x+1}\, dx - \int_3^x \sqrt{x+1}\, dx$

$= \int_0^x \sqrt{x+1}\, dx + \int_x^3 \sqrt{x+1}\, dx$

$= \int_0^3 \sqrt{x+1}\, dx$

46. (a) $\frac{d}{dx}[F(x) + C]$ should equal $f(x)$.

(b) The slope field should help you visualize the solution curve $y = F(x)$.

(c) The graphs of $y_1 = F(x)$ and $y_2 = \int_0^x f(t)\,dt$ should differ only by a vertical shift C.

(d) A table of values for $y_1 - y_2$ should show that $y_1 - y_2 = C$ for any value of x in the appropriate domain.

(e) The graph of f should be the same as the graph of NDER of $F(x)$.

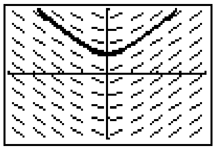
(f) First, we need to find $F(x)$. Let $u = x^2 + 1$, $du = 2x\,dx$.

$$\int \frac{x}{\sqrt{x^2+1}}\,dx = \int \frac{1}{2}u^{-1/2}\,du$$
$$= u^{1/2}$$
$$= \sqrt{x^2+1} + C$$

Therefore, we may let $F(x) = \sqrt{x^2+1}$.

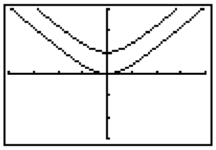
a) $$\frac{d}{dx}(\sqrt{x^2+1} + C) = \frac{1}{2\sqrt{x^2+1}}(2x)$$
$$= \frac{x}{\sqrt{x^2+1}} = f(x)$$

b)



[−4, 4] by [−3, 3]

c)

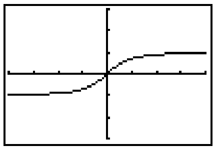


[−4, 4] by [−3, 3]

d)

x	0	1	2	3	4
y_1	1.000	1.414	2.236	3.162	4.123
y_2	0.000	0.414	1.236	2.162	3.123
$y_1 - y_2$	1	1	1	1	1

e)



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47. Let $u = x^4 + 9$, $du = 4x^3\,dx$.

(a) $$\int_0^1 \frac{x^3\,dx}{\sqrt{x^4+9}} = \int_9^{10} \frac{1}{4}u^{-1/2}\,du = \frac{1}{2}u^{1/2}\Big]_9^{10}$$
$$= \frac{1}{2}\sqrt{10} - \frac{1}{2}\sqrt{9}$$
$$= \frac{1}{2}\sqrt{10} - \frac{3}{2} \approx 0.081$$

(b) $$\int \frac{x^3}{x^4+9}\,dx = \int \frac{1}{4}u^{-1/2}\,du$$
$$= \frac{1}{2}u^{1/2} + C$$
$$= \frac{1}{2}\sqrt{x^4+9} + C$$
$$\int_0^1 \frac{x^3}{x^4+9}\,dx = \frac{1}{2}\sqrt{x^4+9}\Big]_0^1$$
$$= \frac{1}{2}\sqrt{10} - \frac{1}{2}\sqrt{9}$$
$$= \frac{1}{2}\sqrt{10} - \frac{3}{2} \approx 0.081$$

48. Let $u = 1 - \cos 3x$, $du = 3\sin 3x\,dx$.

(a) $$\int_{\pi/6}^{\pi/3} (1 - \cos 3x)\sin 3x\,dx = \int_1^2 \frac{1}{3}u\,du = \frac{1}{6}u^2\Big]_1^2$$
$$= \frac{1}{6}(2)^2 - \frac{1}{6}(1)^2 = \frac{1}{2}$$

(b) $$\int (1 - \cos 3x)\sin 3x\,dx = \int \frac{1}{3}u\,du$$
$$= \frac{1}{6}u^2 + C$$
$$= \frac{1}{6}(1 - \cos 3x)^2 + C$$
$$\int_{\pi/6}^{\pi/3} (1 - \cos 3x)\sin 3x\,dx = \frac{1}{6}(1 - \cos 3x)^2\Big]_{\pi/6}^{\pi/3}$$
$$= \frac{1}{6}(2)^2 - \frac{1}{6}(1)^2 = \frac{1}{2}$$

49. We show that $f'(x) = \tan x$ and $f(3) = 5$, where

$$f(x) = \ln\left|\frac{\cos 3}{\cos x}\right| + 5.$$
$$f'(x) = \frac{d}{dx}\left(\ln\left|\frac{\cos 3}{\cos x}\right| + 5\right)$$
$$= \frac{d}{dx}(\ln|\cos 3| - \ln|\cos x| + 5)$$
$$= -\frac{d}{dx}\ln|\cos x|$$
$$= -\frac{1}{\cos x}(-\sin x) = \tan x$$
$$f(3) = \ln\left|\frac{\cos 3}{\cos 3}\right| + 5 = (\ln 1) + 5 = 5$$

50. (a) $u = \cot 2\theta$, $du = -2\csc^2 2\theta\, d\theta$

$$\int \csc^2 2\theta \cot 2\theta\, d\theta = -\frac{1}{2}\int u\, du = -\frac{1}{2}\cdot\frac{u^2}{2} + C = -\frac{u^2}{4} + C = -\frac{1}{4}\cot^2 2\theta + C$$

$F_1(\theta) = -\frac{1}{4}\cot^2 2\theta$

(b) $u = \csc 2\theta$, $du = -2\csc 2\theta \cot 2\theta\, d\theta$

$$\int \csc^2 2\theta \cot 2\theta\, du = -\frac{1}{2}\int u\, du = -\frac{1}{2}\cdot\frac{u^2}{2} + C = -\frac{u^2}{4} + C = -\frac{1}{4}\csc^2 2\theta + C$$

$F_2(\theta) = -\frac{1}{4}\csc^2 2\theta$

(c) $F_1'(\theta) = \left(-\frac{1}{2}\cot 2\theta\right)(-2\csc^2 2\theta) = \csc^2 2\theta \cot 2\theta$

$F_2'(\theta) = \left(-\frac{1}{2}\csc 2\theta\right)(-2\csc 2\theta\cot 2\theta)$

$= \csc^2 2\theta \cot 2\theta$

(d) $F_1(\theta) = F_2(\theta) + b$

$-\frac{1}{4}\cot^2 2\theta = -\frac{1}{4}\csc^2 2\theta + b$

$b = \frac{1}{4}(\csc^2 2\theta - \cot^2 2\theta)$

$= \frac{1}{4}\left(\frac{1 - \cos^2 2\theta}{\sin^2 2\theta}\right) = \frac{1}{4}\left(\frac{\sin^2 2\theta}{\sin^2 2\theta}\right) = \frac{1}{4}$

51. (a) $u = \sin x$, $du = \cos x\, dx$

$$\int 2\sin x\cos x\, dx = \int 2u\, du = u^2 + C = \sin^2 x + C$$

(b) $u = \cos x$, $du = -\sin x\, dx$

$$\int 2\sin x \cos x\, dx = \int(-2u)\, du = -u^2 + C = -\cos^2 x + C$$

(c) $u = 2x$, $du = 2\, dx$

$$\int 2\sin x\cos x\, dx = \int \sin 2x\, dx = \int \frac{1}{2}\sin u\, du = -\frac{1}{2}\cos u + C = -\frac{1}{2}\cos 2x + C$$

(d) $\frac{d}{dx}(\sin^2 x + C) = 2\sin x\cos x$

$\frac{d}{dx}(-\cos^2 x + C) = (-2\cos x)(-\sin x) = 2\sin x\cos x$

$\frac{d}{dx}\left(-\frac{1}{2}\cos 2x + C\right) = \left(\frac{1}{2}\sin 2x\right)(2)$

$= \sin 2x$

$= 2\sin x\cos x$

■ Section 6.3 Integration by Parts
(pp. 323–329)

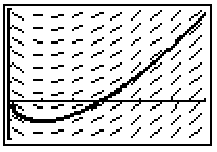
Exploration 1 Evaluating and Checking Integrals

1. $u = \ln x \Rightarrow du = \frac{dx}{x}$ and $dv = dx \Rightarrow v = x$. Thus,

$$\int \ln x\, dx = \int u\, dv = uv - \int v\, du = x\ln x - \int dx = x\ln x - x + C$$

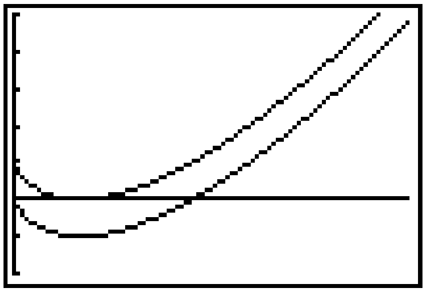
2. $\frac{d}{dx}(x\ln x - x) = \ln x + x\left(\frac{1}{x}\right) - 1 = \ln x$

3. The slope field of $\frac{dy}{dx} = \ln x$ shows the direction of the curve as it is graphed from left to right across the window.



[0, 6] by [−2, 5]

4. The graph of $y_2 = x \ln x - x$ appears to be a vertical shift of the graph of $y_1 = \int_1^x \ln t\, dt$ (down 1 unit). Thus, y_2 appears to be an antiderivative of $\ln x$ which supports $x \ln x - x + C$ as the set of all antiderivatives of $\ln x$.



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Quick Review 6.3

1. $\frac{dy}{dx} = (x^3)(\cos 2x)(2) + (\sin 2x)(3x^2)$

$= 2x^3 \cos 2x + 3x^2 \sin 2x$

2. $\frac{dy}{dx} = (e^{2x})\left(\frac{3}{3x+1}\right) + \ln(3x+1)(2e^{2x})$

$= \frac{3e^{2x}}{3x+1} + 2e^{2x}\ln(3x+1)$

3. $\frac{dy}{dx} = \frac{1}{1+(2x)^2} \cdot 2$

$= \frac{2}{1+4x^2}$

4. $\frac{dy}{dx} = \frac{1}{\sqrt{1-(x+3)^2}}$

5. $y = \tan^{-1} 3x$

$\tan y = 3x$

$x = \frac{1}{3}\tan y$

6. $y = \cos^{-1}(x+1)$

$\cos y = x + 1$

$x = \cos y - 1$

7. $\int_0^1 \sin \pi x\, dx = -\frac{1}{\pi}\cos \pi x\Big]_0^1$

$= -\frac{1}{\pi}\cos \pi + \frac{1}{\pi}\cos 0$

$= -\frac{1}{\pi}(-1) + \frac{1}{\pi} = \frac{2}{\pi}$

8. $\frac{dy}{dx} = e^{2x}$

$dy = e^{2x}\, dx$

Integrate both sides.

$\int dy = \int e^{2x}\, dx$

$y = \frac{1}{2}e^{2x} + C$

9. $\frac{dy}{dx} = x + \sin x$

$dy = (x + \sin x)dx$

Integrate both sides.

$\int dy = \int (x + \sin x)\, dx$

$y = \frac{1}{2}x^2 - \cos x + C$

$y(0) = -1 + C = 2$

$C = 3$

$y = \frac{1}{2}x^2 - \cos x + 3$

10. $\frac{d}{dx}\left(\frac{1}{2}e^x(\sin x - \cos x)\right)$

$= \frac{1}{2}e^x(\cos x + \sin x) + (\sin x - \cos x)\frac{1}{2}e^x$

$= \frac{1}{2}e^x \cos x + \frac{1}{2}e^x \sin x + \frac{1}{2}e^x \sin x - \frac{1}{2}e^x \cos x$

$= e^x \sin x$

Section 6.3 Exercises

1. Let $u = x$ $\quad dv = \sin x\, dx$

$du = dx$ $\quad v = -\cos x$

$\int x \sin x\, dx = -x\cos x + \int \cos x\, dx$

$= -x \cos x + \sin x + C$

Check: $\frac{d}{dx}(-x\cos x + \sin x + C)$

$= (-x)(-\sin x) + (\cos x)(-1) + \cos x$

$= x \sin x$

2. Let $u = x^2$ $\quad dv = \cos x\, dx$

$du = 2x\, dx$ $\quad v = \sin x$

$\int x^2 \cos x\, dx = x^2 \sin x - 2\int x \sin x\, dx$

Using the result from Exercise 1,

$= x^2 \sin x - 2[-x\cos x + \sin x] + C$

$= 2x \cos x + (x^2 - 2)\sin x + C$

Check: $\frac{d}{dx}[2x\cos x + (x^2-2)\sin x + C]$

$= (2x)(-\sin x) + (2\cos x)(1) + (x^2 - 2)(\cos x)$

$+ (\sin x)(2x)$

$= x^2 \cos x$

3. Let $u = \ln y \qquad dv = y\,dy$

$du = \frac{1}{y}\,dy \qquad v = \frac{1}{2}y^2$

$\int y \ln y\,dy = \frac{1}{2}y^2 \ln y - \int \frac{1}{2}y^2 \cdot \frac{1}{y}\,dy$

$= \frac{1}{2}y^2 \ln y - \frac{1}{2}\int y\,dy$

$= \frac{1}{2}y^2 \ln y - \frac{1}{4}y^2 + C$

Check: $\frac{d}{dy}\left[\frac{1}{2}y^2 \ln y - \frac{1}{4}y^2 + C\right]$

$= \left(\frac{1}{2}y^2\right)\left(\frac{1}{y}\right) + (\ln y)(y) - \frac{1}{2}y$

$= y \ln y$

4. Let $u = \tan^{-1} y \qquad dv = dy$

$du = \frac{1}{1+y^2}\,dy \qquad v = y$

$\int \tan^{-1} y\,dy = y \tan^{-1} y - \int \frac{y}{1+y^2}\,dy$

Let $w = 1 + y^2$

$dw = 2y\,dy$

$y \tan^{-1} y - \int \frac{y}{1+y^2}\,dy = y \tan^{-1} y - \frac{1}{2}\int \frac{1}{w}\,dw$

$= y \tan^{-1} y - \frac{1}{2} \ln |w| + C$

$= y \tan^{-1} y - \frac{1}{2} \ln (1 + y^2) + C$

Check: $\frac{d}{dy}\left[y \tan^{-1} y - \frac{1}{2} \ln (1 + y^2) + C\right]$

$= \frac{y}{1+y^2} + \tan^{-1} y - \frac{1}{2}\left(\frac{1}{1+y^2}\right)(2y) = \tan^{-1} y$

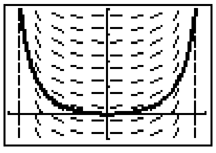
5. Let $u = x \qquad dv = \sec^2 x\,dx$

$du = dx \qquad v = \tan x$

$\int x \sec^2 x\,dx = x \tan x - \int \tan x\,dx$

$= x \tan x - \int \frac{\sin x}{\cos x}\,dx$

$= x \tan x + \ln |\cos x| + C$



[−1.5, 1.5] by [−1, 4]

6. Let $u = \sin^{-1} \theta \qquad dv = d\theta$

$du = \frac{1}{\sqrt{1-\theta^2}} \qquad v = \theta$

$\int \sin^{-1} \theta\,d\theta = \theta \sin^{-1} \theta - \int \theta \frac{1}{\sqrt{1-\theta^2}}\,d\theta$

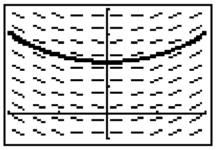
Let $w = 1 - \theta^2$

$dw = -2\theta\,d\theta$

$\theta \sin^{-1} \theta - \int \theta \frac{1}{\sqrt{1-\theta^2}}\,d\theta = \theta \sin^{-1} \theta + \frac{1}{2}\int \frac{1}{\sqrt{w}}\,dw$

$= \theta \sin^{-1} \theta + w^{1/2} + C$

$= \theta \sin^{-1} \theta + \sqrt{1-\theta^2} + C$



[−1, 1] by [−0.5, 2]

7. Let $u = t^2 \qquad dv = \sin t\,dt$

$du = 2t\,dt \qquad v = -\cos t$

$\int t^2 \sin t\,dt = -t^2 \cos t + 2\int (\cos t)(t)\,dt$

Let $u = t \qquad dv = \cos t\,dt$

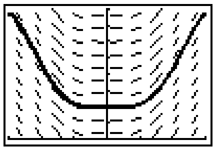
$du = dt \qquad v = \sin t$

$-t^2 \cos t + 2\int t \cos t\,dt$

$= -t^2 \cos t + 2t \sin t - 2\int \sin t\,dt$

$= -t^2 \cos t + 2t \sin t + 2 \cos t + C$

$= (2 - t^2) \cos t + 2t \sin t + C$



[−3, 3] by [0, 8]

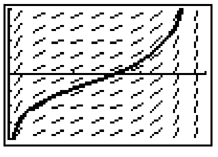
8. Let $u = t \qquad dv = \csc^2 t\,dt$

$du = dt \qquad v = -\cot t$

$\int t \csc^2 t\,dt = -t \cot t + \int \cot t\,dt$

$= -t \cot t + \int \frac{\cos t}{\sin t}\,dt$

$= -t \cot t + \ln |\sin t| + C$



[0, 3] by [−4, 4]

9. Let $u = \ln x \qquad dv = x^3\,dx$

$du = \frac{1}{x}\,dx \qquad v = \frac{1}{4}x^4$

$$\int x^3 \ln x\,dx = \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^4\left(\frac{1}{x}\right)dx$$
$$= \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3\,dx$$
$$= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C$$

10. Use tabular integration with $f(x) = x^4$ and $g(x) = e^{-x}$.

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^4	(+)	e^{-x}
$4x^3$	(–)	$-e^{-x}$
$12x^2$	(+)	e^{-x}
$24x$	(–)	$-e^{-x}$
24	(+)	e^{-x}
0		$-e^{-x}$

$$\int x^4 e^{-x}\,dx$$
$$= -x^4e^{-x} - 4x^3e^{-x} - 12x^2e^{-x} - 24xe^{-x} - 24e^{-x} + C$$
$$= -(x^4 + 4x^3 + 12x^2 + 24x + 24)e^{-x} + C$$

11. Let $u = x^2 - 5x \qquad dv = e^x\,dx$

$du = (2x - 5)\,dx \qquad v = e^x$

$$\int (x^2 - 5x)e^x\,dx = (x^2 - 5x)e^x - \int e^x(2x - 5)\,dx$$

Let $u = 2x - 5 \qquad dv = e^x\,dx$

$du = 2\,dx \qquad v = e^x$

$$(x^2 - 5x)e^x - \int e^x(2x - 5)\,dx$$
$$= (x^2 - 5x)e^x - (2x - 5)e^x + \int 2e^x\,dx$$
$$= (x^2 - 5x)e^x - (2x - 5)e^x + 2e^x + C$$
$$= (x^2 - 7x + 7)e^x + C$$

12. Use tabular integration with $f(x) = x^3$ and $g(x) = e^{-2x}$.

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^3	(+)	e^{-2x}
$3x^2$	(–)	$-\frac{1}{2}e^{-2x}$
$6x$	(+)	$\frac{1}{4}e^{-2x}$
6	(–)	$-\frac{1}{8}e^{-2x}$
0		$\frac{1}{16}e^{-2x}$

$$\int x^3e^{-2x}\,dx$$
$$= -\frac{1}{2}x^3e^{-2x} - \frac{3}{4}x^2e^{-2x} - \frac{3}{4}xe^{-2x} - \frac{3}{8}e^{-2x} + C$$
$$= -\left(\frac{x^3}{2} + \frac{3x^2}{4} + \frac{3x}{4} + \frac{3}{8}\right)e^{-2x} + C$$

13. Let $u = e^y \qquad dv = \sin y\,dy$

$du = e^y\,dy \qquad v = -\cos y$

$$\int e^y \sin y\,dy = -e^y \cos y + \int \cos y\, e^y\,dy$$

Let $u = e^y \qquad dv = \cos y\,dy$

$du = e^y\,dy \qquad v = \sin y$

$$\int e^y \sin y\,dy = -e^y \cos y + e^y \sin y - \int \sin y e^y\,dy$$
$$2\int e^y \sin y\,dy = -e^y \cos y + e^y \sin y$$
$$\int e^y \sin y\,dy = \frac{1}{2}e^y(\sin y - \cos y) + C$$

