**(b)**

| Year of $Q$ | Slope of $PQ$ |
|---|---|
| 1995 | $\dfrac{20.1 - 2.7}{2000 - 1995} = 3.48$ |
| 1996 | $\dfrac{20.1 - 4.8}{2000 - 1996} = 3.825$ |
| 1997 | $\dfrac{20.1 - 7.8}{2000 - 1997} = 4.1$ |
| 1998 | $\dfrac{20.1 - 11.2}{2000 - 1998} = 4.45$ |
| 1999 | $\dfrac{20.1 - 15.2}{2000 - 1999} = 4.9$ |

**(c)** Approximately 5 billion dollars per year.

**(d)** $y = 0.3214x^2 - 1.3471x + 1.3857$

$$\lim_{h \to 0} \frac{y(10 + h) - y(10)}{h} = \lim_{h \to 0} \frac{[0.3214(10 + h)^2 - 1.3471(10 + h) + 1.3857] - [0.3214(10)^2 - 1.3471(10) + 1.3857]}{h}$$

$$= \lim_{h \to 0} \frac{0.3214(20h + h^2) - 1.3471h}{h}$$

$$= 0.3214(20) - 1.3471$$

$$\approx 5.081$$

The predicted rate of change in 2000 is about
5.081 billion dollars per year.

# Chapter 3
## Derivatives

■ **Section 3.1** Derivative of a Function (pp. 95–104)

**Exploration 1    Reading the Graphs**

**1.** The graph in Figure 3.3b represents the rate of change of the depth of the water in the puddle with respect to time. Since $y$ is measured in inches and $x$ is measured in days, the derivative $\dfrac{dy}{dx}$ would be measured in inches per day. Those are the units that should be used along the $y$-axis in Figure 3.3b.

**2.** The water in the ditch is 1 inch deep at the start of the first day and rising rapidly. It continues to rise, at a gradually decreasing rate, until the end of the second day, when it achieves a maximum depth of 5 inches. During days 3, 4, 5, and 6, the water level goes down, until it reaches a depth of 1 inch at the end of day 6. During the seventh day it rises again, almost to a depth of 2 inches.

**3.** The weather appears to have been wettest at the beginning of day 1 (when the water level was rising fastest) and driest at the end of day 4 (when the water level was declining the fastest).

**4.** The highest point on the graph of the derivative shows where the water is rising the fastest, while the lowest point (most negative) on the graph of the derivative shows where the water is declining the fastest.

**5.** The $y$-coordinate of point $C$ gives the maximum depth of the water level in the ditch over the 7-day period, while the $x$-coordinate of $C$ gives the time during the 7-day period that the maximum depth occurred. The derivative of the function changes sign from positive to negative at $C'$, indicating that this is when the water level stops rising and begins falling.

**6.** Water continues to run down sides of hills and through underground streams long after the rain has stopped falling. Depending on how much high ground is located near the ditch, water from the first day's rain could still be flowing into the ditch several days later. Engineers responsible for flood control of major rivers must take this into consideration when they predict when floodwaters will "crest," and at what levels.

## Quick Review 3.1

**1.** $\lim\limits_{h\to 0}\dfrac{(2+h)^2-4}{4}=\lim\limits_{h\to 0}\dfrac{(4+4h+h^2)-4}{h}$

$\qquad\qquad\qquad\quad =\lim\limits_{h\to 0}4+h$

$\qquad\qquad\qquad\quad =4+0=4$

**2.** $\lim\limits_{x\to 2^+}\dfrac{x+3}{2}=\dfrac{2+3}{2}=\dfrac{5}{2}$

**3.** Since $\dfrac{|y|}{y}=-1$ for $y<0$, $\lim\limits_{y\to 0^-}\dfrac{|y|}{y}=-1$.

**4.** $\lim\limits_{x\to 4}\dfrac{2x-8}{\sqrt{x}-2}=\lim\limits_{x\to 4}\dfrac{2(\sqrt{x}+2)(\sqrt{x}-2)}{\sqrt{x}-2}$

$\quad =\lim\limits_{h\to 4}2(\sqrt{x}+2)=2(\sqrt{4}+2)=8$

**5.** The vertex of the parabola is at $(0, 1)$. The slope of the line

through $(0, 1)$ and another point $(h, h^2+1)$ on the parabola

is $\dfrac{(h^2+1)-1}{h-0}=h$. Since $\lim\limits_{h\to 0}h=0$, the slope of the line

tangent to the parabola at its vertex is 0.