14. Let $u = e^{-y} \qquad dv = \cos y\,dy$

$du = -e^{-y}\,dy \qquad v = \sin y$

$$\int e^{-y}\cos y\,dy = e^{-y}\sin y + \int \sin y\, e^{-y}\,dy$$

Let $u = e^{-y} \qquad dv = \sin y\,dy$

$du = -e^{-y}\,dy \qquad v = -\cos y$

$$\int e^{-y} \cos y\,dy = e^{-y} \sin y - e^{-y}\cos y - \int e^{-y} \cos y\,dy$$
$$2\int e^{-y} \cos y\,dy = e^{-y} \sin y - e^{-y} \cos y$$
$$\int e^{-y} \cos y\,dy = \frac{1}{2}e^{-y}(\sin y - \cos y) + C$$

15. Use tabular integration with $f(x) = x^2$ and $g(x) = \sin 2x$.

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^2	(+)	$\sin 2x$
$2x$	(–)	$-\frac{1}{2}\cos 2x$
2	(+)	$-\frac{1}{4}\sin 2x$
0		$\frac{1}{8}\cos 2x$

$$\int x^2 \sin 2x\,dx = -\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C$$

$$= \left(\frac{1-2x^2}{4}\right)\cos 2x + \frac{x}{2}\sin 2x + C$$

$$\int_0^{\pi/2} x^2 \sin 2x\,dx = \left[\left(\frac{1-2x^2}{4}\right)\cos 2x + \frac{x}{2}\sin 2x\right]_0^{\pi/2}$$

$$= \left(\frac{1-2\left(\frac{\pi}{2}\right)^2}{4}\right)(-1) + 0 - \frac{1}{4}(1) - 0$$

$$= \frac{\pi^2}{8} - \frac{1}{2} \approx 0.734$$

Check: $\text{NINT}\left(x^2 \sin 2x, x, 0, \frac{\pi}{2}\right) \approx 0.734$

16. Use tabular integration with $f(x) = x^3$ and $g(x) = \cos 2x$.

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^3	(+)	$\cos 2x$
$3x^2$	(–)	$\frac{1}{2}\sin 2x$
$6x$	(+)	$-\frac{1}{4}\cos 2x$
6	(–)	$-\frac{1}{8}\sin 2x$
0		$\frac{1}{16}\cos 2x$

$$\int x^3 \cos 2x\,dx = \frac{1}{2}x^3\sin 2x + \frac{3}{4}x^2\cos 2x - \frac{3}{4}x\sin 2x - \frac{3}{8}\cos 2x$$

$$= \left(\frac{x^3}{2} - \frac{3x}{4}\right)\sin 2x + \left(\frac{3x^2}{4} - \frac{3}{8}\right)\cos 2x + C$$

$$\int_0^{\pi/2} x^3 \cos 2x\,dx = \left[\left(\frac{x^3}{2} - \frac{3x}{4}\right)\sin 2x + \left(\frac{3x^2}{4} - \frac{3}{8}\right)\cos 2x\right]_0^{\pi/2}$$

$$= 0 + \left(\frac{3\pi^2}{16} - \frac{3}{8}\right)(-1) - 0 - \left(-\frac{3}{8}\right)(1)$$

$$= \frac{3}{4} - \frac{3\pi^2}{16} \approx -1.101$$

Check: $\text{NINT}\left(x^3 \cos 2x, x, 0, \frac{\pi}{2}\right) \approx -1.101$

17. Let $u = e^{2x}$ $\quad dv = \cos 3x\, dx$

$du = 2e^{2x}\, dx \quad v = \frac{1}{3}\sin 3x$

$$\int e^{2x}\cos 3x\, dx = (e^{2x})\left(\frac{1}{3}\sin 3x\right) - \int\left(\frac{1}{3}\sin 3x\right)(2e^{2x}\, dx)$$

$$= \frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\int e^{2x}\sin 3x\, dx$$

Let $u = e^{2x}$ $\quad dv = \sin 3x\, dx$

$du = 2e^{2x}\, dx \quad v = -\frac{1}{3}\cos 3x$

$$\int e^{2x}\cos 3x\, dx = \frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\left[(e^{2x})\left(-\frac{1}{3}\cos 3x\right) - \int\left(-\frac{1}{3}\cos 3x\right)(2e^{2x}\, dx)\right]$$

$$= \frac{1}{9}e^{2x}(3\sin 3x + 2\cos 3x) - \frac{4}{9}\int e^{2x}\cos 3x\, dx$$

$$\frac{13}{9}\int e^{2x}\cos 3x\, dx = \frac{1}{9}e^{2x}(3\sin 3x + 2\cos 3x)$$

$$\int e^{2x}\cos 3x\, dx = \frac{1}{13}e^{2x}(3\sin 3x + 2\cos 3x)$$

$$\int_{-2}^{3} e^{2x}\cos 3x\, dx = \left[\frac{1}{13}e^{2x}(3\sin 3x + 2\cos 3x)\right]_{-2}^{3}$$

$$= \frac{1}{13}[e^{6}(3\sin 9 + 2\cos 9) - e^{-4}(3\sin(-6) + 2\cos(-6))]$$

$$= \frac{1}{13}[e^{6}(2\cos 9 + 3\sin 9) - e^{-4}(2\cos 6 - 3\sin 6)]$$

$$\approx -18.186$$

Check: NINT($e^{2x}\cos 3x$, x, -2, 3) ≈ -18.186

18. Let $u = e^{-2x}$ $\quad dv = \sin 2x\, dx$

$du = -2e^{-2x}\, dx \quad v = -\frac{1}{2}\cos 2x$

$$\int e^{-2x}\sin 2x\, dx = (e^{-2x})\left(-\frac{1}{2}\cos 2x\right) - \int\left(-\frac{1}{2}\cos 2x\right)(-2e^{-2x}\, dx)$$

$$= -\frac{1}{2}e^{-2x}\cos 2x - \int e^{-2x}\cos 2x\, dx$$

Let $u = e^{-2x}$ $\quad dv = \cos 2x\, dx$

$du = -2e^{-2x} \quad v = \frac{1}{2}\sin 2x$

$$\int e^{-2x}\sin 2x\, dx = -\frac{1}{2}e^{-2x}\cos 2x - \left[(e^{-2x})\left(\frac{1}{2}\sin 2x\right) - \int\left(\frac{1}{2}\sin 2x\right)(-2e^{-2x}\, dx)\right]$$

$$= -\frac{1}{2}e^{-2x}(\cos 2x + \sin 2x) - \int e^{-2x}\sin 2x\, dx$$

$$2\int e^{-2x}\sin 2x\, dx = -\frac{1}{2}e^{-2x}(\cos 2x + \sin 2x) + C$$

$$\int e^{-2x}\sin 2x\, dx = -\frac{e^{-2x}}{4}(\cos 2x + \sin 2x) + C$$

$$\int_{-3}^{2} e^{-2x}\sin 2x\, dx = \left[-\frac{e^{-2x}}{4}(\cos 2x + \sin 2x)\right]_{-3}^{2}$$

$$= -\frac{e^{-4}}{4}(\cos 4 + \sin 4) + \frac{e^{6}}{4}[\cos(-6) + \sin(-6)]$$

$$= -\frac{e^{-4}}{4}(\cos 4 + \sin 4) + \frac{e^{6}}{4}(\cos 6 - \sin 6)$$

$$\approx 125.028$$

Check: NINT($e^{-2x}\sin 2x$, x, -3, 2) ≈ 125.028

19. $y = \int x^2 e^{4x}\,dx$

Let $u = x^2$ $\quad dv = e^{4x}dx$

$du = 2x\,dx$ $\quad v = \frac{1}{4}e^{4x}$

$$y = (x^2)\left(\frac{1}{4}e^{4x}\right) - \int\left(\frac{1}{4}e^{4x}\right)(2x\,dx)$$
$$= \frac{1}{4}x^2e^{4x} - \frac{1}{2}\int xe^{4x}dx$$

Let $u = x$ $\quad dv = e^{4x}\,dx$

$du = dx$ $\quad v = \frac{1}{4}e^{4x}$

$$y = \frac{1}{4}x^2e^{4x} - \frac{1}{2}\left[(x)\left(\frac{1}{4}e^{4x}\right) - \int\left(\frac{1}{4}e^{4x}\right)dx\right]$$
$$y = \frac{1}{4}x^2e^{4x} - \frac{1}{8}xe^{4x} + \frac{1}{32}e^{4x} + C$$
$$y = \left(\frac{x^2}{4} - \frac{x}{8} + \frac{1}{32}\right)e^{4x} + C$$

20. $y = \int x^2 \ln x\,dx$

Let $u = \ln x$ $\quad dv = x^2\,dx$

$du = \frac{1}{x}\,dx$ $\quad v = \frac{1}{3}x^3$

$$y = (\ln x)\left(\frac{1}{3}x^3\right) - \int\left(\frac{1}{3}x^3\right)\left(\frac{1}{x}\,dx\right)$$
$$y = \frac{1}{3}x^3\ln x - \frac{1}{3}\int x^2\,dx$$
$$y = \frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + C$$

21. $y = \int \theta \sec^{-1}\theta\,d\theta$

Let $u = \sec^{-1}\theta$ $\quad dv = \theta\,d\theta$

$du = \frac{1}{\theta\sqrt{\theta^2 - 1}}\,du$ $\quad v = \frac{1}{2}\theta^2$

Note that we are told $\theta > 1$, so no absolute value is needed in the expression for du.

$$y = (\sec^{-1}\theta)\left(\frac{1}{2}\theta^2\right) - \int\left(\frac{1}{2}\theta^2\right)\left(\frac{1}{\theta\sqrt{\theta^2-1}}\,d\theta\right)$$
$$y = \frac{\theta^2}{2}\sec^{-1}\theta - \frac{1}{4}\int\frac{2|\theta|\,d\theta}{\sqrt{\theta^2-1}}$$

Let $w = \theta^2 - 1$, $dw = 2\theta\,d\theta$

$$y = \frac{\theta^2}{2}\sec^{-1}\theta - \frac{1}{4}\int w^{-1/2}\,dw$$
$$y = \frac{\theta^2}{2}\sec^{-1}\theta - \frac{1}{2}w^{1/2} + C$$
$$y = \frac{\theta^2}{2}\sec^{-1}\theta - \frac{1}{2}\sqrt{\theta^2-1} + C$$

22. $y = \int \theta \sec\theta \tan\theta\,d\theta$

Let $u = \theta$ $\quad dv = \sec\theta\tan\theta\,d\theta$

$du = d\theta$ $\quad v = \sec\theta$

$$y = \theta\sec\theta - \int\sec\theta\,d\theta$$
$$y = \theta\sec\theta - \ln|\sec\theta + \tan\theta| + C$$

Note: In the last step, we used the result of Exercise 29 in Section 6.2.

23. Let $u = x$ $\quad dv = \sin x\,dx$

$du = dx$ $\quad v = -\cos x$

$$\int x\sin x\,dx = -x\cos x + \int\cos x\,dx$$
$$= -x\cos x + \sin x + C$$

(a) $\int_0^\pi |x\sin x|\,dx = \int_0^\pi x\sin x\,dx$
$$= \left[-x\cos x + \sin x\right]_0^\pi$$
$$= -\pi(-1) + 0 + 0(1) - 0$$
$$= \pi$$

(b) $\int_\pi^{2\pi} |x\sin x|\,dx = -\int_\pi^{2\pi} x\sin x\,dx$
$$= \left[x\cos x - \sin x\right]_\pi^{2\pi}$$
$$= 2\pi(1) - 0 - \pi(-1) + 0$$
$$= 3\pi$$

(c) $\int_0^{2\pi} |x\sin x|\,dx = \int_0^\pi |x\sin x|\,dx + \int_\pi^{2\pi} |x\sin x|\,dx$
$$= \pi + 3\pi = 4\pi$$

24. We begin by evaluating $\int(x^2 + x + 1)e^{-x}\,dx$.

Let $u = x^2 + x + 1$ $\quad dv = e^{-x}\,dx$

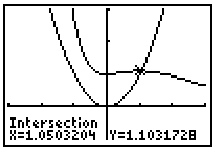
$du = (2x + 1)\,dx$ $\quad v = -e^{-x}$

$$\int(x^2 + x + 1)e^{-x}\,dx$$
$$= -(x^2 + x + 1)e^{-x} + \int(2x + 1)e^{-x}\,dx$$

Let $u = 2x + 1$ $\quad dv = e^{-x}\,dx$

$du = 2\,dx$ $\quad v = -e^{-x}$

$$\int(x^2 + x + 1)e^{-x}\,dx$$
$$= -(x^2 + x + 1)e^{-x} - (2x + 1)e^{-x} + \int 2e^{-x}\,dx$$
$$= -(x^2 + x + 1)e^{-x} - (2x + 1)e^{-x} - 2e^{-x} + C$$
$$= -(x^2 + 3x + 4)e^{-x} + C$$



[−3, 3] by [−3, 3]

The graph shows that the two curves intersect at $x = k$, where $k \approx 1.050$. The area we seek is

$$\int_0^k(x^2 + x + 1)e^{-x} - \int_0^k x^2\,dx$$
$$= \left[-(x^2 + 3x + 4)e^{-x}\right]_0^k - \left[\frac{1}{3}x^3\right]_0^k$$
$$\approx (-2.888 + 4) - (0.386 - 0)$$
$$\approx 0.726$$

25. First, we evaluate $\int e^{-t}\cos t\,dt$.

Let $\quad u = e^{-t} \qquad dv = \cos t\,dt$

$\quad du = -e^{-t}\,dt \qquad v = \sin t$

$$\int e^{-t}\cos t\,dt = e^{-t}\sin t + \int \sin t\, e^{-t}\,dt$$

Let $u = e^{-t} \qquad dv = \sin t\,dt$

$\quad du = -e^{-t}\,dt \qquad v = -\cos t$

$$\int e^{-t}\cos t\,dt = e^{-t}\sin t - e^{-t}\cos t - \int e^{-t}\cos t\,dt$$

$$2\int e^{-t}\cos t\,dt = e^{-t}(\sin t - \cos t) + C$$

$$\int e^{-t}\cos t\,dt = \frac{1}{2}e^{-t}(\sin t - \cos t) + C$$

Now we find the average value of $y = 2e^{-t}\cos t$ for $0 \le t \le 2\pi$.

$$\begin{aligned}\text{Average value } &= \frac{1}{2\pi}\int_0^{2\pi} 2e^{-t}\cos t\,dt \\ &= \frac{1}{\pi}\int_0^{2\pi} e^{-t}\cos t\,dt \\ &= \frac{1}{2\pi}e^{-t}(\sin t - \cos t)\Big]_0^{2\pi} \\ &= \frac{1}{2\pi}[e^{-2\pi}(-1) - e^0(-1)] \\ &= \frac{1 - e^{-2\pi}}{2\pi} \approx 0.159\end{aligned}$$

26. **(a)** Let $\quad u = x \qquad dv = e^x\,dx$

$\quad du = dx \qquad v = e^x$

$$\begin{aligned}\int xe^x\,dx &= xe^x - \int e^x\,dx \\ &= xe^x - e^x + C \\ &= (x-1)e^x + C\end{aligned}$$

(b) Using the result from part (a):

Let $\quad u = x^2 \qquad dv = e^x\,dx$

$\quad du = 2x\,dx \qquad v = e^x$

$$\begin{aligned}\int x^2 e^x\,dx &= x^2 e^x - \int 2x\, e^x\,dx \\ &= x^2e^x - 2(x-1)e^x + C \\ &= (x^2 - 2x + 2)e^x + C\end{aligned}$$

(c) Using the result from part (b):

Let $\quad u = x^3 \qquad dv = e^x\,dx$

$\quad du = 3x^2\,dx \qquad v = e^x$

$$\begin{aligned}\int x^3 e^x\,dx &= x^3 e^x - \int 3x^2 e^x\,dx \\ &= x^3e^x - 3(x^2 - 2x + 2)e^x + C \\ &= (x^3 - 3x^2 + 6x - 6)e^x + C\end{aligned}$$

(d) $\left[x^n - \frac{d}{dx}x^n + \frac{d^2}{dx^2}x^n - \cdots + (-1)^n\frac{d^n}{dx^n}x^n\right]e^x + C$

or $[x^n - nx^{n-1} + n(n-1)x^{n-2} -$

$\cdots + (-1)^{n-1}(n-1)!x + (-1)^n(n!)]e^x + C$

(e) Use mathematical induction or argue based on tabular integration.

Alternately, show that the derivative of the answer to part (d) is x^ne^x:

$$\begin{aligned}&\frac{d}{dx}\Big[\Big(x^n - nx^{n-1} + n(n-1)x^{n-2} - \\ &\qquad \cdots + (-1)^{n-1}(n!)x + (-1)^n n!\Big)e^x + C\Big] \\ &= [x^n - nx^{n-1} + n(n-1)x^{n-2} - \\ &\qquad \cdots + (-1)^{n-1}(n!)x + (-1)^n n!]e^x + \\ &\qquad e^x\frac{d}{dx}[x^n - nx^{n-1} + n(n-1)x^{n-2} - \\ &\qquad \cdots + (-1)^{n-1}(n!)x + (-1)^n n!] \\ &= [x^n - nx^{n-1} + n(n-1)x^{n-2} - \\ &\qquad \cdots + (-1)^{n-1}(n!)x + (-1)^n n!]e^x \\ &\qquad + [nx^{n-1} - n(n-1)x^{n-2} \\ &\qquad + n(n-1)(n-2)x^{n-3} - \\ &\qquad \cdots + (-1)^{n-1}n!]e^x \\ &= x^ne^x\end{aligned}$$

27. Let $w = \sqrt{x}$. Then $dw = \frac{dx}{2\sqrt{x}}$, so $dx = 2\sqrt{x}\,dw = 2w\,dw$.

$$\int \sin\sqrt{x}\,dx = \int(\sin w)(2w\,dw) = 2\int w\sin w\,dw$$

Let $\quad u = w \qquad dv = \sin w\,dw$

$\quad du = dw \qquad v = -\cos w$

$$\begin{aligned}\int w\sin w\,dw &= -w\cos w + \int \cos w\,dw \\ &= -w\cos w + \sin w + C\end{aligned}$$

$$\begin{aligned}\int \sin\sqrt{x}\,dx &= 2\int w\sin w\,dw \\ &= -2w\cos w + 2\sin w + C \\ &= -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C\end{aligned}$$

28. Let $w = \sqrt{3x+9}$. Then $dw = \frac{1}{2\sqrt{3x+9}}(3)\,dx$, so $dx = \frac{2}{3}\sqrt{3x+9}\,dw = \frac{2}{3}w\,dw$.

$\int e^{\sqrt{3x+9}}\,dx = \int (e^w)\left(\frac{2}{3}w\,dw\right) = \frac{2}{3}\int w\,e^w\,dw$

Let $u = w \quad dv = e^w\,dw$

$du = dw \quad v = e^w$

$$\int w\,e^w\,dw = w\,e^w - \int e^w\,dw = w\,e^w - e^w = (w-1)e^w$$

$$\int e^{\sqrt{3x+9}}\,dx = \frac{2}{3}\int w\,e^w\,dw = \frac{2}{3}(w-1)e^w = \frac{2}{3}(\sqrt{3x+9}-1)e^{\sqrt{3x+9}} + C$$

29. Let $w = x^2$. Then $dw = 2x\,dx$.

$\int x^7 e^{x^2}\,dx = \int (x^2)^3 e^{x^2}\,x\,dx = \frac{1}{2}\int w^3\,e^w\,dw.$

Use tabular integration with $f(x) = w^3$ and $g(w) = e^w$.