**6.** Use the graph of $f$ in the window $[-6, 6]$ by $[-4, 4]$ to find
that $(0, 2)$ is the coordinate of the high point and $(2, -2)$ is
the coordinate of the low point. Therefore, $f$ is increasing
on $(-\infty, 0]$ and $[2, \infty)$.

**7.** $\lim\limits_{x\to 1^+}f(x)=\lim\limits_{x\to 1^+}(x-1)^2=(1-1)^2=0$

$\lim\limits_{x\to 1^-}f(x)=\lim\limits_{x\to 1^-}(x+2)=1+2=3$

**8.** $\lim\limits_{h\to 0^+}f(1+h)=\lim\limits_{x\to 1^+}f(x)=0$

**9.** No, the two one-sided limits are different (see Exercise 7).

**10.** No, $f$ is discontinuous at $x=1$ because $\lim\limits_{x\to 1}f(x)$ does not

exist.

## Section 3.1 Exercises

**1. (a)** The tangent line has slope 5 and passes through $(2, 3)$.
$y=5(x-2)+3$
$y=5x-7$

**(b)** The normal line has slope $-\dfrac{1}{5}$ and passes through

$(2, 3)$.

$y=-\dfrac{1}{5}(x-2)+3$

$y=-\dfrac{1}{5}x+\dfrac{17}{5}$

**2.** $f'(3)=\lim\limits_{h\to 0}\dfrac{f(3+h)-f(3)}{h}$

$\qquad =\lim\limits_{h\to 0}\dfrac{\dfrac{1}{3+h}-\dfrac{1}{3}}{h}$

$\qquad =\lim\limits_{h\to 0}\dfrac{3-(3+h)}{3h(3+h)}$

$\qquad =\lim\limits_{h\to 0}\dfrac{-1}{3(3+h)}=-\dfrac{1}{9}$

**3.** $f'(3)=\lim\limits_{x\to 3}\dfrac{f(x)-f(3)}{x-3}$

$\qquad =\lim\limits_{x\to 3}\dfrac{\dfrac{1}{x}-\dfrac{1}{3}}{x-3}$

$\qquad =\lim\limits_{x\to 3}\dfrac{3-x}{(x-3)(x)(3)}$

$\qquad =\lim\limits_{x\to 3}-\dfrac{1}{3x}=-\dfrac{1}{9}$

**4.** $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$\qquad =\lim\limits_{h\to 0}\dfrac{[3(x+h)-12]-(3x-12)}{h}$

$\qquad =\lim\limits_{h\to 0}\dfrac{3h}{h}=\lim\limits_{h\to 0}3=3$

**5.** $\dfrac{dy}{dx}=\lim\limits_{h\to 0}\dfrac{y(x+h)-y(x)}{h}$

$\qquad =\lim\limits_{h\to 0}\dfrac{7(x+h)-7x}{h}$

$\qquad =\lim\limits_{h\to 0}\dfrac{7h}{h}=\lim\limits_{h\to 0}7=7$

**6.** Let $f(x)=x^2$.

$\dfrac{d}{dx}(x^2)=f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$\qquad\qquad\quad =\lim\limits_{h\to 0}\dfrac{(x+h)^2-x^2}{h}$

$\qquad\qquad\quad =\lim\limits_{h\to 0}\dfrac{x^2+2xh+h^2-x^2}{h}$

$\qquad\qquad\quad =\lim\limits_{h\to 0}(2x+h)=2x$

**7.** The graph of $y=f_1(x)$ is decreasing for $x<0$ and
increasing for $x>0$, so its derivative is negative for $x<0$
and positive for $x>0$. (b)

**8.** The graph of $y=f_2(x)$ is always increasing, so its derivative
is always $\geq 0$. (a)

**9.** The graph of $y=f_3(x)$ oscillates between increasing and
decreasing, so its derivative oscillates between positive and
negative. (d)

**10.** The graph of $y=f_4(x)$ is decreasing, then increasing, then
decreasing, and then increasing, so its derivative is
negative, then positive, then negative, and then positive. (c)

**11.** $\dfrac{dy}{dx} = \lim_{h \to 0} \dfrac{y(x+h) - y(x)}{h}$

$= \lim_{h \to 0} \dfrac{[2(x+h)^2 - 13(x+h) + 5] - (2x^2 - 13x + 5)}{h}$

$= \lim_{h \to 0} \dfrac{2x^2 + 4xh + 2h^2 - 13x - 13h + 5 - 2x^2 + 13x - 5}{h}$

$= \lim_{h \to 0} \dfrac{4xh + 2h^2 - 13h}{h}$

$= \lim_{h \to 0} (4x + 2h - 13) = 4x - 13$

At $x = 3$, $\dfrac{dy}{dx} = 4(3) - 13 = -1$, so the tangent line has

slope $-1$ and passes through $(3, y(3)) = (3, -16)$.