$f(w)$ and its derivatives		$g(w)$ and its integrals
w^3	(+)	e^w
$3w^2$	(−)	e^w
$6w$	(+)	e^w
6	(−)	e^w
0		e^w

$$\int w^3\,e^w\,dw = w^3\,e^w - 3w^2e^w + 6w\,e^w - 6e^w + C = (w^3 - 3w^2 + 6w - 6)e^w + C$$

$$\int x^7 e^{x^2}\,dx = \frac{1}{2}\int w^3\,e^w\,dw = \frac{1}{2}(w^3 - 3w^2 + 6w - 6)e^w + C = \frac{(x^6 - 3x^4 + 6x^2 - 6)e^{x^2}}{2} + C$$

30. Let $y = \ln r$. Then $dy = \frac{1}{r}\,dr$, and so $dr = r\,dy = e^y\,dy$.

Using the result of Exercise 13, we have:

$$\int \sin(\ln r)\,dr = \int (\sin y)e^y\,dy = \frac{1}{2}e^y(\sin y - \cos y) + C = \frac{1}{2}e^{\ln r}[\sin(\ln r) - \cos(\ln r)] + C = \frac{r}{2}[\sin(\ln r) - \cos(\ln r)] + C$$

31. Let $u = x^n \qquad dv = \cos x\,dx$

$du = nx^{n-1}dx \qquad v = \sin x$

$$\int x^n \cos x\,dx = x^n \sin x - \int (\sin x)(nx^{n-1}dx) = x^n \sin x - n\int x^{n-1}\sin x\,dx$$

32. Let $u = x^n \qquad dv = \sin x\,dx$

$du = nx^{n-1}\,dx \qquad v = -\cos x$

$$\int x^n \sin x\,dx = (x^n)(-\cos x) - \int (-\cos x)(nx^{n-1}\,dx) = -x^n\cos x + n\int x^{n-1}\cos x\,dx$$

33. Let $u = x^n \qquad dv = e^{ax}\,dx$

$du = nx^{n-1}\,dx \qquad v = \frac{1}{a}e^{ax}$

$$\int x^n e^{ax}\,dx = (x^n)\left(\frac{1}{a}e^{ax}\right) - \int \left(\frac{1}{a}e^{ax}\right)(nx^{n-1}\,dx) = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx,\ a \neq 0$$

34. Let $u = (\ln x)^n \qquad dv = dx$

$du = \frac{n(\ln x)^{n-1}}{x}\,dx \qquad v = x$

$$\int (\ln x)^n\,dx = (\ln x)^n(x) - \int x\left[\frac{n(\ln x)^{n-1}}{x}\right]dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$$

35. (a) Let $y = f^{-1}(x)$. Then $x = f(y)$, so $dx = f'(y)\,dy$.

Hence, $\int f^{-1}(x)\,dx = \int (y)[f'(y)\,dy] = \int y\,f'(y)\,dy$

(b) Let $u = y \qquad dv = f'(y)dy$

$du = dy \qquad v = f(y)$

$$\int y\,f'(y)\,dy = y\,f(y) - \int f(y)\,dy = f^{-1}(x)(x) - \int f(y)\,dy$$

Hence, $\int f^{-1}(x)\,dx = \int y\,f'(y)\,dy = x\,f^{-1}(x) - \int f(y)\,dy.$

36. Let $u = f^{-1}(x) \qquad dv = dx$

$du = \left(\frac{d}{dx}f^{-1}(x)\right)dx \qquad v = x$

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - \int x\left(\frac{d}{dx}f^{-1}(x)\right)dx$$

37. (a) Using $y = f^{-1}(x) = \sin^{-1}x$ and $f(y) = \sin y$, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, we have:

$$\int \sin^{-1}x\,dx = x\sin^{-1}x - \int \sin y\,dy = x\sin^{-1}x + \cos y + C = x\sin^{-1}x + \cos(\sin^{-1}x) + C$$

37. continued

(b) $\int \sin^{-1} x\, dx = x \sin^{-1} x - \int x\left(\frac{d}{dx} \sin^{-1} x\right) dx$

$= x \sin^{-1} x - \int x \frac{1}{\sqrt{1-x^2}}\, dx$

$u = 1 - x^2,\ du = -2x\, dx$

$= x \sin^{-1} x + \frac{1}{2}\int u^{-1/2}\, du$

$= x \sin^{-1} x + u^{1/2} + C$

$= x \sin^{-1} x + \sqrt{1-x^2} + C$

(c) $\cos(\sin^{-1} x) = \sqrt{1-x^2}$

38. (a) Using $y = f^{-1}(x) = \tan^{-1} x$ and $f(y) = \tan y$, $-\frac{\pi}{2} < y < \frac{\pi}{2}$, we have:

$\int \tan^{-1} x\, dx = x \tan^{-1} x - \int \tan y\, dy$

$= x \tan^{-1} x - \ln |\sec y| + C$

(Section 6.2, Example 7)

$= x \tan^{-1} x + \ln |\cos y| + C$

$= x \tan^{-1} x + \ln |\cos(\tan^{-1} x)|\, dx + C$

(b) $\int \tan^{-1} x\, dx = x \tan^{-1} x - \int x\left(\frac{d}{dx} \tan^{-1} x\right) dx$

$= x \tan^{-1} x - \int x\left(\frac{1}{1+x^2}\right) dx$

$u = 1 + x^2,\ du = 2x\, dx$

$= x \tan^{-1} x - \frac{1}{2}\int u^{-1}\, du$

$= x \tan^{-1} x - \frac{1}{2}\ln |u| + C$

$= x \tan^{-1} x - \frac{1}{2}\ln(1+x^2) + C$

(c) $\ln |\cos(\tan^{-1} x)| = \ln \left|\frac{1}{\sqrt{1+x^2}}\right| = -\frac{1}{2}\ln(1+x^2)$

39. (a) Using $y = f^{-1}(x) = \cos^{-1} x$ and $f(y) = \cos x,\ 0 \le x \le \pi$, we have:

$\int \cos^{-1} x\, dx = x \cos^{-1} x - \int \cos y\, dy$

$= x \cos^{-1} x - \sin y + C$

$= x \cos^{-1} x - \sin(\cos^{-1} x) + C$

(b) $\int \cos^{-1} x\, dx = x \cos^{-1} x - \int x\left(\frac{d}{dx} \cos^{-1} x\right) dx$

$= x \cos^{-1} x - \int x\left(-\frac{1}{\sqrt{1-x^2}}\right) dx$

$u = 1 - x^2,\ du = -2x\, dx$

$= x \cos^{-1} x - \frac{1}{2}\int u^{-1/2}\, du$

$= x \cos^{-1} x - u^{1/2} + C$

$= x \cos^{-1} x - \sqrt{1-x^2} + C$

(c) $\sin(\cos^{-1} x) = \sqrt{1-x^2}$

40. (a) Using $y = f^{-1}(x) = \log_2 x$ and $f(y) = 2^y$, we have

$\int \log_2 x\, dx = x \log_2 x - \int 2^y\, dy$

$= x \log_2 x - \frac{2^y}{\ln 2} + C$

$= x \log_2 x - \frac{1}{\ln 2}\, 2^{\log_2 x}$

(b) $\int \log_2 x\, dx = x \log_2 x - \int x\left(\frac{d}{dx} \log_2 x\right) dx$

$= x \log_2 x - \int x\left(\frac{1}{x \ln 2}\right) dx$

$= x \log_2 x - \int \frac{dx}{\ln 2}$

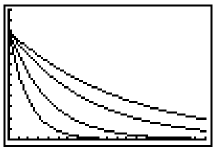
$= x \log_2 x - \left(\frac{1}{\ln 2}\right)x + C$

(c) $2^{\log_2 x} = x$

■ Section 6.4 Exponential Growth and Decay (pp. 330–341)

Exploration 1 Slowing Down More Slowly

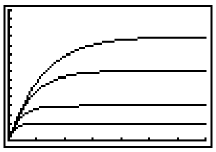
1. As m increases the velocity of the object represented by the graph slows down more slowly. That is, the y-coordinates of the graphs decrease to 0 more slowly as m increases.



[0, 20] by [0, 120]

2. As we saw in Section 5.1, $s(t) = \int_0^t v(u)\, du$ gives the distance traveled by the object over the time interval $[0, t]$. Since $s(0) = \int_0^0 v(u)\, du = 0$, the integral gives the distance traveled by the object at time t.

3. The total distance traveled is about 200 units for $m = 1$, about 400 units for $m = 2$, about 800 units for $m = 4$, and about 1200 units for $m = 6$.



[0, 70] by [0, 1500]

Quick Review 6.4

1. $a = e^b$
2. $c = \ln d$
3. $\ln(x+3) = 2$
   $x + 3 = e^2$
   $x = e^2 - 3$
4. $100e^{2x} = 600$
   $e^{2x} = 6$
   $2x = \ln 6$
   $x = \frac{1}{2}\ln 6$

5. $0.85^x = 2.5$

$\ln 0.85^x = \ln 2.5$

$x \ln 0.85 = \ln 2.5$

$x = \dfrac{\ln 2.5}{\ln 0.85} \approx -5.638$

6. $2^{k+1} = 3^k$

$\ln 2^{k+1} = \ln 3^k$

$(k+1)\ln 2 = k \ln 3$

$\ln 2 = k(\ln 3 - \ln 2)$

$k = \dfrac{\ln 2}{\ln 3 - \ln 2} \approx 1.710$

7. $1.1^t = 10$

$\ln 1.1^t = \ln 10$

$t \ln 1.1 = \ln 10$

$t = \dfrac{\ln 10}{\ln 1.1} = \dfrac{1}{\log 1.1} \approx 24.159$

8. $e^{-2t} = \dfrac{1}{4}$

$-2t = \ln\left(\dfrac{1}{4}\right)$

$t = -\dfrac{1}{2}\ln\left(\dfrac{1}{4}\right) = \dfrac{1}{2}\ln 4 = \ln 2$

9. $\ln(y+1) = 2x + 3$

$y + 1 = e^{2x+3}$

$y = -1 + e^{2x+3}$

10. $\ln|y+2| = 3t - 1$

$|y+2| = e^{3t-1}$

$y + 2 = \pm e^{3t-1}$

$y = -2 \pm e^{3t-1}$

Section 6.4 Exercises

1. $y(t) = y_0 e^{kt}$

$y(t) = 100e^{1.5t}$

2. $y(t) = y_0 e^{kt}$

$y(t) = 200e^{-05t}$

3. $y(t) = y_0 e^{kt}$

$y(t) = 50e^{kt}$

$y(5) = 100 = 50e^{5k}$

$2 = e^{5k}$

$\ln 2 = 5k$

$k = 0.2 \ln 2$

Solution: $y(t) = 50e^{(0.2 \ln 2)t}$ or $y(t) = 50 \cdot 2^{0.2t}$

4. $y(t) = y_0 e^{kt}$

$y(t) = 60e^{kt}$

$y(10) = 30 = 60e^{10k}$

$\dfrac{1}{2} = e^{10k}$

$\ln \dfrac{1}{2} = 10k$

$k = 0.1 \ln \dfrac{1}{2} = -0.1 \ln 2$

Solution: $y(t) = 60e^{-(0.1 \ln 2)t}$ or $y(t) = 60 \cdot 2^{-t/10}$

5. Doubling time:

$A(t) = A_0 e^{rt}$

$2000 = 1000e^{0.086t}$

$2 = e^{0.086t}$

$\ln 2 = 0.086t$

$t = \dfrac{\ln 2}{0.086} \approx 8.06$ yr

Amount in 30 years:

$A = 1000e^{(0.086)(30)} \approx \$13{,}197.10$

6. Annual rate:

$A(t) = A_0 e^{rt}$

$4000 = 2000e^{(r)(15)}$

$2 = e^{15r}$

$\ln 2 = 15r$

$r = \dfrac{\ln 2}{15} \approx 0.0462 = 4.62\%$

Amount in 30 years:

$A(t) = A_0 e^{rt}$

$A = 2000e^{[(\ln 2)/15](30)}$

$= 2000e^{2 \ln 2}$

$= 2000 \cdot 2^2$

$= \$8000$

7. Initial deposit:

$A(t) = A_0 e^{rt}$

$2898.44 = A_0 e^{(0.0525)(30)}$

$A_0 = \dfrac{2898.44}{e^{1.575}} \approx \$600.00$

Doubling time:

$A(t) = A_0 e^{rt}$

$1200 = 600e^{0.0525t}$

$2 = e^{0.0525t}$

$\ln 2 = 0.0525t$

$t = \dfrac{\ln 2}{0.0525} \approx 13.2$ years

8. Annual rate:

$A(t) = A_0 e^{rt}$

$10{,}405.37 = 1200e^{(r)(30)}$

$\dfrac{104.0537}{12} = e^{30r}$

$\ln \dfrac{104.0537}{12} = 30r$

$r = \dfrac{1}{30}\ln \dfrac{104.0537}{12} \approx 0.072 = 7.2\%$

Doubling time:

$A(t) = A_0 e^{rt}$

$2400 = 1200e^{0.072t}$

$2 = e^{0.072t}$

$\ln 2 = 0.072t$

$t = \dfrac{\ln 2}{0.072} \approx 9.63$ years

9. **(a)** Annually:

$$2 = 1.0475^t$$
$$\ln 2 = t \ln 1.0475$$
$$t = \frac{\ln 2}{\ln 1.0475} \approx 14.94 \text{ years}$$

(b) Monthly:

$$2 = \left(1 + \frac{0.0475}{12}\right)^{12t}$$
$$\ln 2 = 12t \ln\left(1 + \frac{0.0475}{12}\right)$$
$$t = \frac{\ln 2}{12 \ln\left(1 + \frac{0.0475}{12}\right)} \approx 14.62 \text{ years}$$

(c) Quarterly:

$$2 = \left(1 + \frac{0.0475}{4}\right)^{4t}$$
$$\ln 2 = 4t \ln 1.011875$$
$$t = \frac{\ln 2}{4 \ln 1.011875} \approx 14.68 \text{ years}$$

(d) Continuously:

$$2 = e^{0.0475t}$$
$$\ln 2 = 0.0475t$$
$$t = \frac{\ln 2}{0.0475} \approx 14.59 \text{ years}$$

10. **(a)** Annually:

$$2 = 1.0825^t$$
$$\ln 2 = t \ln 1.0825$$
$$t = \frac{\ln 2}{\ln 1.0825} \approx 8.74 \text{ years}$$

(b) Monthly:

$$2 = \left(1 + \frac{0.0825}{12}\right)^{12t}$$
$$\ln 2 = 12t \ln\left(1 + \frac{0.0825}{12}\right)$$
$$t = \frac{\ln 2}{12 \ln\left(1 + \frac{0.0825}{12}\right)} \approx 8.43 \text{ years}$$

(c) Quarterly:

$$2 = \left(1 + \frac{0.0825}{4}\right)^{4t}$$
$$\ln 2 = 4t \ln 1.020625$$
$$t = \frac{\ln 2}{4 \ln 1.020625} \approx 8.49 \text{ years}$$

(d) Continuously:

$$2 = e^{0.0825t}$$
$$\ln 2 = 0.0825t$$
$$t = \frac{\ln 2}{0.0825} \approx 8.40 \text{ years}$$

11. **(a)** Since there are 48 half-hour doubling times in 24 hours, there will be $2^{48} \approx 2.8 \times 10^{14}$ bacteria.

(b) The bacteria reproduce fast enough that even if many are destroyed there are still enough left to make the person sick.

12. Using $y = y_0 e^{kt}$, we have

$10{,}000 = y_0 e^{3k}$ and $40{,}000 = y_0 e^{5k}$.

Hence $\frac{40{,}000}{10{,}000} = \frac{y_0 e^{5k}}{y_0 e^{3k}}$, which gives

$e^{2k} = 4$, or $k = \ln 2$. Solving $10{,}000 = y_0 e^{3\ln 2}$, we have

$y_0 = 1250$. There were 1250 bacteria initially.

We could solve this more quickly by noticing that the population increased by a factor of 4, i.e. doubled twice, in 2 hrs, so the doubling time is 1 hr. Thus in 3 hrs the population would have doubled 3 times, so the initial population was $\frac{10{,}0000}{2^3} = 1250$.

13.

$$0.9 = e^{-0.18t}$$
$$\ln 0.9 = -0.18t$$
$$t = -\frac{\ln 0.9}{0.18} \approx 0.585 \text{ days}$$

14. **(a)** Half-life $= \frac{\ln 2}{k} = \frac{\ln 2}{0.005} \approx 138.6$ days

(b)

$$0.05 = e^{-0.005t}$$
$$\ln 0.05 = -0.005t$$
$$t = -\frac{\ln 0.05}{0.005} \approx 599.15 \text{ days}$$

The sample will be useful for about 599 days.

15. Since $y_0 = y(0) = 2$, we have:

$$y = 2e^{kt}$$
$$5 = 2e^{(k)(2)}$$
$$\ln 5 = \ln 2 + 2k$$
$$k = \frac{\ln 5 - \ln 2}{2} = 0.5 \ln 2.5$$

Function: $y = 2e^{(0.5 \ln 2.5)t}$ or $y \approx 2e^{0.4581t}$

16. Since $y_0 = y(0) = 1.1$, we have:

$$y = 1.1e^{kt}$$
$$3 = 1.1e^{(k)(-3)}$$
$$\ln 3 = \ln 1.1 - 3k$$
$$k = \frac{1}{3}(\ln 1.1 - \ln 3)$$

Function: $y = 1.1e^{(\ln 1.1 - \ln 3)t/3}$ or $y \approx 1.1e^{-0.3344t}$

17. At time $t = \frac{3}{k}$, the amount remaining is $y_0e^{-kt} = y_0e^{-k(3/k)} = y_0e^{-3} \approx 0.0499y_0$. This is less than 5% of the original amount, which means that over 95% has decayed already.

18. $T - T_s = (T_0 - T_s)\, e^{-kt}$

$35 - 65 = (T_0 - 65)e^{-(k)(10)}$

$50 - 65 = (T_0 - 65)e^{-(k)(20)}$

Dividing the first equation by the second, we have:

$2 = e^{10k}$

$k = \frac{1}{10}\ln 2$

Substituting back into the first equation, we have:

$-30 = (T_0 - 65)e^{-[(\ln 2)/10](10)}$

$-30 = (T_0 - 65)\left(\frac{1}{2}\right)$

$-60 = T_0 - 65$

$5 = T_0$

The beam's initial temperature is 5°F.

19. **(a)** First, we find the value of k.

$T - T_s = (T_0 - T_s)e^{-kt}$

$60 - 20 = (90 - 20)e^{-(k)(10)}$

$\frac{4}{7} = e^{-10k}$

$k = -\frac{1}{10}\ln\frac{4}{7}$

When the soup cools to 35°, we have:

$35 - 20 = (90 - 20)e^{[(1/10)\ln(4/7)]t}$

$15 = 70e^{[(1/10)\ln(4/7)]t}$

$\ln\frac{3}{14} = \left(\frac{1}{10}\ln\frac{4}{7}\right)t$

$$t = \frac{10\ln\left(\frac{3}{14}\right)}{\ln\left(\frac{4}{7}\right)} \approx 27.53 \text{ min}$$

It takes a total of about 27.53 minutes, which is an additional 17.53 minutes after the first 10 minutes.

(b) Using the same value of k as in part (a), we have:

$T - T_s = (T_0 - T_s)e^{-kt}$

$35 - (-15) = [90 - (-15)]e^{[(1/10)\ln(4/7)]t}$

$50 = 105e^{[(1/10)\ln(4/7)]t}$

$\ln\frac{10}{21} = \left(\frac{1}{10}\ln\frac{4}{7}\right)t$

$$t = \frac{10\ln\left(\frac{10}{21}\right)}{\ln\left(\frac{4}{7}\right)} \approx 13.26$$

It takes about 13.26 minutes

20. First, we find the value of k.

Taking "right now" as $t = 0$, 60° above room temperature means $T_0 - T_s = 60$. Thus, we have

$T - T_s = (T_0 - T_s)e^{-kt}$

$70 = 60e^{(-k)(-20)}$

$\frac{7}{6} = e^{20k}$

$k = \frac{1}{20}\ln\frac{7}{6}$

(a) $T - T_s = (T_0 - T_s)e^{-kt} = 60e^{(-(1/20)\ln(7/6))(15)} \approx 53.45$

It will be about 53.45°C above room temperature.

(b) $T - T_s = (T_0 - T_s)e^{-kt} = 60e^{(-(1/20)\ln(7/6))(120)} \approx 23.79$

It will be about 23.79° above room temperature.

(c) $T - T_s = (T_0 - T_s)e^{-kt}$

$10 = 60e^{(-(1/20)\ln(7/6))t}$

$\ln\frac{1}{6} = \left(-\frac{1}{20}\ln\frac{7}{6}\right)t$

$t = -\frac{20\ln(1/6)}{\ln(7/6)} \approx 232.47 \text{ min}$

It will take about 232.47 min or 3.9 hr.

21. Use $k = \frac{\ln 2}{5700}$ (see Example 3).

$e^{-kt} = 0.445$

$-kt = \ln 0.445$

$t = -\frac{\ln 0.445}{k} = -\frac{5700\ln 0.445}{\ln 2} \approx 6658 \text{ years}$

Crater Lake is about 6658 years old.

22. Use $k = \frac{\ln 2}{5700}$ (see Example 3).

(a) $e^{-kt} = 0.17$

$-kt = \ln 0.17$

$t = -\frac{\ln 0.17}{k} = -\frac{5700\ln 0.17}{\ln 2} \approx 14{,}571 \text{ years}$

The animal died about 14,571 years before A.D. 2000, in 12,571 B.C.

(b) $e^{-kt} = 0.18$

$-kt = \ln 0.18$

$t = -\frac{\ln 0.18}{k} = -\frac{5700\ln 0.18}{\ln 2} \approx 14{,}101 \text{ years}$

The animal died about 14,101 years before A.D. 2000, in 12,101 B.C.

22. continued

(c)
$$e^{-kt} = 0.16$$
$$-kt = \ln 0.16$$
$$t = -\frac{\ln 0.16}{k} = -\frac{5700 \ln 0.16}{\ln 2} \approx 15{,}070 \text{ years}$$

The animal died about 15,070 years before A.D. 2000, in 13,070 B.C.

23. Note that the total mass is $66 + 7 = 73$ kg.

$v = v_0 e^{-(k/m)t}$

$v = 9e^{-3.9t/73}$

(a) $s(t) = \int 9e^{-3.9t/73} dt = -\frac{2190}{13} e^{-3.9t/73} + C$

Since $s(0) = 0$ we have $C = \frac{2190}{13}$ and

$$\lim_{t\to\infty} s(t) = \lim_{t\to\infty} \frac{2190}{13}(1 - e^{-3.9t/73}) = \frac{2190}{13} \approx 168.5$$

The cyclist will coast about 168.5 meters.

(b)
$$1 = 9e^{-3.9t/73}$$
$$\frac{3.9t}{73} = \ln 9$$
$$t = \frac{73 \ln 9}{3.9} \approx 41.13 \text{ sec}$$

It will take about 41.13 seconds.

24. $v = v_0 e^{-(k/m)t}$

$v = 9e^{-(59{,}000/51{,}000{,}000)t}$

$v = 9e^{-59t/51{,}000}$

(a) $s(t) = \int 9e^{-59t/51{,}000}\, dt = -\frac{459{,}000}{59} e^{59t/51{,}000} + C$

Since $s(0) = 0$, we have $C = \frac{459{,}000}{59}$ and

$$\lim_{t\to\infty} s(t) = \lim_{t\to\infty} \frac{459{,}000}{59}(1 - e^{-59t/51{,}000})$$
$$= \frac{459{,}000}{59} \approx 7780 \text{ m}$$

The ship will coast about 7780 m, or 7.78 km.

(b)
$$1 = 9e^{-59t/51{,}000}$$
$$\frac{59t}{51{,}000} = \ln 9$$
$$t = \frac{51{,}000 \ln 9}{59} \approx 1899.3 \text{ sec}$$

It will take about 31.65 minutes.

25.
$$y = y_0 e^{-kt}$$
$$800 = 1000e^{-(k)(10)}$$
$$0.8 = e^{-10k}$$
$$k = -\frac{\ln 0.8}{10}$$

At $t = 10 + 14 = 24$ h:

$$y = 1000e^{-(-\ln 0.8/10)24}$$
$$= 1000e^{2.4 \ln 0.8} \approx 585.4 \text{ kg}$$

About 585.4 kg will remain.

26.
$$0.2 = e^{-0.1t}$$
$$\ln 0.2 = -0.1t$$
$$t = -10 \ln 0.2 \approx 16.09 \text{ yr}$$

It will take about 16.09 years.

27. (a)
$$\frac{dp}{dn} = kp$$
$$\frac{dp}{p} = k\,dh$$
$$\int \frac{dp}{p} = \int k\,dh$$
$$\ln|p| = kh + C$$
$$e^{\ln|p|} = e^{kh+C}$$
$$|p| = e^C e^{kh}$$
$$p = Ae^{kh}$$

Initial condition: $p = p_0$ when $h = 0$

$$p_0 = Ae^0$$
$$A = p_0$$

Solution: $p = p_0 e^{kh}$

Using the giving altitude-pressure data, we have

$p_0 = 1013$ millibars, so:

$$p = 1013e^{kh}$$
$$90 = 1013e^{(k)(20)}$$
$$\frac{90}{1013} = e^{20k}$$
$$k = \frac{1}{20} \ln \frac{90}{1013} \approx -0.121 \text{ km}^{-1}$$

Thus, we have $p \approx 1013e^{-0.121h}$

(b) At 50 km, the pressure is

$1013e^{((1/20)\ln(90/1013))(50)} \approx 2.383$ millibars.

(c)
$$900 = 1013e^{kh}$$
$$\frac{900}{1013} = e^{kh}$$
$$h = \frac{1}{k} \ln \frac{900}{1013} = \frac{20 \ln(900/1013)}{\ln(90/1013)} \approx 0.977 \text{ km}$$

The pressure is 900 millibars at an altitude of about 0.977 km.

28. By the Law of Exponential Change, $y = 100e^{-0.6t}$. At $t = 1$ hour, the amount remaining will be $100e^{-0.6(1)} \approx 54.88$ grams.

29. (a) By the Law of Exponential Change, the solution is $V = V_0 e^{-(1/40)t}$.

(b)
$$0.1 = e^{-(1/40)t}$$
$$\ln 0.1 = -\frac{t}{40}$$
$$t = -40 \ln 0.1 \approx 92.1 \text{ sec}$$

It will take about 92.1 seconds.

30. (a) $A(t) = A_0 e^t$
It grows by a factor of e each year.

(b) $3 = e^t$
$\ln 3 = t$
It will take $\ln 3 \approx 1.1$ yr.

(c) In one year your account grows from A_0 to $A_0 e$, so you can earn $A_0 e - A_0$, or $(e - 1)$ times your initial amount. This represents an increase of about 172%.

31. (a) $s(t) = \int v_0 e^{-(k/m)t}\, dt = -\frac{v_0 m}{k} e^{-(k/m)t} + C$

Initial condition: $s(0) = 0$

$$0 = -\frac{v_0 m}{k} + C$$
$$\frac{v_0 m}{k} = C$$
$$s(t) = -\frac{v_0 m}{k} e^{-(k/m)t} + \frac{v_0 m}{k}$$
$$= \frac{v_0 m}{k}\left(1 - e^{-(k/m)t}\right)$$

(b) $\lim_{t\to\infty} s(t) = \lim_{t\to\infty} \frac{v_0 m}{k}\left(1 - e^{-(k/m)t}\right) = \frac{v_0 m}{k}$

32. (a) $90 = e^{(r)(100)}$

$\ln 90 = 100r$

$r = \frac{\ln 90}{100} \approx 0.045$ or 4.5%

(b) $131 = e^{(r)(100)}$

$\ln 131 = 100r$

$r = \frac{\ln 131}{100} \approx 0.049$ or 4.9%

33. $\frac{v_0 m}{k} = \text{coasting distance}$

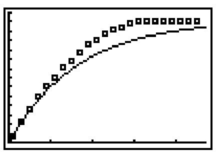
$$\frac{(0.80)(49.90)}{k} = 1.32$$
$$k = \frac{998}{33}$$

We know that $\frac{v_0 m}{k} = 1.32$ and $\frac{k}{m} = \frac{998}{33(49.9)} = \frac{20}{33}$.

Using Equation 3, we have:

$$s(t) = \frac{v_0 m}{k}(1 - e^{-(k/m)t})$$
$$= 1.32(1 - e^{-20t/33})$$
$$\approx 1.32(1 - e^{-0.606t})$$

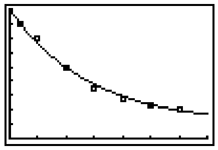
A graph of the model is shown superimposed on a graph of the data.



[0, 4.7] by [0, 1.4]

34. (a) $T - T_s = 79.47(0.932)^t$

(b) $T = 10 + 79.47(0.932)^t$



[0, 35] by [0, 90]

(c) Solving $T = 12$ and using the exact values from the regression equation, we obtain $t \approx 52.5$ sec.

(d) Substituting $t = 0$ into the equation we found in part (b), the temperature was approximately 89.47°C.

35. (a) $\frac{dT}{dt} = -k(T - T_s)$

$$\frac{dT}{T - T_s} = -k\, dt$$
$$\int \frac{dT}{T - T_s} = -k\, dt$$
$$\ln |T - T_s| = -kt + C$$
$$|T - T_s| = e^{-kt+C}$$
$$T - T_s = \pm e^C e^{-kt}$$
$$T - T_s = Ae^{-kt}$$

Initial condition: $T = T_0$ when $t = 0$

$$T_0 - T_s = Ae^{-(k)(0)}$$
$$T_0 - T_s = A$$

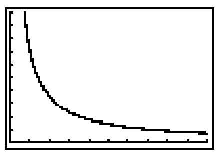
Solution: $T - T_s = (T_0 - T_s)e^{-kt}$

(b) $\lim_{t\to\infty} T = \lim_{t\to\infty} [T_s + (T_0 - T_s)e^{-kt}] = T_s$
Horizontal asymptote: $T = T_s$

36. (a) $2y_0 = y_0 e^{rt}$

$$2 = e^{rt}$$
$$\ln 2 = rt$$
$$t = \frac{\ln 2}{r}$$

(b)



[0, 0.1] by [0, 100]

(c) $\ln 2 \approx 0.69$, so the doubling time is $\frac{0.69}{r}$ which is almost the same as the rules.

(d) $\frac{70}{5} = 14$ years or $\frac{72}{5} = 14.4$ years

36. continued

(e) $3y_0 = y_0 e^{rt}$

$3 = e^{rt}$

$\ln 3 = rt$

$t = \frac{\ln 3}{r}$

Since $\ln 3 \approx 1.099$, a suitable rule is $\frac{108}{100r}$ or $\frac{108}{i}$.

(We choose 108 instead of 110 because 108 has more factors.)

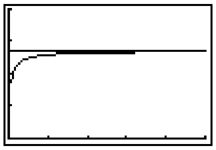
37. (a)

x	$\left(1 + \frac{1}{x}\right)^x$
10	2.5937
100	2.7048
1000	2.7169
10,000	2.7181
100,000	2.7183

$e \approx 2.7183$

Graphical support:

$y_1 = \left(1 + \frac{1}{x}\right)^x, y_2 = e$



[0, 50] by [0, 4]

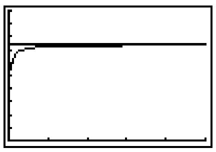
(b) $r = 2$

x	$\left(1 + \frac{2}{x}\right)^x$
10	6.1917
100	7.2446
1000	7.3743
10,000	7.3876
100,000	7.3889

$e^2 \approx 7.389$

Graphical support:

$y_1 = \left(1 + \frac{2}{x}\right)^x, y_2 = e^2$



[0, 500] by [0, 10]

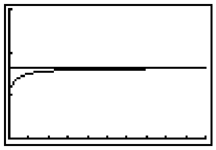
$r = 0.5$

x	$\left(1 + \frac{0.5}{x}\right)^x$
10	1.6289
100	1.6467
1000	1.6485
10,000	1.6487
100,000	1.6487

$e^{0.5} \approx 1.6487$

Graphical support:

$y_1 = \left(1 + \frac{0.5}{x}\right)^x, y_2 = e^{0.5}$



[0, 10] by [0, 3]

(c) As we compound more times, the increment of time between compounding approaches 0. Continuous compounding is based on an instantaneous rate of change which is a limit of average rates as the increment in time approaches 0.

38. (a) To simplify calculations somewhat, we may write:

$$v(t) = \sqrt{\frac{mg}{k}}\,\frac{e^{at} - e^{-at}}{e^{at} + e^{-at}}\,\frac{e^{at}}{e^{at}}$$
$$= \sqrt{\frac{mg}{k}}\,\frac{e^{2at} - 1}{e^{2at} + 1}$$
$$= \sqrt{\frac{mg}{k}}\,\frac{(e^{2at} + 1) - 2}{e^{2at} + 1}$$
$$= \sqrt{\frac{mg}{k}}\left(1 - \frac{2}{e^{2at} + 1}\right)$$

The left side of the differential equation is:

$$m\,\frac{dv}{dt} = m\,\sqrt{\frac{mg}{k}}(2)(e^{2at} + 1)^{-2}\,(2ae^{2at})$$
$$= 4ma\,\sqrt{\frac{mg}{k}}\,(e^{2at} + 1)^{-2}\,(e^{2at})$$
$$= 4m\,\sqrt{\frac{gk}{m}}\,\sqrt{\frac{mg}{k}}\,(e^{2at} + 1)^{-2}(e^{2at})$$
$$= \frac{4mge^{2at}}{(e^{2at} + 1)^2}$$

The right side of the differential equation is:

$$\begin{aligned} mg - kv^2 &= mg - k\left(\frac{mg}{k}\right)\left(1 - \frac{2}{e^{2at}+1}\right)^2 \\ &= mg\left[1 - \left(1 - \frac{2}{e^{2at}+1}\right)^2\right] \\ &= mg\left(1 - 1 + \frac{4}{e^{2at}+1} - \frac{4}{(e^{2at}+1)^2}\right) \\ &= mg\,\frac{4(e^{2at}+1) - 4}{(e^{2at}+1)^2} \\ &= \frac{4\,mg\,e^{2at}}{(e^{2at}+1)^2} \end{aligned}$$

Since the left and right sides are equal, the differential equation is satisfied.

And $v(0) = \sqrt{\frac{mg}{k}}\,\frac{e^0 - e^0}{e^0 + e^0} = 0$, so the initial condition is also satisfied.

(b) $$\begin{aligned} \lim_{t\to\infty} v(t) &= \lim_{t\to\infty}\left(\sqrt{\frac{mg}{k}}\,\frac{e^{at}-e^{-at}}{e^{at}+e^{-at}} \cdot \frac{e^{-at}}{e^{-at}}\right) \\ &= \lim_{t\to\infty}\left(\sqrt{\frac{mg}{k}}\,\frac{1-e^{-2at}}{1+e^{-2at}}\right) \\ &= \sqrt{\frac{mg}{k}}\left(\frac{1-0}{1+0}\right) = \sqrt{\frac{mg}{k}} \end{aligned}$$

The limiting velocity is $\sqrt{\frac{mg}{k}}$.

(c) $\sqrt{\frac{mg}{k}} = \sqrt{\frac{160}{0.005}} \approx 179$ ft/sec

The limiting velocity is about 179 ft/sec,

or about 122 mi/hr.

■ **Section 6.5** Population Growth
(pp. 342–349)

Quick Review 6.5

1. All real numbers

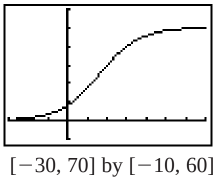
2. $\lim_{x\to\infty} f(x) = \frac{50}{1+0} = 50$

$\lim_{x\to-\infty} f(x) = 0$

3. $y = 0$, $y = 50$

4. In both f' and f'', the denominator will be a power of $1 + 5e^{-0.1x}$, which is never 0. Thus, the domains of both are all real numbers.

5.

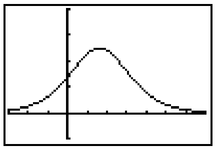


[−30, 70] by [−10, 60]

$f(x)$ has no zeros.

6. Use NDER $f(x)$, or calculate the derivative as follows.

$$f'(x) = \frac{d}{dx}\frac{50}{1 + 5e^{-0.1x}}$$
$$= \frac{(1 + 5e^{-0.1x})(0) - (50)(5e^{-0.1x})(-0.1)}{(1 + 5e^{-0.1x})^2}$$
$$= \frac{25e^{-0.1x}}{(1 + 5e^{-0.1x})^2}$$



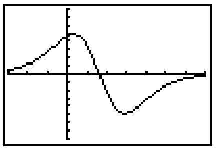
$[-30, 70]$ by $[-0.5, 2]$

(a) $(-\infty, \infty)$

(b) None

7. Use NDER(NDER $f(x)$), or calculate the second derivatives as follows.

$$f''(x) = \frac{d}{dx}\frac{25e^{-0.1x}}{(1 + 5e^{-0.1x})^2}$$
$$= \frac{(1 + 5e^{-0.1x})^2(25e^{-0.1x})(-0.1) - (25e^{-0.1x})(2)(1 + 5e^{-0.1x})(5e^{-0.1x})(-0.1)}{(1 + 5e^{-0.1x})^4}$$
$$= \frac{-2.5e^{-0.1x}[(1 + 5e^{-0.1x}) - 2(5e^{-0.1x}]}{(1 + 5e^{-0.1x})^3}$$
$$= \frac{12.5e^{-0.2x} - 2.5e^{-0.1x}}{(1 + 5e^{-0.1x})^3}$$



$[-30, 70]$ by $[-0.08, 0.08]$

Locate the inflection point using graphical methods, or analytically as follows.

$$f''(x) = 0$$
$$\frac{12.5e^{-0.2x} - 2.5e^{-0.1x}}{(1 + 5e^{-0.1x})^3} = 0$$
$$2.5e^{-0.1x}(5e^{-0.1x} - 1) = 0$$
$$e^{-0.1x} = \frac{1}{5}$$
$$-0.1x = -\ln 5$$
$$x = 10 \ln 5 \approx 16.094$$

(a) Since $f''(x) > 0$ for $x < 10 \ln 5$, the graph of f is concave up on the interval $(-\infty, 10 \ln 5)$, or approximately $(-\infty, 16.094)$.

(b) Since $f''(x) < 0$ for $x > 10 \ln 5$, the graph of f is concave down on the interval $(10 \ln 5, \infty)$, or approximately $(16.094, \infty)$.

8. Using the result of the previous exercise, the inflection point occurs at $x = 10 \ln 5$.

Since $f(10 \ln 5) = \frac{50}{1 + 5e^{-\ln 5}} = 25$,

the point of inflection is $(10 \ln 5, 25)$, or approximately $(16.094, 25)$.

9. $\frac{x - 12}{x^2 - 4x} = \frac{A}{x} + \frac{B}{x - 4}$

$x - 12 = A(x - 4) + Bx$

$x - 12 = (A + B)x - 4A$

Since $A + B = 1$ and $-4A = -12$, we have $A = 3$ and

$B = -2$.

10. $\frac{2x+16}{x^2+x-6} = \frac{A}{x+3} + \frac{B}{x-2}$

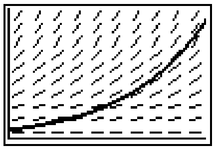
$2x + 16 = A(x-2) + B(x+3)$

When $x = -3$, the equation becomes $10 = -5A$, and when $x = 2$, the equation becomes $20 = 5B$. Thus, $A = -2$ and $B = 4$.

Section 6.5 Exercises

1. (a) $\frac{dP}{dt} = 0.025P$

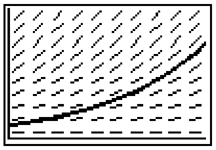
(b) Using the Law of Exponential Change from Section 6.4, the formula is $P = 75{,}000e^{0.025t}$

(c) 

[0, 100] by [0, 1,000,000]

2. (a) $\frac{dP}{dt} = 0.019P$

(b) Using the Law of Exponential Change from Section 6.4, the formula is $P = 110{,}000e^{0.019t}$.

(c) 

[0, 100] by [0, 1,000,000]

3. (a) $\frac{dP}{dt} = \frac{k}{M}P(M-P)$

$\frac{dP}{dt} = \frac{0.05}{200}P(200-P)$

$\frac{dP}{dt} = 0.00025P(200-P)$

(b) $P = \frac{M}{1+Ae^{-kt}}$

$P = \frac{200}{1+Ae^{-0.05t}}$

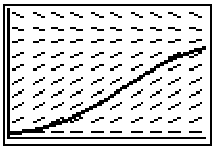
Initial condition: $P(0) = 10$

$10 = \frac{200}{1+Ae^0}$

$1 + A = \frac{200}{10} = 20$

$A = 19$

Formula: $P = \frac{200}{1+19e^{-0.05t}}$

(c) 

[0, 100] by [0, 250]

4. (a) $\frac{dP}{dt} = \frac{k}{m}P(M-P)$

$\frac{dP}{dt} = \frac{0.02}{150}P(150-P)$

$\frac{dP}{dt} = \frac{1}{7500}P(150-P)$

(b) $P = \frac{M}{1+Ae^{-kt}}$

$P = \frac{150}{1+Ae^{-0.02t}}$

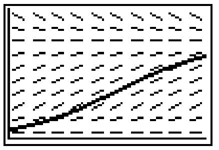
Initial condition: $P(0) = 15$

$15 = \frac{150}{1+Ae^0}$

$1 + A = \frac{150}{15} = 10$

$A = 9$

Formula: $P = \frac{150}{1+9e^{-0.02t}}$

(c) 

[0, 200] by [0, 200]

5. The growth rate is -0.3 or -30%.

6. The growth rate is 0.075 or 7.5%.

7. $\frac{dP}{dt} = 0.04P - 0.0004P^2$

$= 0.0004P(100-P)$

$= \frac{0.04}{100}P(100-P)$

$= \frac{k}{M}P(M-P)$

Thus, $k = 0.04$ and the carrying capacity is $M = 100$.

8. $\frac{50}{P}\frac{dP}{dt} = 2 - \frac{P}{250}$

$\frac{1}{P}\frac{dP}{dt} = \frac{1}{50}\left(2 - \frac{P}{250}\right)$

$= \frac{1}{25}\left(1 - \frac{P}{500}\right)$

$= 0.04\left(1 - \frac{P}{500}\right)$

$= k\left(1 - \frac{P}{M}\right)$

Thus, $k = 0.04$ and the carrying capacity is $M = 500$.

9. Choose the slope field that shows slopes that increase as y increases. (d)

10. Choose the slope field that matches a logistic differential equation with $M = 100$. (b)

11. Choose the only slope field whose slopes vary with x as well as with y. (c)

12. Choose the slope field that matches a logistic differential equation with $M = 150$. (a)

13. **(a)** $P(t) = \frac{1000}{1 + e^{4.8-0.7t}}$

$= \frac{1000}{1 + e^{4.8}e^{-0.7t}}$

$= \frac{M}{1 + Ae^{-kt}}$

This is a logistic growth model with $k = 0.7$ and $M = 1000$.

(b) $P(0) = \frac{1000}{1 + e^{4.8}} \approx 8$

Initially there are 8 rabbits.

14. **(a)** $P(t) = \frac{200}{1 + e^{5.3-t}}$

$= \frac{200}{1 + e^{5.3}e^{-t}}$

$= \frac{M}{1 + Ae^{-kt}}$

This is a logistic growth model with $k = 1$ and $M = 200$.

(b) $P(0) = \frac{200}{1 + e^{5.3}} \approx 1$

Initially 1 student has the measles.

15. **(a)** Note that $\frac{dP}{dT} = \frac{1 \text{ person}}{14 \text{ sec}} \cdot \frac{365 \cdot 24 \cdot 3600 \text{ sec}}{1 \text{ yr}}$

$\approx 2{,}252{,}571$ people per year.

The relative growth rate is

$\frac{\frac{dP}{dT}}{P} \approx \frac{2{,}252{,}571}{257{,}313{,}431} \approx 0.00875$ or 0.875%

(b) The population after 8 years will be approximately $P_0e^{rt} = 257{,}313{,}431e^{8r}$

$\approx 275{,}980{,}017$, where r is the unrounded rate from part (a).

16. **(a)** Let t be the number of years.

$1000 = 10{,}000(0.8)^t$

$0.1 = 0.8^t$

$\ln 0.1 = t \ln 0.8$

$t = \frac{\ln 0.1}{\ln 0.8} \approx 10.32$

It will take about 10.32 years.

(b) Let $f(t) = 10{,}000(0.8)^t$. So that $f(t)$ will round to less than 1, we actually require $f(t) < 0.5$.

$0.5 = 10{,}000(0.8)^t$

$0.00005 = 0.8^t$

$\ln 0.00005 = t \ln 0.8$

$t = \frac{\ln 0.00005}{\ln 0.8} \approx 44.38$

It will take about 44.4 years.

17. **(a)** $\frac{dP}{dt} = 0.0015P(150 - P)$

$= \frac{0.225}{150}P(150 - P)$

$= \frac{k}{M}P(M - P)$

Thus, $k = 0.225$ and $M = 150$.

$P = \frac{M}{1 + Ae^{-kt}}$

$= \frac{150}{1 + Ae^{-0.225t}}$

Initial condition: $P(0) = 6$

$6 = \frac{150}{1 + Ae^0}$

$1 + A = 25$

$A = 24$

Formula: $P = \frac{150}{1 + 24e^{-0.225t}}$

(b) $100 = \frac{150}{1 + 24e^{-0.225t}}$

$1 + 24e^{-0.225t} = \frac{3}{2}$

$24e^{-0.225t} = \frac{1}{2}$

$e^{-0.225t} = \frac{1}{48}$

$-0.225t = -\ln 48$

$t = \frac{\ln 48}{0.225} \approx 17.21$ weeks

$125 = \frac{150}{1 + 24e^{-0.225t}}$

$1 + 24e^{-0.225t} = \frac{6}{5}$

$24e^{-0.225t} = \frac{1}{5}$

$e^{-0.225t} = \frac{1}{120}$

$-0.225t = -\ln 120$

$t = \frac{\ln 120}{0.225} \approx 21.28$

It will take about 17.21 weeks to reach 100 guppies, and about 21.28 weeks to reach 125 guppies.

18. (a)
$$\frac{dP}{dt} = 0.0004P(250 - P) = \frac{0.1}{250}P(250 - P) = \frac{k}{M}P(M - P)$$

Thus, $k = 0.1$ and $M = 250$.

$$P = \frac{M}{1 + Ae^{-kt}} = \frac{250}{1 + Ae^{-0.1t}}$$

Initial condition: $P(0) = 28$, where $t = 0$ represents the year 1970.

$$28 = \frac{250}{1 + Ae^0}$$
$$28(1 + A) = 250$$
$$A = \frac{250}{28} - 1 = \frac{111}{14} \approx 7.9286$$

Formula: $P(t) = \dfrac{250}{1 + 111e^{-0.1t}/14}$, or approximately

$$P(t) = \frac{250}{1 + 7.9286e^{-0.1t}}$$

(b) The population $P(t)$ will round to 250 when

$$P(t) \geq 249.5.$$
$$249.5 = \frac{250}{1 + 111e^{-0.1t}/14}$$
$$249.5\left(1 + \frac{111e^{-0.1t}}{14}\right) = 250$$
$$\frac{(249.5)(111e^{-0.1t})}{14} = 0.5$$
$$e^{-0.1t} = \frac{14}{55{,}389}$$
$$-0.1t = \ln\frac{14}{55{,}389}$$
$$t = 10(\ln 55{,}389 - \ln 14) \approx 82.8$$

It will take about 83 years.

19. (a) $y = y_0e^{-0.01(t/1000)} = y_0e^{-0.00001t}$

(b)
$$0.9 = e^{-0.00001t}$$
$$\ln 0.9 = -0.00001t$$
$$t = -100{,}000 \ln 0.9 \approx 10{,}536$$

It will take about 10,536 years.

(c) $y = y_0e^{-(0.00001)(20{,}000)} \approx 0.819y_0$

The tooth size will be about 81.9% of our present tooth size.

20. First find the time to grow from 5000 bees to 10,000 bees.

$$\frac{dP}{dt} = \frac{1}{4}P$$
$$P(t) = 5000e^{0.25t}$$
$$10{,}000 = 5000e^{0.25t}$$
$$2 = e^{0.25t}$$
$$\ln 2 = 0.25t$$
$$4 \ln 2 = t$$

Now find the time to grow from 10,000 bees to 25,000 bees.

$$\frac{dP}{dt} = \frac{1}{12}P$$
$$P(t) = 10{,}000e^{t/12}$$
$$25{,}000 = 10{,}000e^{t/12}$$
$$2.5 = e^{t/12}$$
$$\ln 2.5 = \frac{t}{12}$$
$$12 \ln 2.5 = t$$

The total time required is $4 \ln 2 + 12 \ln 2.5 \approx 13.8$ years.

21. (a)
$$\frac{dx}{dt} = 1000 + 0.10x$$
$$\int \frac{dx}{1000 + 0.1x} = \int dt$$
$$10 \ln |1000 + 0.1x| = t + C$$
$$\ln |1000 + 0.1x| = 0.1(t + C)$$
$$1000 + 0.1x = \pm e^{0.1(t+C)}$$
$$0.1x = -1000 \pm e^{0.1C}\, e^{0.1t}$$
$$x = -10{,}000 \pm 10e^{0.1C}\, e^{0.1t}$$
$$x = -10{,}000 + Ae^{0.1t}$$

Initial condition: $x(0) = 1000$

$$1000 = -10{,}000 + Ae^0$$
$$11{,}000 = A$$

Solution: $x = 11{,}000e^{0.1t} - 10{,}000$

(b)
$$100{,}000 = 11{,}000e^{0.1t} - 10{,}000$$
$$10 = e^{0.1t}$$
$$\ln 10 = 0.1t$$
$$t = 10 \ln 10 \approx 23 \text{ yr}$$

It will take about 23 years.

22. (a) Using the Law of Exponential Change in Section 6.4, the solution is $p(x) = p_0e^{-x/100}$, where $p_0 = p(0)$.
Initial condition: $p(100) = 20.09$
$20.09 = p_0e^{-1}$
$20.09e = p_0$
Solution: $p(x) = (20.09e)e^{-x/100}$
or $p(x) = 20.09e^{1-0.01x}$

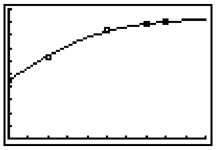
(b) $p(10) = 20.09e^{0.9} \approx \$49.41$
$p(90) = 20.09e^{0.1} \approx \$22.20$

(c) $r(x) = xp(x) = 20.09x\, e^{1-0.01x}$
$r'(x) = 20.09[(x)(e^{1-0.01x})(-0.01) + (e^{1-0.01x})(1)]$
$= 20.09e^{1-0.01x}(1 - 0.01x)$

The derivative is zero at $x = 100$, positive for $x < 100$, and negative for $x > 100$, so $r(x)$ has its maximum value at $x = 100$.

23. **(a)** Note that the given years correspond to $x = 0$, $x = 20$, $x = 50$, $x = 70$, and $x = 80$.

$$y = \frac{18.70}{1 + 1.075e^{-0.0422x}}$$

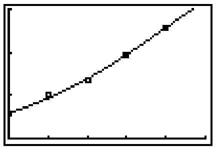


[0, 100] by [0, 20]

(b) Carrying capacity $= \lim_{x \to \infty} y = 18.70$, representing 18.7 million people.

(c) Using NDER twice and solving graphically, we find that $y'' = 0$ when $x \approx 1.7$, corresponding to the year 1912. The population at this time was about $y(1.7) \approx 9.35$ million.

24. **(a)** $y = \dfrac{24.76}{1 + 7.195e^{-0.0513x}}$



[0, 50] by [0, 15]

(b) Carrying capacity $= \lim_{x \to \infty} y = 24.76$, representing 24.76 million people.

(c) Using NDER twice and solving graphically, we find that $y'' = 0$ when $x \approx 38.44$, corresponding to the year 1988. The population at this time was about $y(38.44) \approx 12.38$ million.

25.

$$\begin{aligned}
\frac{dP}{dt} &= \frac{k}{M}P(M - P) \\
\frac{M\,dP}{P(M - P)} &= k\,dt \\
\frac{(M - P) + P}{P(M - P)}\,dP &= k\,dt \\
\left(\frac{1}{P} + \frac{1}{M - P}\right)dP &= k\,dt \\
\ln|P| - \ln|M - P| &= kt + C \\
\ln\left|\frac{P}{M - P}\right| &= kt + C \\
\ln\left|\frac{M - P}{P}\right| &= -kt - C \\
\frac{M - P}{P} &= \pm e^{-C}e^{-kt} \\
\frac{M - P}{P} &= Ae^{-kt} \\
M - P &= APe^{-kt} \\
M &= P(1 + Ae^{-kt}) \\
P &= \frac{M}{1 + Ae^{-kt}}
\end{aligned}$$

26. **(a)** $y = \dfrac{16.90}{1 + 5.132e^{-0.0666x}}$

(b) Carrying capacity $= \lim_{x \to \infty} y = 16.90$ representing 16.9 million people.

27. $\dfrac{dy}{dx} = (\cos x)e^{\sin x}$

$\int dy = \int (\cos x)e^{\sin x}\,dx$

$\int dy = \int e^u\,du$

$y = e^u + C$

$y = e^{\sin x} + C$

Initial value: $y(0) = 0$

$0 = e^{\sin 0} + C$

$-1 = C$

Solution: $y = e^{\sin x} - 1$

28. $\dfrac{dy}{dx} = -2(y - 3)$

$\int \dfrac{dy}{y - 3} = -2\int dx$

$\ln|y - 3| = -2x + C$

$y - 3 = \pm e^{-2x+C}$

$y = 3 + Ae^{-2x}$

Initial condition: $y(0) = 5$

$5 = 3 + Ae^0$

$2 = A$

Solution: $y = 3 + 2e^{-2x}$

29. $\dfrac{dy}{dx} = \dfrac{x}{y}$

$\int y\,dy = \int x\,dx$

$\dfrac{y^2}{2} = \dfrac{x^2}{2} + C$

Initial condition:

$y(0) = 2$

$\dfrac{2^2}{2} = \dfrac{0^2}{2} + C$

$2 = C$

This gives $\dfrac{y^2}{2} = \dfrac{x^2}{2} + 2$, or $y^2 = x^2 + 4$.

But this equation represents two functions, $y = \pm\sqrt{x^2 + 4}$.

The solution of the initial value problem is the function that satisfies the initial condition, namely $y = \sqrt{x^2 + 4}$.

30. $\frac{dy}{dx} = y\sqrt{x}$

$$\int \frac{dy}{y} = \int x^{1/2}\,dx$$

$$\ln|y| = \frac{2}{3}x^{3/2} + C$$

$$|y| = e^{(2/3)x^{3/2}+C}$$

$$y = \pm e^C e^{(2/3)x^{3/2}}$$

$$y = Ae^{(2/3)x^{3/2}}$$

Initial condition: $y(0) = 1$

$$1 = Ae^0$$

$$1 = A$$

Solution: $y = e^{(2/3)x^{3/2}}$

31. (a) Note that $k > 0$ and $M > 0$, so the sign of $\frac{dP}{dt}$ is the same as the sign of $(M - P)(P - m)$. For $m < P < M$, both $M - P$ and $P - m$ are positive, so the product is positive. For $P < m$ or $P > M$, the expressions $M - P$ and $P - m$ have opposite signs, so the product is negative.

(b) $\frac{dP}{dt} = \frac{k}{M}(M - P)(P - m)$

$$\frac{dP}{dt} = \frac{k}{1200}(1200 - P)(P - 100)$$

$$\frac{1200}{(1200 - P)(P - 100)}\frac{dP}{dt} = k$$

$$\frac{1100}{(1200 - P)(P - 100)}\frac{dP}{dt} = \frac{11}{12}k$$

$$\frac{(P - 100) + (1200 - P)}{(1200 - P)(P - 100)}\frac{dP}{dt} = \frac{11}{12}k$$

$$\left(\frac{1}{1200 - P} + \frac{1}{P - 100}\right)\frac{dP}{dt} = \frac{11}{12}k$$

$$\int\left(\frac{1}{1200 - P} + \frac{1}{P - 100}\right)dP = \frac{11}{12}k\,dt$$

$$-\ln|1200 - P| + \ln|P - 100| = \frac{11}{12}kt + C$$

$$\ln\left|\frac{P - 100}{1200 - P}\right| = \frac{11}{12}kt + C$$

$$\frac{P - 100}{1200 - P} = \pm e^C e^{11kt/12}$$

$$\frac{P - 100}{1200 - P} = Ae^{11kt/12}$$

$$P - 100 = 1200Ae^{11kt/12} - APe^{11kt/12}$$

$$P(1 + Ae^{11kt/12}) = 1200Ae^{11kt/12} + 100$$

$$P = \frac{1200Ae^{11kt/12} + 100}{1 + Ae^{11kt/12}}$$

(c) $300 = \frac{1200Ae^0 + 100}{1 + Ae^0}$

$$300(1 + A) = 1200A + 100$$

$$300 - 100 = 1200A - 300A$$

$$200 = 900A$$

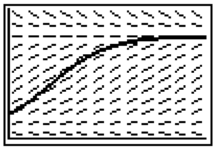
$$A = \frac{2}{9}$$

$$P(t) = \frac{1200(2/9)e^{11kt/12} + 100}{1 + (2/9)e^{11kt/12}}$$

$$P(t) = \frac{1200(2)e^{11kt/12} + 100(9)}{9 + 2e^{11k/12}}$$

$$P(t) = \frac{300(8e^{11kt/12} + 3)}{9 + 2e^{11kt/12}}$$

(d)



[0, 75] by [0, 1500]

Note that the slope field is given by $\frac{dP}{dt} = \frac{0.1}{1200}(1200 - P)(P - 100)$.

(e) $\frac{dP}{dt} = \frac{k}{M}(M - P)(P - m)$

$$\frac{M}{(M - P)(P - m)}\frac{dP}{dt} = k$$

$$\frac{M}{M - m}\frac{M - m}{(M - P)(P - m)}\frac{dP}{dt} = k$$

$$\frac{(P - m) + (M - P)}{(M - P)(P - m)}\frac{dP}{dt} = \frac{M - m}{M}k$$

$$\left(\frac{1}{M - P} + \frac{1}{P - m}\right)\frac{dP}{dt} = \frac{M - m}{M}k$$

$$\int\left(\frac{1}{M - P} + \frac{1}{P - m}\right)dP = \int\frac{M - m}{M}k\,dt$$

$$-\ln|M - P| + \ln|P - m| = \frac{M - m}{M}kt + C$$

$$\ln\left|\frac{P - m}{M - P}\right| = \frac{M - m}{M}kt + C$$

$$\frac{P - m}{M - P} = \pm e^C e^{(M-m)kt/M}$$

$$\frac{P - m}{M - P} = Ae^{(M-m)kt/M}$$

$$P - m = (M - P)Ae^{(M-m)kt/M}$$

$$P(1 + Ae^{(M-m)kt/M}) = AMe^{(M-m)kt/M} + m$$

$$P = \frac{AMe^{(M-m)kt/M} + m}{1 + Ae^{(M-m)kt/M}}$$

$$P(0) = \frac{AMe^0 + m}{1 + Ae^0} = \frac{AM + m}{1 + A}$$

$$P(0)(1 + A) = AM + m$$

$$A(P(0) - M) = m - P(0)$$

$$A = \frac{m - P(0)}{P(0) - M} = \frac{P(0) - m}{M - P(0)}$$

Therefore, the solution to the differential equation is

$$P = \frac{AMe^{(M-m)kt/M} + m}{1 + Ae^{(M-m)kt/M}} \text{ where } A = \frac{P(0) - m}{M - P(0)}.$$

32. (a)
$$\frac{dp}{dt} = k(t)p$$
$$\int \frac{dp}{p} = \int k(t)\,dt$$
$$\ln|p| = \int_0^t k(u)\,du + C$$
$$p = \pm e^C e^{\int_0^t k(u)\,du}$$
$$p = Ae^{\int_0^t k(u)\,du}$$

Initial condition: $p(0) = p_0$.

$$p_0 = Ae^{\int_0^0 k(u)\,du}$$
$$p_0 = Ae^0$$
$$p_0 = A$$

Solution: $p(t) = p_0 e^{\int_0^t k(u)\,du}$

(b)
$$\int_0^9 k(u)\,du = \int_0^9 \frac{0.04}{1+u}\,du$$
$$= 0.04 \ln(1+u)\Big]_0^9$$
$$= 0.04(\ln 10 - \ln 1)$$
$$= 0.04 \ln 10$$
$$p(9) = p_0 e^{\int_0^9 k(u)\,du}$$
$$= 100e^{0.04 \ln 10} \approx 109.65$$

After 9 years during which the inflation rate is $\frac{0.04}{1+t}$ per year, the price of an item which originally cost \$100 will be increased to \$109.65.

(c) $p(9) = p_0 e^{\int_0^9 0.04\,du} = 100e^{0.04(9)} \approx 143.33$

The price will be \$143.33.

(d)
$$\int_0^9 k(u)\,du = \int_0^9 (0.04 + 0.004u)\,du$$
$$= \Big[0.04u + 0.002u^2\Big]_0^9 = 0.522$$
$$p(9) = p_0 e^{\int_0^9 k(u)\,du} = 100e^{0.522} \approx 168.54$$

33. (a)
$$\frac{dP}{dt} = kP^2$$
$$\int P^{-2}\,dP = \int k\,dt$$
$$-P^{-1} = kt + C$$
$$P = -\frac{1}{kt + C}$$

Initial condition: $P(0) = P_0$

$$P_0 = -\frac{1}{C}$$
$$C = -\frac{1}{P_0}$$

Solution: $P = -\frac{1}{kt - (1/P_0)} = \frac{P_0}{1 - kP_0 t}$

(b) There is a vertical asymptote at $t = \frac{1}{kP_0}$

■ Section 6.6 Numerical Methods
(pp. 350–356)

Quick Review 6.6

1. $f'(x) = 3x^2 - 3$
$f'(2) = 3(2)^2 - 3 = 9$

2. $L(x) = f(2) + f'(2)(x-2)$
$= 2 + 9(x-2)$
$= 9x - 16$

3. $f'(x) = \sec^2 x$
$$f'\left(\frac{\pi}{4}\right) = \sec^2 \frac{\pi}{4} = (\sqrt{2})^2 = 2$$

4.
$$L(x) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right)$$
$$= 1 + 2\left(x - \frac{\pi}{4}\right)$$
$$= 2x + 1 - \frac{\pi}{2}$$

5. $f'(x) = 0.2x - 5x^{-2}$
$f'(4) = 0.2(4) - 5(4)^{-2} = 0.4875$

6. $L(x) = f(4) + f'(4)(x-4)$
$= 2.85 + 0.4875(x-4)$
$= 0.4875x + 0.9$

7. $L(4.1) = 0.4875(4.1) + 0.9 = 2.89875$

$f(4.1) = 0.1(4.1)^2 + \frac{5}{4.1} \approx 2.900512$

(a) $|L(4.1) - f(4.1)| \approx 0.001762$

(b) $\frac{|L(4.1) - f(4.1)|}{f(4.1)} \approx 0.00061 = 0.061\%$

8. $L(4.2) = 0.4875(4.2) + 0.9 = 2.9475$

$f(4.2) = 0.1(4.2)^2 + \frac{5}{4.2} \approx 2.954476$

(a) $|L(4.2) - f(4.2)| \approx 0.006976$

(b) $\frac{|L(4.2) - f(4.2)|}{f(4.2)} \approx 0.00236 = 0.236\%$

9. $L(4.5) = 0.4875(4.5) + 0.9 = 3.09375$

$f(4.5) = 0.1(4.5)^2 + \frac{5}{4.5} \approx 3.136111$

(a) $|L(4.5) - f(4.5)| \approx 0.042361$

(b) $\frac{|L(4.5) - f(4.5)|}{f(4.5)} \approx 0.01351 = 1.351\%$

10. $L(3.5) = 0.4875(3.5) + 0.9 = 2.60625$

$f(3.5) = 0.1(3.5)^2 + \frac{5}{3.5} \approx 2.653571$

(a) $|L(3.5) - f(3.5)| \approx 0.047321$

(b) $\frac{|L(3.5) - f(3.5)|}{f(3.5)} \approx 0.01783 = 1.783\%$

Section 6.6 Exercises

1. Check the differential equation:

$$y' = \frac{d}{dx}(x - 1 + 2e^{-x}) = 1 + 2e^{-x}(-1) = 1 - 2e^{-x}$$

$$x - y = x - (x - 1 + 2e^{-x}) = 1 - 2e^{-x}$$

Therefore, $y' = x - y$.

Check the initial condition:

$$y(0) = 0 - 1 + 2e^{-(0)} = -1 + 2 = 1$$

2. Check the differential equation:

$$y' = \frac{d}{dx}(x - 1 - e^{-x}) = 1 - e^{-x}(-1) = 1 + e^{-x}$$

$$x - y = x - (x - 1 - e^{-x}) = 1 + e^{-x}$$

Therefore, $y' = x - y$.

Check the initial condition:

$$y(0) = 0 - 1 - e^{-(0)} = -1 - 1 = -2$$

3. Check the differential equation:

$$y' = \frac{d}{dx}\left(\frac{e^{2x} - 2\sin x - \cos x}{5}\right) = \frac{2e^{2x} - 2\cos x + \sin x}{5}$$

$$\begin{aligned} 2y + \sin x &= 2\left(\frac{e^{2x} - 2\sin x - \cos x}{5}\right) + \sin x \\ &= \frac{2e^{2x} - 4\sin x - 2\cos x + 5\sin x}{5} \\ &= \frac{2e^{2x} - 2\cos x + \sin x}{5} \end{aligned}$$

Therefore, $y' = 2y + \sin x$

Check the initial condition:

$$y(0) = \frac{e^{2(0)} - 2\sin 0 - \cos 0}{5} = \frac{1 - 1}{5} = 0$$

4. Check the differential equation:

$$y' = \frac{d}{dx}(e^x - e^{2x} - 1) = e^x - 2e^{2x}$$

$$y - e^{2x} + 1 = (e^x - e^{2x} - 1) - e^{2x} + 1 = e^x - 2e^{2x}$$

Therefore, $y' = y - e^{2x} + 1$.

Check the initial condition:

$$y(0) = e^0 - e^{2(0)} - 1 = -1$$

5. Note that we are finding an exact solution to the initial value problem discussed in Examples 1–4.

$$\frac{dy}{dx} = 1 + y$$

$$\int \frac{dy}{1 + y} = \int dx$$

$$\ln|1 + y| = x + C$$

$$|1 + y| = e^{x+C}$$

$$1 + y = \pm e^{x+C}$$

$$y = \pm e^C e^x - 1$$

$$y = Ae^x - 1$$

Initial condition: $y(0) = 1$
$1 = Ae^0 - 1$
$2 = A$
Solution: $y = 2e^x - 1$

6.

$$\frac{dy}{dx} = x(1 - y)$$

$$\int \frac{dy}{1 - y} = \int x\,dx$$

$$-\ln|1 - y| = \frac{1}{2}x^2 + C$$

$$|1 - y| = e^{-(x^2/2) - C}$$

$$1 - y = \pm e^{-(x^2/2) - C}$$

$$y = \pm e^{-C} e^{-x^2/2} + 1$$

$$y = Ae^{-x^2/2} + 1$$

Initial condition: $y(-2) = 0$
$0 = Ae^{-(-2)^2/2} + 1$
$0 = Ae^{-2} + 1$
$-e^2 = A$
Solution: $y = -e^2 e^{-x^2/2} + 1$ or $y = -e^{-(x^2/2)+2} + 1$

7.

$$\frac{dy}{dx} = 2y(x + 1)$$

$$\frac{dy}{y} = 2(x + 1)\,dx$$

$$\int \frac{dy}{y} = \int (2x + 2)\,dx$$

$$\ln|y| = x^2 + 2x + C$$

$$|y| = e^{x^2+2x+C}$$

$$y = \pm e^C e^{x^2+2x}$$

$$y = A\,e^{x^2+2x}$$

Initial condition: $y(-2) = 2$
$2 = Ae^{(-2)^2+2(-2)}$
$2 = A$
Solution: $y = 2e^{x^2+2x}$

8.

$$\frac{dy}{dx} = y^2(1 + 2x)$$

$$\int y^{-2}\,dy = \int (1 + 2x)\,dx$$

$$-y^{-1} = x + x^2 + C$$

$$y = -\frac{1}{x^2 + x + C}$$

Initial condition: $y(-1) = -1$

$$-1 = -\frac{1}{(-1)^2 + (-1) + C}$$

$$-1 = -\frac{1}{C}$$

$C = 1$

Solution: $y = -\dfrac{1}{x^2 + x + 1}$

9. To find the approximate values, set $y_1 = 2y + \sin x$ and use EULERT with initial values $x = 0$ and $y = 0$ and step size 0.1 for 10 points. The exact values are given by $y = \frac{1}{5}(e^{2x} - 2\sin x - \cos x)$.

x	y (Euler)	y (exact)	Error
0	0	0	0
0.1	0	0.0053	0.0053
0.2	0.0100	0.0229	0.0129
0.3	0.0318	0.0551	0.0233
0.4	0.0678	0.1051	0.0374
0.5	0.1203	0.1764	0.0561
0.6	0.1923	0.2731	0.0808
0.7	0.2872	0.4004	0.1132
0.8	0.4090	0.5643	0.1553
0.9	0.5626	0.7723	0.2097
1.0	0.7534	1.0332	0.2797

10. To find the approximate values, set $y_1 = x - y$ and use EULERT with initial values $x = 0$ and $y = -2$ and step size 0.1 for 10 points. The exact values are given by $y = x - 1 - e^{-x}$.

x	y (Euler)	y (exact)	Error
0	-2	-2	0
0.1	-1.8000	-1.8048	0.0048
0.2	-1.6100	-1.6187	0.0087
0.3	-1.4290	-1.4408	0.0118
0.4	-1.2561	-1.2703	0.0142
0.5	-1.0905	-1.1065	0.0160
0.6	-0.9314	-0.9488	0.0174
0.7	-0.7783	-0.7966	0.0183
0.8	-0.6305	-0.6493	0.0189
0.9	-0.4874	-0.5066	0.0191
1.0	-0.3487	-0.3679	0.0192

11. To find the approximate values, set $y_1 = 2y(x + 1)$ and use IMPEULT with initial values $x = -2$ and $y = 2$ and step size 0.1 for 20 points. The exact values are given by $y = 2e^{x^2+2x}$.

x	y (improved Euler)	y (exact)	Error
-2	2	2	0
-1.9	1.6560	1.6539	0.0021
-1.8	1.3983	1.3954	0.0030
-1.7	1.2042	1.2010	0.0032
-1.6	1.0578	1.0546	0.0032
-1.5	0.9478	0.9447	0.0031
-1.4	0.8663	0.8634	0.0029
-1.3	0.8077	0.8050	0.0027
-1.2	0.7683	0.7658	0.0025
-1.1	0.7456	0.7432	0.0024
-1.0	0.7381	0.7358	0.0023
-0.9	0.7455	0.7432	0.0023
-0.8	0.7682	0.7658	0.0024
-0.7	0.8075	0.8050	0.0024
-0.6	0.8659	0.8634	0.0025
-0.5	0.9473	0.9447	0.0026
-0.4	1.0572	1.0546	0.0026
-0.3	1.2036	1.2010	0.0026
-0.2	1.3976	1.3954	0.0022
-0.1	1.6553	1.6539	0.0014
0	1.9996	2	0.0004

12. To find the approximate values, set $y_1 = x(1 - y)$ and use IMPEULT with initial values $x = -2$ and $y = 0$ and step size 0.1 for 20 points. The exact values are given by $y = -e^{-(x^2/2)+2} + 1$.

x	$y\left(\begin{array}{c}\text{improved}\\\text{Euler}\end{array}\right)$	y (exact)	Error
−2	0	0	0
−1.9	−0.2140	−0.2153	0.0013
−1.8	−0.4593	−0.4623	0.0029
−1.7	−0.7371	−0.7419	0.0049
−1.6	−1.0473	−1.0544	0.0071
−1.5	−1.3892	−1.3989	0.0097
−1.4	−1.7607	−1.7732	0.0125
−1.3	−2.1585	−2.1740	0.0155
−1.2	−2.5780	−2.5966	0.0186
−1.1	−3.0131	−3.0350	0.0219
−1.0	−3.4565	−3.4817	0.0252
−0.9	−3.9000	−3.9283	0.0284
−0.8	−4.3341	−4.3656	0.0315
−0.7	−4.7491	−4.7834	0.0344
−0.6	−5.1348	−5.1719	0.0370
−0.5	−5.4815	−5.5208	0.0394
−0.4	−5.7796	−5.8210	0.0413
−0.3	−6.0210	−6.0639	0.0430
−0.2	−6.1986	−6.2427	0.0441
−0.1	−6.3073	−6.3522	0.0449
0	−6.3438	−6.3891	0.0452

13. To find the approximate values, set $y_1 = x - y$ and use EULERT and IMPEULT with initial values $x = 0$ and $y = 1$ and step size 0.1 for 20 points. The exact values are given by $y = x - 1 + 2e^{-x}$.

x	y (Euler)	y (improved Euler)	y (exact)	Error (Euler)	Error (improved Euler)
0	1	1	1	0	0
0.1	0.9000	0.9100	0.9097	0.0097	0.0003
0.2	0.8200	0.8381	0.8375	0.0175	0.0006
0.3	0.7580	0.7824	0.7816	0.0236	0.0008
0.4	0.7122	0.7416	0.7406	0.0284	0.0010
0.5	0.6810	0.7142	0.7131	0.0321	0.0011
0.6	0.6629	0.6988	0.6976	0.0347	0.0012
0.7	0.6566	0.6944	0.6932	0.0366	0.0012
0.8	0.6609	0.7000	0.6987	0.0377	0.0013
0.9	0.6748	0.7145	0.7131	0.0383	0.0013
1.0	0.6974	0.7371	0.7358	0.0384	0.0013
1.1	0.7276	0.7671	0.7657	0.0381	0.0013
1.2	0.7649	0.8037	0.8024	0.0375	0.0013
1.3	0.8084	0.8463	0.8451	0.0367	0.0013
1.4	0.8575	0.8944	0.8932	0.0357	0.0012
1.5	0.9118	0.9475	0.9463	0.0345	0.0012
1.6	0.9706	1.0050	1.0038	0.0332	0.0012
1.7	1.0335	1.0665	1.0654	0.0318	0.0011
1.8	1.1002	1.1317	1.1306	0.0304	0.0011
1.9	1.1702	1.2002	1.1991	0.0290	0.0010
2.0	1.2432	1.2716	1.2707	0.0275	0.0010

14. To find the approximate values, set $y_1 = y - e^{2x} + 1$ and use EULERT and IMPEULT with initial values $x = 0$ and $y = -1$ and step size 0.1 for 20 points. The exact values are given by $y = e^x - e^{2x} - 1$.

x	y (Euler)	$y\left(\begin{matrix}\text{improved}\\\text{Euler}\end{matrix}\right)$	y (exact)	Error (Euler)	Error $\left(\begin{matrix}\text{improved}\\\text{Euler}\end{matrix}\right)$
0	−1	−1	−1	0	0
0.1	−1.1000	−1.1161	−1.1162	0.0162	0.0002
0.2	−1.2321	−1.2700	−1.2704	0.0383	0.0004
0.3	−1.4045	−1.4715	−1.4723	0.0677	0.0007
0.4	−1.6272	−1.7325	−1.7337	0.1065	0.0012
0.5	−1.9125	−2.0678	−2.0696	0.1571	0.0018
0.6	−2.2756	−2.4954	−2.4980	0.2224	0.0026
0.7	−2.7351	−3.0378	−3.0414	0.3063	0.0037
0.8	−3.3142	−3.7224	−3.7275	0.4133	0.0050
0.9	−4.0409	−4.5832	−4.5900	0.5492	0.0068
1.0	−4.9499	−5.6616	−5.6708	0.7209	0.0092
1.1	−6.0838	−7.0087	−7.0208	0.9370	0.0121
1.2	−7.4947	−8.6872	−8.7031	1.2084	0.0159
1.3	−9.2465	−10.7738	−10.7944	1.5480	0.0206
1.4	−11.4175	−13.3628	−13.3894	1.9719	0.0267
1.5	−14.1037	−16.5696	−16.6038	2.5001	0.0342
1.6	−17.4227	−20.5358	−20.5795	3.1568	0.0437
1.7	−21.5182	−25.4345	−25.4902	3.9720	0.0556
1.8	−26.5664	−31.4781	−31.5486	4.9822	0.0705
1.9	−32.7829	−38.9262	−39.0153	6.2324	0.0891
2.0	−40.4313	−48.0970	−48.2091	7.7778	0.1121

15. (a)

$$\frac{dy}{dx} = 2y^2(x-1)$$
$$\frac{dy}{y^2} = 2(x-1)dx$$
$$\int y^{-2}\,dy = \int (2x-2)\,dx$$
$$-y^{-1} = x^2 - 2x + C$$

Initial value: $y(2) = -\frac{1}{2}$

$2 = 2^2 - 2(2) + C$

$2 = C$

Solution: $-y^{-1} = x^2 - 2x + 2$ or $y = -\dfrac{1}{x^2 - 2x + 2}$

$y(3) = -\dfrac{1}{3^2 - 2(3) + 2} = -\dfrac{1}{5} = -0.2$

(b) To find the approximation, set $y_1 = 2y^2(x-1)$ and use EULERT with initial values $x = 2$ and $y = -\frac{1}{2}$ and step size 0.2 for 5 points. This gives $y(3) \approx -0.1851$; error ≈ 0.0149.

(c) Use step size 0.1 for 10 points. This gives $y(3) \approx -0.1929$; error ≈ 0.0071.

(d) Use step size 0.05 for 20 points. This gives $y(3) \approx -0.1965$; error ≈ 0.0035.

16. (a) $\frac{dy}{dx} = y - 1$

$\int \frac{dy}{y-1} = \int dx$

$\ln|y - 1| = x + C$

$|y - 1| = e^{x+C}$

$y - 1 = \pm e^C e^x$

$y = Ae^x + 1$

Initial condition: $y(0) = 3$

$3 = Ae^0 + 1$

$2 = A$

Solution: $y = 2e^x + 1$

$y(1) = 2e + 1 \approx 6.4366$

(b) To find the approximation, set $y_1 = y - 1$ and use EULERT with initial values $x = 0$ and $y = 3$ and step size 0.2 for 5 points. This gives $y(1) \approx 5.9766$; error ≈ 0.4599.

(c) Use step size 0.1 for 10 points. This gives $y(1) \approx 6.1875$; error ≈ 0.2491.

(d) Use step size 0.05 for 20 points. This gives $y(1) \approx 6.3066$; error ≈ 0.1300.

17. The exact solution is $y = -\frac{1}{x^2 - 2x + 2}$, so $y(3) = -0.2$.

(a) To find the approximation, set $y_1 = 2y^2(x - 1)$ and use IMPEULT with initial values $x = 2$ and $y = -\frac{1}{2}$ and step size 0.2 for 5 points. This gives $y(3) \approx -0.2024$; error ≈ 0.0024.

(b) Use step size 0.1 for 10 points. This gives $y(3) \approx -0.2005$; error ≈ 0.0005.

(c) Use step size 0.05 for 20 points. This gives $y(3) = -0.2001$; error ≈ 0.0001.

(d) As the step size decreases, the accuracy of the method increases and so the error decreases.

18. The exact solution is $y = 2e^x + 1$, so $y(1) = 2e + 1 \approx 6.4366$.

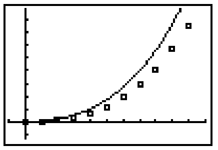
(a) To find the approximation, set $y_1 = y - 1$ and use IMPEULT with initial values $x = 0$ and $y = 3$ and step size 0.2 for 5 points. This gives $y(1) \approx 6.4054$; error ≈ 0.0311.

(b) Use step size 0.1 for 10 points. This gives $y(1) \approx 6.4282$; error ≈ 0.0084

(c) Use step size 0.05 for 20 points. This gives $y_1 \approx 6.4344$; error ≈ 0.0022.

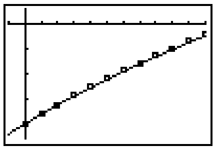
(d) As the step size decreases, the accuracy of the method increases and so the error decreases.

19. Set $y_1 = 2y + \sin x$ and use EULERG with initial values $x = 0$ and $y = 0$ and step size 0.1. The exact solution is $y = \frac{1}{5}(e^{2x} - 2\sin x - \cos x)$.



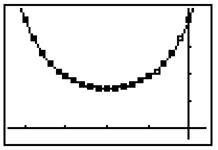
$[-0.1, 1.1]$ by $[-0.13, 0.88]$

20. Set $y_1 = x - y$ and use EULERG with initial values $x = 0$ and $y = -2$ and step size 0.1. The exact solution is $y = x - 1 - e^{-x}$.



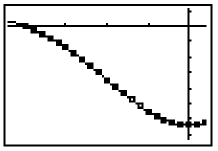
$[-0.1, 1.1]$ by $[-2.3, 0.3]$

21. Set $y_1 = 2y(x + 1)$ and use IMPEULG with initial values $x = -2$ and $y = 2$ and step size 0.1. The exact solution is $y = 2e^{x^2+2x}$.



$[-2.2, 0.2]$ by $[-0.2, 2.2]$

22. Set $y_1 = x(1 - y)$ and use IMPEULG with initial values $x = -2$ and $y = 0$ and step size 0.1. The exact solution is $y = -e^{-(x^2/2)+2} + 1$.



$[-2.2, 0.2]$ by $[-7.3, 1.1]$

23. To find the approximate values, set $y_1 = x + y$ and use EULERT with initial values $x = 0$ and $y = 1$ and step size -0.1 for 10 points. The exact values are given by $y = 2e^x - x - 1$.

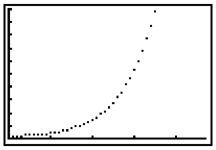
x	y (Euler)	y (exact)	Error
0	1	1.0	0
−0.1	0.9000	0.9097	0.0097
−0.2	0.8200	0.8375	0.0175
−0.3	0.7580	0.7816	0.0236
−0.4	0.7122	0.7406	0.0284
−0.5	0.6810	0.7131	0.0321
−0.6	0.6629	0.6976	0.0347
−0.7	0.6566	0.6932	0.0366
−0.8	0.6609	0.6987	0.0377
−0.9	0.6748	0.7131	0.0383
−1.0	0.6974	0.7358	0.0384

24. To find the approximate values, set $y_1 = x + y$ and use IMPEULT with initial values $x = 0$ and $y = 1$ and step size -0.1 for 10 points. The exact values are given by $y = 2e^x - x - 1$.

x	$y\left(\begin{matrix}\text{improved}\\\text{Euler}\end{matrix}\right)$	y (exact)	Error
0	1	1.0	0
−0.1	0.9100	0.9097	0.0003
−0.2	0.8381	0.8375	0.0006
−0.3	0.7824	0.7816	0.0008
−0.4	0.7416	0.7406	0.0010
−0.5	0.7142	0.7131	0.0011
−0.6	0.6988	0.6976	0.0012
−0.7	0.6944	0.6932	0.0012
−0.8	0.7000	0.6987	0.0013
−0.9	0.7145	0.7131	0.0013
−1.0	0.7371	0.7358	0.0013

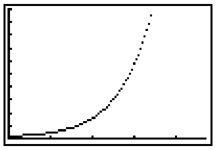
25. Set $y_1 = y + e^x - 2$ and EULERG, with initial values $x = 0$ and $y = 2$ and step sizes 0.1 and 0.05.

(a)



[0, 4.7] by [0, 100]

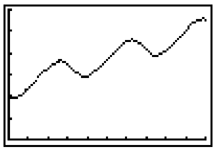
(b)



[0, 4.7] by [0, 100]

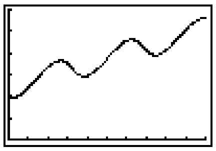
26. Set $y_1 = \cos(2x - y)$ and use EULERG with initial values $x = 0$ and $y = 2$ and step sizes 0.1 and 0.05.

(a)



[0, 10] by [0, 6]

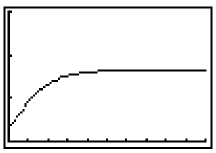
(b)



[0, 10] by [0, 6]

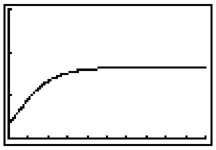
27. Set $y_1 = y\left(\frac{1}{2} - \ln|y|\right)$ and use IMPEULG with initial values $x = 0$ and $y = \frac{1}{3}$ and step size 0.1 and 0.05.

(a)



[0, 10] by [0, 3]

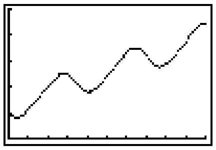
(b)



[0, 10] by [0, 3]

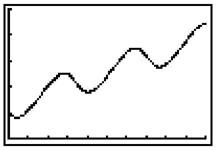
28. Set $y = \sin(2x - y)$ and use IMPEULG with initial values $x = 0$ and $y = 1$ and step sizes 0.1 and 0.05.

(a)



[0, 10] by [0, 5]

(b)



[0, 10] by [0, 5]

29. To find the approximate values, let $y_1 = y$ and use EULERT with initial values $x = 0$ and $y = 1$ and step size 0.05 for 20 points. This gives $y(1) \approx 2.6533$.

Since the exact solution to the initial value problem is $y = e^x$, the exact value of $y(1)$ is e.

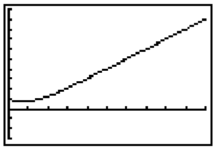
30. To find the approximate values, let $y_1 = 3y$ and use IMPEULT with initial values $x = 0$ and $y = 1$ and step size 0.05 for 20 points. This gives $y(1) \approx 19.8845$.

Since the exact solution to the initial value problem is $y = e^{3x}$, the exact value of $y(1)$ is e^3.

31. To find the approximate values, let $y_1 = 1 + y$ and use RUNKUTT with initial values $x = 0$ and $y = 1$ and step size 0.1 for 10 points. The exact values are given by $y = 2e^x - 1$.

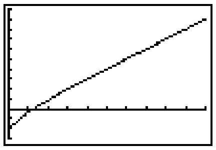
x	y (Runge-Kutta)	y (exact)	Error
0	1	1	0
0.1	1.2103	1.2103	0.0000002
0.2	1.4428	1.4428	0.0000004
0.3	1.6997	1.6997	0.0000006
0.4	1.9836	1.9836	0.0000009
0.5	2.2974	2.2974	0.0000013
0.6	2.6442	2.6442	0.0000017
0.7	3.0275	3.0275	0.0000022
0.8	3.4511	3.4511	0.0000027
0.9	3.9192	3.9192	0.0000034
1.0	4.4366	4.4366	0.0000042

32. (a) Set $y_1 = x - y$ and use RUNKUTT with initial values $x = 0$ and $y = 1$ and step size 0.1.



[0, 10] by [−3, 10]

(b) Use RUNKUTT with initial values $x = 0$ and $y = -2$ and step size 0.1.



[0, 10] by [−3, 10]

Chapter 6 Review Exercises

(pp. 358 – 361)

1. $\int_0^{\pi/3} \sec^2 \theta \, d\theta = \tan \theta \Big]_0^{\pi/3} = \tan \frac{\pi}{3} - \tan 0 = \sqrt{3}$

2. $\int_1^2 \left(x + \frac{1}{x^2}\right)dx = \left[\frac{1}{2}x^2 - x^{-1}\right]_1^2$

$= \left(\frac{1}{2}(4) - \frac{1}{2}\right) - \left(\frac{1}{2} - 1\right)$

$= \frac{3}{2} + \frac{1}{2}$

$= \frac{4}{2} = 2$

3. Let $u = 2x + 1$

$du = 2\,dx$

$\frac{1}{2}\,du = dx$

$\int_0^1 \frac{36}{(2x+1)^3}\,dx = 18\int_1^3 \frac{1}{u^3}\,du$

$= 18\left(-\frac{1}{2}\right)u^{-2}\Big]_1^3$

$= -9\left(\frac{1}{9} - 1\right)$

$= -9\left(-\frac{8}{9}\right)$

$= 8$

4. Let $u = 1 - x^2$

$du = -2x\,dx$

$-du = 2x\,dx$

$\int_{-1}^1 2x \sin(1 - x^2)\,dx = -\int_0^0 \sin u\,du = 0$

5. Let $u = \sin x$

$du = \cos x\,dx$

$\int_0^{\pi/2} 5 \sin^{3/2} x \cos x\,dx = \int_0^1 5u^{3/2}\,du$

$= 5 \cdot \frac{2}{5}u^{5/2}\Big]_0^1$

$= 2(1 - 0)$

$= 2$

6. $\int_{1/2}^4 \frac{x^2 + 3x}{x}\,dx = \int_{1/2}^4 (x + 3)\,dx \ (x \neq 0)$

$= \left(\frac{1}{2}x^2 + 3x\right)\Big]_{1/2}^4$

$= \left(\frac{1}{2}(16) + 3(4)\right) - \left(\frac{1}{2}\left(\frac{1}{4}\right) + \frac{3}{2}\right)$

$= 20 - \left(\frac{1}{8} + \frac{12}{8}\right)$

$= 20 - \frac{13}{8}$

$= \frac{147}{8}$

7. Let $u = \tan x$

$du = \sec^2 x\,dx$

$\int_0^{\pi/4} e^{\tan x} \sec^2 x\,dx = \int_0^1 e^u\,du$

$= e^u\Big]_0^1$

$= e^1 - e^0$

$= e - 1$

8. Let $u = \ln r$

$du = \frac{1}{r}\,dr$

$\int_1^e \frac{\sqrt{\ln r}}{r}\,dr = \int_0^1 u^{1/2}\,du$

$= \frac{2}{3}u^{3/2}\Big]_0^1$

$= \frac{2}{3}(1 - 0)$

$= \frac{2}{3}$

9. Let $u = 2 - \sin x$

$du = -\cos x\,dx$

$-du = \cos x\,dx$

$\int \frac{\cos x}{2 - \sin x}\,dx = -\int \frac{1}{u}\,du$

$= -\ln |u| + C$

$= -\ln |2 - \sin x| + C$

10. Let $u = 3x + 4$

$$du = 3\,dx$$

$$\frac{1}{3}\,du = dx$$

$$\int \frac{dx}{\sqrt[3]{3x+4}} = \frac{1}{3}\int u^{-1/3}\,du = \frac{1}{3}\cdot\frac{3}{2}u^{2/3} + C = \frac{1}{2}(3x+4)^{2/3} + C$$

11. Let $u = t^2 + 5$

$$du = 2t\,dt$$

$$\frac{1}{2}\,du = t\,dt$$

$$\int \frac{t\,dt}{t^2+5} = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|t^2+5| + C = \frac{1}{2}\ln(t^2+5) + C$$

12. Let $u = \frac{1}{\theta}$

$$du = -\frac{1}{\theta^2}\,d\theta$$

$$\int \frac{1}{\theta^2}\sec\frac{1}{\theta}\tan\frac{1}{\theta}\,d\theta = -\int \sec u \tan u\,du = -\sec u + C = -\sec\frac{1}{\theta} + C$$

13. Let $u = \ln y$

$$du = \frac{1}{y}\,dy$$

$$\int \frac{\tan(\ln y)}{y}\,dy = \int \tan u\,du = \int \frac{\sin u}{\cos u}\,du$$

Let $w = \cos u$

$$dw = -\sin u\,du$$

$$= -\int \frac{1}{w}\,dw = \ln|w| + C = -\ln|\cos u| + C = -\ln|\cos(\ln y)| + C$$

14. Let $u = e^x$

$$du = e^x\,dx$$

$$\int e^x \sec(e^x)\,dx = \int \sec u\,du = \ln|\sec u + \tan u| + C = \ln|\sec(e^x) + \tan(e^x)| + C$$

15. Let $u = \ln x$

$$du = \frac{1}{x}\,dx$$

$$\int \frac{dx}{x\ln x} = \int \frac{1}{u}\,du = \ln|u| + C = \ln|\ln x| + C$$

16. $$\int \frac{dt}{t\sqrt{t}} = \int \frac{dt}{t^{3/2}} = \int t^{-3/2}\,dt = -2t^{-1/2} + C = -\frac{2}{\sqrt{t}} + C$$

17. Use tabular integration with $f(x) = x^3$ and $g(x) = \cos x$.

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^3	(+)	$\cos x$
$3x^2$	(−)	$\sin x$
$6x$	(+)	$-\cos x$
6	(−)	$-\sin x$
0		$\cos x$

$$\int x^3\cos x\,dx = x^3\sin x + 3x^2\cos x - 6x\sin x - 6\cos x + C$$

18. Let $u = \ln x$ $\quad dv = x^4\,dx$

$$du = \frac{1}{x}\,dx \qquad v = \frac{1}{5}x^5$$

$$\int x^4\ln x\,dx = \frac{1}{5}x^5\ln x - \int \frac{1}{5}x^5\left(\frac{1}{x}\right)dx = \frac{1}{5}x^5\ln x - \frac{1}{5}\int x^4\,dx = \frac{1}{5}x^5\ln x - \frac{1}{25}x^5 + C$$

19. Let $u = e^{3x}$ $\quad dv = \sin x\, dx$

$du = 3e^{3x}\, dx \quad v = -\cos x$

$$\int e^{3x} \sin x\, dx = -e^{3x} \cos x + \int 3 \cos x\, e^{3x}\, dx$$

Integrate by parts again.

Let $u = 3e^{3x}$ $\quad dv = \cos x\, dx$

$du = 9e^{3x}\, dx \quad v = \sin x$

$$\int e^{3x} \sin x\, dx = -e^{3x} \cos x + 3e^{3x} \sin x - \int 9e^{3x} \sin x\, dx$$

$$10\int e^{3x} \sin x\, dx = -e^{3x} \cos x + 3e^{3x} \sin x + C$$

$$\int e^{3x} \sin x\, dx = \frac{1}{10}[-e^{3x} \cos x + 3e^{3x} \sin x] + C$$

$$= \left(\frac{3 \sin x}{10} - \frac{\cos x}{10}\right)e^{3x} + C$$

20. Let $u = x^2$ $\quad dv = e^{-3x}\, dx$

$du = 2x\, dx \quad v = -\frac{1}{3}e^{-3x}$

$$\int x^2 e^{-3x}\, dx = -\frac{1}{3}x^2 e^{-3x} + \frac{2}{3}\int e^{-3x} x\, dx$$

Let $u = x$ $\quad dv = e^{-3x}\, dx$

$du = dx \quad v = -\frac{1}{3}e^{-3x}$

$$= -\frac{1}{3}x^2 e^{-3x} + \frac{2}{3}\left[-\frac{1}{3}xe^{-3x} + \frac{1}{3}\int e^{-3x}\, dx\right]$$

$$= -\frac{1}{3}x^2 e^{-3x} - \frac{2}{9}xe^{-3x} + \frac{2}{9}\int e^{-3x}\, dx$$

$$= -\frac{1}{3}x^2 e^{-3x} - \frac{2}{9}xe^{-3x} - \frac{2}{27}e^{-3x} + C$$

$$= \left(-\frac{x^2}{3} - \frac{2x}{9} - \frac{2}{27}\right)e^{-3x} + C$$

21. $\frac{dy}{dx} = 1 + x + \frac{x^2}{2}$

$dy = \left(1 + x + \frac{x^2}{2}\right) dx$

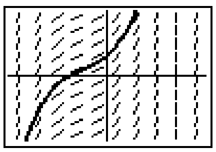
$\int dy = \int\left(1 + x + \frac{x^2}{2}\right) dx$

$y = x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + C$

$y(0) = C = 1$

$y = \frac{x^3}{6} + \frac{x^2}{2} + x + 1$

Graphical support:



$[-4, 4]$ by $[-3, 3]$

22. $\frac{dy}{dx} = \left(x + \frac{1}{x}\right)^2$

$dy = \left(x + \frac{1}{x}\right)^2 dx$

$\int dy = \int\left(x + \frac{1}{x}\right)^2 dx$

$y = \int\left(x^2 + 2 + \frac{1}{x^2}\right) dx$

$y = \frac{1}{3}x^3 + 2x - x^{-1} + C$

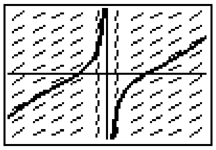
$y(1) = \frac{1}{3} + 2 - 1 + C = 1$

$\frac{4}{3} + C = 1$

$C = -\frac{1}{3}$

$y = \frac{x^3}{3} + 2x - \frac{1}{x} - \frac{1}{3}$

Graphical support:



$[-2, 2]$ by $[-10, 10]$

23. $\frac{dy}{dt} = \frac{1}{t + 4}$

$dy = \frac{1}{t + 4}\, dt$

$\int dy = \int \frac{1}{t + 4}\, dt$

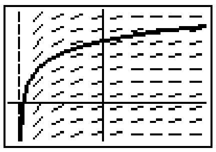
$y = \ln |t + 4| + C$

$y(-3) = \ln(1) + C = 2$

$C = 2$

$y = \ln(t + 4) + 2$

Graphical Support:



$[-4.5, 5]$ by $[-2, 5]$

24. $\frac{dy}{d\theta} = \csc 2\theta \cot 2\theta$

$dy = \csc 2\theta \cot 2\theta\, d\theta$

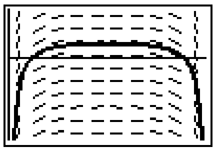
$\int dy = \int \csc 2\theta \cot 2\theta\, d\theta$

$y = -\frac{1}{2}\csc 2\theta + C$

$y\left(\frac{\pi}{4}\right) = -\frac{1}{2} + C = 1$

$C = \frac{3}{2}$

$y = -\frac{1}{2}\csc 2\theta + \frac{3}{2}$



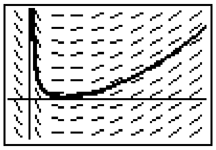
$[0, 1.57]$ by $[-5, 3]$

25.
$$\begin{aligned}
\frac{d(y')}{dx} &= 2x - \frac{1}{x^2} \\
d(y') &= \left(2x - \frac{1}{x^2}\right) dx \\
\int d(y') &= \int\left(2x - \frac{1}{x^2}\right) dx \\
y' &= x^2 + x^{-1} + C \\
y'(1) &= 2 + C = 1 \\
C &= -1 \\
y' &= x^2 + x^{-1} - 1 \\
\int dy &= \int (x^2 + x^{-1} - 1)\, dx \\
y &= \frac{1}{3}x^3 + \ln x - x + C \\
y(1) &= \frac{1}{3} + 0 - 1 + C = 0 \\
-\frac{2}{3} + C &= 0 \\
C &= \frac{2}{3} \\
y &= \frac{x^3}{3} + \ln x - x + \frac{2}{3}
\end{aligned}$$

Graphical support:

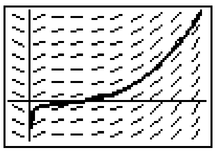
Let $f(x) = \frac{x^3}{3} + \ln x - x + \frac{2}{3}$.

We first show the graph of $y = f'(x) = x^2 + x^{-1} - 1$, $x > 0$, along with the slope field for $y' = f''(x) = 2x - \frac{1}{x^2}$.



[−0.5, 4.21] by [−9, 21]

We now show the graph of $y = f(x)$ along with the slope field for $y' = f'(x) = x^2 + x^{-1} - 1$.

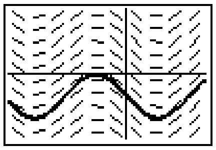


[−0.5, 4.21] by [−9, 21]

26.
$$\begin{aligned}
\frac{d(r'')}{dt} &= -\cos t \\
d(r'') &= -\cos t\, dt \\
\int d(r'') &= \int -\cos t\, dt \\
r'' &= -\sin t + C \\
r''(0) &= C = -1 \\
r'' &= -\sin t - 1 \\
\int d(r') &= \int (-\sin t - 1)\, dt \\
r' &= \cos t - t + C \\
r'(0) &= 1 + C = -1 \\
C &= -2 \\
r' &= \cos t - t - 2 \\
\int dr &= \int (\cos t - t - 2)\, dt \\
r &= \sin t - \frac{t^2}{2} - 2t + C \\
r(0) &= C = -1 \\
r &= \sin t - \frac{t^2}{2} - 2t - 1
\end{aligned}$$

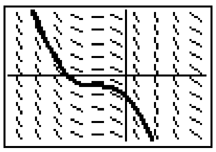
Graphical support:

We first show the graph of $y = r'' = -\sin t - 1$ along with the slope field for $y' = r''' = -\cos t$.



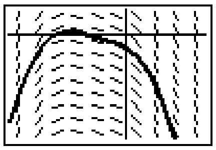
[−6, 4] by [−3, 3]

Next, we show the graph of $y = r' = \cos t - t - 2$ along with the slope field for $y' = r'' = -\sin t - 1$.



[−6, 4] by [−3, 3]

Finally we show the graph of $y = r = \sin t - \frac{t^2}{2} - 2t - 1$ along with the slope field for $y' = r' = \cos t - t - 2$.



[−6, 4] by [−8, 2]

27. $\frac{dy}{dx} = y + 2$

$\frac{dy}{y+2} = dx$

$\int \frac{dy}{y+2} = \int dx$

$\ln |y + 2| = x + C$

$y + 2 = Ce^x$

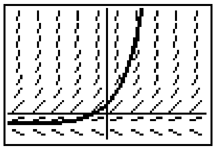
$y = Ce^x - 2$

$y(0) = C - 2 = 2$

$C = 4$

$y = 4e^x - 2$

Graphical support:



$[-5, 5]$ by $[-5, 20]$

28. $\frac{dy}{dx} = (2x + 1)(y + 1)$

$\frac{dy}{y+1} = (2x + 1)\, dx$

$\int \frac{dy}{y+1} = \int (2x + 1)\, dx$

$\ln |y + 1| = x^2 + x + C$

$y + 1 = Ce^{x^2+x}$

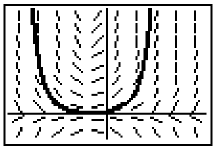
$y = Ce^{x^2+x} - 1$

$y(-1) = C - 1 = 1$

$C = 2$

$y = 2e^{x^2+x} - 1$

Graphical support:



$[-3, 3]$ by $[-10, 40]$

29. $\int -f(x)\, dx = -\int f(x)\, dx$

$= -(1 - \sqrt{x}) + C$

$= -1 + \sqrt{x} + C$

Since $-1 + C$ is an arbitrary constant, we may write the indefinite integral as $\sqrt{x} + C$.

30. $\int [x + f(x)]\, dx = \int x\, dx + \int f(x)\, dx$

$= \frac{x^2}{2} + (1 - \sqrt{x}) + C$

$= \frac{x^2}{2} + 1 - \sqrt{x} + C$

Since $1 + C$ is an arbitrary constant, we may write the indefinite integral as $\frac{x^2}{2} - \sqrt{x} + C$.

31. $\int [2f(x) - g(x)]\, dx = 2\int f(x)\, dx - \int g(x)\, dx$

$= 2(1 - \sqrt{x}) - (x + 2) + C$

$= -2\sqrt{x} - x + C$

32. $\int [g(x) - 4]\, dx = \int g(x)\, dx - \int 4\, dx$

$= (x + 2) - 4x + C$

$= 2 - 3x + C$

Since $2 + C$ is an arbitrary constant, we may write the indefinite integral as $-3x + C$.

33. We seek the graph of a function whose derivative is $\frac{\sin x}{x}$. Graph (b) is increasing on $[-\pi, \pi]$, where $\frac{\sin x}{x}$ is positive, and oscillates slightly outside of this interval. This is the correct choice, and this can be verified by graphing $\text{NINT}\left(\frac{\sin x}{x}, x, 0, x\right)$.

34. We seek the graph of a function whose derivative is e^{-x^2}. Since $e^{-x^2} > 0$ for all x, the desired graph is increasing for all x. Thus, the only possibility is graph (d), and we may verify that this is correct by graphing $\text{NINT}(e^{-x^2}, x, 0, x)$.

35. (iv) The given graph looks like the graph of $y = x^2$, which satisfies $\frac{dy}{dx} = 2x$ and $y(1) = 1$.

36. Yes, $y = x$ is a solution.

37. **(a)** $\frac{dv}{dt} = 2 + 6t$

$\int dv = \int (2 + 6t)\, dt$

$v = 2t + 3t^2 + C$

Initial condition: $v = 4$ when $t = 0$

$4 = 0 + C$

$4 = C$

$v = 2t + 3t^2 + 4$

(b) $\int_0^1 v(t)\, dt = \int_0^1 (2t + 3t^2 + 4)\, dt$

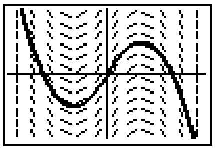
$= \left[t^2 + t^3 + 4t\right]_0^1$

$= 6 - 0$

$= 6$

The particle moves 6 m.

38.



$[-10, 10]$ by $[-10, 10]$

39. Set $y_1 = y + \cos x$ and use EULERT with initial values $x = 0$ and $y = 0$ and step size 0.1 for 20 points.

x	y
0	0
0.1	0.1000
0.2	0.2095
0.3	0.3285
0.4	0.4568
0.5	0.5946
0.6	0.7418
0.7	0.8986
0.8	1.0649
0.9	1.2411
1.0	1.4273
1.1	1.6241
1.2	1.8319
1.3	2.0513
1.4	2.2832
1.5	2.5285
1.6	2.7884
1.7	3.0643
1.8	3.3579
1.9	3.6709
2.0	4.0057

40. Set $y_1 = (2 - y)(2x + 3)$ and use IMPEULT with intial values $x = -3$ and $y = 1$ and step size 0.1 for 20 points.

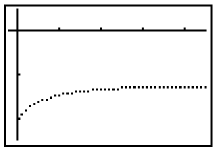
x	y
−3	1
−2.9	0.6680
−2.8	0.2599
−2.7	−0.2294
−2.6	−0.8011
−2.5	−1.4509
−2.4	−2.1687
−2.3	−2.9374
−2.2	−3.7333
−2.1	−4.5268
−2.0	−5.2840
−1.9	−5.9686
−1.8	−6.5456
−1.7	−6.9831
−1.6	−7.2562
−1.5	−7.3488
−1.4	−7.2553
−1.3	−6.9813
−1.2	−6.5430
−1.1	−5.9655
−1.0	−5.2805

41. To estimate $y(3)$, set $y_1 = \frac{x - 2y}{x + 1}$ and use IMPEULT with initial values $x = 0$ and $y = 1$ and step size 0.05 for 60 points. This gives $y(3) \approx 0.9063$.

42. To estimate $y(4)$, set $y_1 = \frac{x^2 - 2y + 1}{x}$ and use EULERT with initial values $x = 1$ and $y = 1$ and step size 0.05 for 60 points. This gives $y(4) \approx 4.4974$.

43. Set $y_1 = e^{-(x+y+2)}$ and use EULERG with initial values $x = 0$ and $y = -2$ and step sizes 0.1 and -0.1.

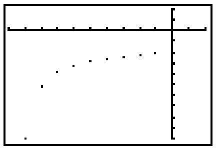
(a)



$[-0.2, 4.5]$ by $[-2.5, 0.5]$

43. continued

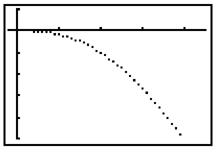
(b) Note that we choose a small interval of x-values because the y-values decrease very rapidly and our calculator cannot handle the calculations for $x \le -1$. (This occurs because the analytic solution is $y = -2 + \ln(2 - e^{-x})$, which has an asymptote at $x = -\ln 2 \approx -0.69$. Obviously, the Euler approximations are misleading for $x \le -0.7$.)



$[-1, 0.2]$ by $[-10, 2]$

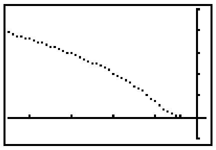
44. Set $y_1 = -\dfrac{x^2 + y}{e^y + x}$ and use IMPEULG with initial values $x = 0$ and $y = 0$ and step sizes 0.1 and -0.1.

(a)



$[-0.2, 4.5]$ by $[-5, 1]$

(b)



$[-4.5, 0.2]$ by $[-1, 5]$

45. (a) Half-life $= \dfrac{\ln 2}{k}$

$$2.645 = \frac{\ln 2}{k}$$

$$k = \frac{\ln 2}{2.645} \approx 0.262059$$

(b) Mean life $= \dfrac{1}{k} \approx 3.81593$ years

46. $T - T_s = (T_0 - T_s)e^{-kt}$

$T - 40 = (220 - 40)e^{-kt}$

Use the fact that $T = 180$ and $t = 15$ to find k.

$$180 - 40 = (220 - 40)e^{-(k)(15)}$$

$$e^{15k} = \frac{180}{140} = \frac{9}{7}$$

$$k = \frac{1}{15}\ln\frac{9}{7}$$

$$T - 40 = (220 - 40)e^{-((1/15)\ln(9/7))t}$$

$$70 - 40 = (220 - 40)e^{-((1/15)\ln(9/7))t}$$

$$e^{((1/15)\ln(9/7))t} = \frac{180}{30} = 6$$

$$\left(\frac{1}{15}\ln\frac{9}{7}\right)t = \ln 6$$

$$t = \frac{15\ln 6}{\ln(9/7)} \approx 107 \text{ min}$$

It took a total of about 107 minutes to cool from 220°F to 70°F. Therefore, the time to cool from 180°F to 70°F was about 92 minutes.

47. $T - T_s = (T_0 - T_s)e^{-kt}$

We have the system:

$$\begin{cases} 39 - T_s = (46 - T_s)e^{-10k} \\ 33 - T_s = (46 - T_s)e^{-20k} \end{cases}$$

Thus, $\dfrac{39 - T_s}{46 - T_s} = e^{-10k}$ and $\dfrac{33 - T_s}{46 - T_s} = e^{-20k}$.

Since $(e^{-10k})^2 = e^{-20k}$, this means:

$$\left(\frac{39 - T_s}{46 - T_s}\right)^2 = \frac{33 - T_s}{46 - T_s}$$

$$(39 - T_s)^2 = (33 - T_s)(46 - T_s)$$

$$1521 - 78T_s + T_s^2 = 1518 - 79T_s + T_s^2$$

$$T_s = -3$$

The refrigerator temperature was -3°C.

48. Use the method of Example 3 in Section 6.4.

$$e^{-kt} = 0.995$$

$$-kt = \ln 0.995$$

$$t = -\frac{1}{k}\ln 0.995 = -\frac{5700}{\ln 2}\ln 0.995 \approx 41.2$$

The painting is about 41.2 years old.

49. Use the method of Example 3 in Section 6.4.

Since 90% of the carbon-14 has decayed, 10% remains.

$$e^{-kt} = 0.1$$

$$-kt = \ln 0.1$$

$$t = -\frac{1}{k}\ln 0.1 = -\frac{5700}{\ln 2}\ln 0.1 \approx 18{,}935$$

The charcoal sample is about 18,935 years old.

50. Use $t = 1988 - 1924 = 64$ years.

$$250\,e^{rt} = 7500$$

$$e^{rt} = 30$$

$$rt = \ln 30$$

$$r = \frac{\ln 30}{t} = \frac{\ln 30}{64} \approx 0.053$$

The rate of appreciation is about 0.053, or 5.3%.

51. Using the Law of Exponential Change in Section 6.4 with appropriate changes of variables, the solution to the differential equation is $L(x) = L_0e^{-kx}$, where $L_0 = L(0)$ is the surface intensity. We know $0.5 = e^{-18k}$, so $k = \frac{\ln 0.5}{-18}$ and our equation becomes $L(x) = L_0e^{(\ln 0.5)(x/18)} = L_0\left(\frac{1}{2}\right)^{x/18}$. We now find the depth where the intensity is one-tenth of the surface value.

$$0.1 = \left(\frac{1}{2}\right)^{x/18}$$
$$\ln 0.1 = \frac{x}{18}\ln\left(\frac{1}{2}\right)$$
$$x = \frac{18 \ln 0.1}{\ln 0.5} \approx 59.8 \text{ ft}$$

You can work without artificial light to a depth of about 59.8 feet.

52. (a)
$$\frac{dy}{dt} = \frac{kA}{V}(c - y)$$
$$\int \frac{dy}{c - y} = \int \frac{kA}{V}\,dt$$
$$-\ln|c - y| = \frac{kA}{V}t + C$$
$$\ln|c - y| = -\frac{kA}{V}t - C$$
$$|c - y| = e^{-(kA/V)t - C}$$
$$c - y = \pm e^{-(kA/V)t - C}$$
$$y = c \pm e^{-(kA/V)t - C}$$
$$y = c + De^{-(kA/V)t}$$

Initial condition $y = y_0$ when $t = 0$

$$y_0 = c + D$$
$$y_0 - c = D$$

Solution: $y = c + (y_0 - c)e^{-(kA/V)t}$

(b) $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} [c + (y_0 - c)e^{-(kA/V)t}] = c$

53. (a) $P(t) = \frac{150}{1 + e^{4.3-t}} = \frac{150}{1 + e^{4.3}e^{-t}}$

This is $P = \frac{M}{1 + Ae^{-kt}}$ where $M = 150$, $A = e^{4.3}$, and $k = 1$. Therefore, it is a solution of the logistic differential equation.

$\frac{dP}{dt} = \frac{k}{M}P(M - P)$, or $\frac{dP}{dt} = \frac{1}{150}P(150 - P)$. The carrying capacity is 150.

(b) $P(0) = \frac{150}{1 + e^{4.3}} \approx 2$

Initially there were 2 infected students.

(c)
$$\frac{150}{1 + e^{4.3-t}} = 125$$
$$\frac{6}{5} = 1 + e^{4.3-t}$$
$$\frac{1}{5} = e^{4.3-t}$$
$$-\ln 5 = 4.3 - t$$
$$t = 4.3 + \ln 5 \approx 5.9 \text{ days}$$

It took about 6 days.

54. Use the Fundamental Theorem of Calculus.

$$y' = \frac{d}{dx}\left(\int_0^x \sin t^2\,dt\right) + \frac{d}{dx}(x^3 + x + 2)$$
$$= (\sin x^2) + (3x^2 + 1)$$
$$y'' = \frac{d}{dx}(\sin x^2 + 3x^2 + 1)$$
$$= (\cos x^2)(2x) + 6x$$
$$= 2x\cos(x^2) + 6x$$

Thus, the differential equation is satisfied.

Verify the initial conditions:

$y'(0) = (\sin 0^2) + 3(0)^2 + 1 = 1$

$y(0) = \int_0^0 \sin(t^2)\,dt + 0^3 + 0 + 2 = 2$

55.
$$\frac{dP}{dt} = 0.002P\left(1 - \frac{P}{800}\right)$$
$$\frac{dP}{dt} = 0.002P\left(\frac{800 - P}{800}\right)$$
$$\frac{800}{P(800 - P)}\,dP = 0.002\,dt$$
$$\frac{(800 - P) + P}{P(800 - P)}\,dP = 0.002\,dt$$
$$\int\left(\frac{1}{P} + \frac{1}{800 - P}\right)dP = 0.002\,dt$$
$$\ln|P| - \ln|800 - P| = 0.002t + C$$
$$\ln\left|\frac{P}{800 - P}\right| = 0.002t + C$$
$$\ln\left|\frac{800 - P}{P}\right| = -0.002t - C$$
$$\left|\frac{800 - P}{P}\right| = e^{-0.002t - C}$$
$$\frac{800 - P}{P} = \pm e^{-C}e^{-0.002t}$$
$$\frac{800}{P} - 1 = Ae^{-0.002t}$$
$$P = \frac{800}{1 + Ae^{-0.002t}}$$

Initial condition: $P(0) = 50$

$$50 = \frac{800}{1 + Ae^0}$$
$$1 + A = 16$$
$$A = 15$$

Solution: $P = \frac{800}{1 + 15e^{-0.002t}}$

56. Method 1–Compare graph of $y_1 = x^2 \ln x$ with $y_2 = \text{NDER}\left(\frac{x^3 \ln x}{3} - \frac{x^3}{9}\right)$. The graphs should be the same.
Method 2–Compare graph of $y_1 = \text{NINT}(x^2 \ln x)$ with $y_2 = \frac{x^3 \ln x}{3} - \frac{x^3}{9}$. The graphs should be the same or differ only by a vertical translation.

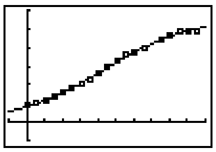
57. **(a)**
$$20{,}000 = 10{,}000(1.063)^t$$
$$2 = 1.063^t$$
$$\ln 2 = t \ln 1.063$$
$$t = \frac{\ln 2}{\ln 1.063} \approx 11.345$$
It will take about 11.3 years.

(b)
$$20{,}000 = 10{,}000e^{0.063t}$$
$$2 = e^{0.063t}$$
$$\ln 2 = 0.063t$$
$$t = \frac{\ln 2}{0.063} \approx 11.002$$
It will take about 11.0 years.

58. **(a)** $f'(x) = \frac{d}{dx}\int_0^x u(t)\,dt = u(x)$
$g'(x) = \frac{d}{dx}\int_3^x u(t)\,dt = u(x)$

(b)
$$C = f(x) - g(x)$$
$$= \int_0^x u(t)\,dt - \int_3^x u(t)\,dt$$
$$= \int_0^x u(t)\,dt + \int_x^3 u(t)\,dt$$
$$= \int_0^3 u(t)\,dt$$

59. **(a)** $y = \frac{56.0716}{1 + 5.894e^{-0.0205x}}$

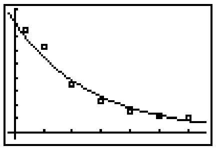


$[-20, 200]$ by $[-10, 60]$

(b) The carrying capacity is about 56.0716 million people.

(c) Use NDER twice to solve $y'' = 0$. The solution is $x \approx 86.52$, representing (approximately) the year 1887. The population at this time was approximately $P(86.52) \approx 28.0$ million people.

60. **(a)** $T = 79.961(0.9273)^t$

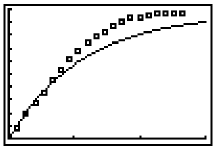


$[-1, 33]$ by $[-5, 90]$

(b) Solving $T(t) = 40$ graphically, we obtain $t \approx 9.2$ sec. The temperature will reach 40° after about 9.2 seconds.

(c) When the probe was removed, the temperature was about $T(0) \approx 79.96$°C.

61.
$$\frac{v_0 m}{k} = \text{coasting distance}$$
$$\frac{(0.86)(30.84)}{k} = 0.97$$
$$k \approx 27.343$$
$$s(t) = \frac{v_0 m}{k}(1 - e^{-(k/m)t})$$
$$s(t) = 0.97(1 - e^{-(27.343/30.84)t})$$
$$s(t) = 0.97(1 - e^{-0.8866t})$$
A graph of the model is shown superimposed on a graph of the data.



$[0, 3]$ by $[0,1]$

Chapter 7
Applications of Definite Integrals

■ Section 7.1 Integral as Net Change (pp. 363–374)

Exploration 1 Revisiting Example 2

1. $s(t) = \int\left(t^2 - \frac{8}{(t+1)^2}\right)dt = \frac{t^3}{3} + \frac{8}{t+1} + C$
$s(0) = \frac{0^3}{3} + \frac{8}{0+1} + C = 9 \Rightarrow C = 1$
Thus, $s(t) = \frac{t^3}{3} + \frac{8}{t+1} + 1$.

2. $s(1) = \frac{1^3}{3} + \frac{8}{1+1} + 1 = \frac{16}{3}$. This is the same as the answer we found in Example 2a.

3. $s(5) = \frac{5^3}{3} + \frac{8}{5+1} + 1 = 44$. This is the same answer we found in Example 2b.