$y = -1(x - 3) - 16$

$y = -x - 13$

**12.** Let $f(x) = x^3$.

$f'(1) = \lim_{h \to 0} \dfrac{f(1+h) - f(1)}{h}$

$= \lim_{h \to 0} \dfrac{(1+h)^3 - 1^3}{h}$

$= \lim_{h \to 0} \dfrac{1 + 3h + 3h^2 + h^3 - 1}{h}$

$= \lim_{h \to 0} (3 + 3h + h^2) = 3$

**(a)** The tangent line has slope 3 and passes through $(1, 1)$.
    Its equation is $y = 3(x - 1) + 1$, or $y = 3x - 2$.

**(b)** The normal line has slope $-\dfrac{1}{3}$ and passes through

    $(1, 1)$. Its equation is $y = -\dfrac{1}{3}(x - 1) + 1$, or

    $y = -\dfrac{1}{3}x + \dfrac{4}{3}$.

**13.** Since the graph of $y = x \ln x - x$ is decreasing for
    $0 < x < 1$ and increasing for $x > 1$, its derivative is
    negative for $0 < x < 1$ and positive for $x > 1$. The only
    one of the given functions with this property is $y = \ln x$.
    Note also that $y = \ln x$ is undefined for $x < 0$, which
    further agrees with the given graph. (ii)

**14.** Each of the functions $y = \sin x$, $y = x$, $y = \sqrt{x}$ has the

    property that $y(0) = 0$ but the graph has nonzero slope (or

    undefined slope) at $x = 0$, so none of these functions can be

    its own derivative. The function $y = x^2$ is not its own

    derivative because $y(1) = 1$ but

    $y'(1) = \lim_{h \to 0} \dfrac{(1+h)^2 - 1^2}{h} = \lim_{h \to 0} \dfrac{2h + h^2}{h}$

    $= \lim_{h \to 0} (2 + h) = 2.$

    This leaves only $e^x$, which can plausibly be its own
    derivative because both the function value and the slope
    increase from very small positive values to very large
    values as we move from left to right along the graph. (iv)

**15. (a)** The amount of daylight is increasing at the fastest rate
    when the slope of the graph is largest. This occurs
    about one-fourth of the way through the year, sometime
    around April 1. The rate at this time is approximately
    $\dfrac{4 \text{ hours}}{24 \text{ days}}$ or $\dfrac{1}{6}$ hour per day.

**(b)** Yes, the rate of change is zero when the tangent to the
    graph is horizontal. This occurs near the beginning of
    the year and halfway through the year, around
    January 1 and July 1.

**(c)** Positive: January 1 through July 1
    Negative: July 1 through December 31

**16.** The slope of the given graph is zero at $x \approx -2$ and at
    $x \approx 1$, so the derivative graph includes $(-2, 0)$ and $(1, 0)$.
    The slopes at $x = -3$ and at $x = 2$ are about 5 and the
    slope at $x = -0.5$ is about $-2.5$, so the derivative graph
    includes $(-3, 5)$, $(2, 5)$, and $(-0.5, -2.5)$. Connecting the
    points smoothly, we obtain the graph shown.
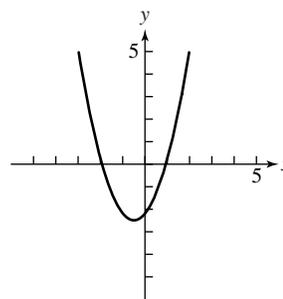


**17. (a)** Using Figure 3.10a, the number of rabbits is largest
    after 40 days and smallest from about 130 to 200 days.
    Using Figure 3.10b, the derivative is 0 at these times.

**(b)** Using Figure 3.10b, the derivative is largest after
    20 days and smallest after about 63 days. Using
    Figure 3.10a, there were 1700 and about 1300 rabbits,
    respectively, at these times.

**18. (a)** The slope from $x = -4$ to $x = 0$ is $\dfrac{2 - 0}{0 - (-4)} = \dfrac{1}{2}$.

    The slope from $x = 0$ to $x = 1$ is $\dfrac{-2 - 2}{1 - 0} = -4$.

    The slope from $x = 1$ to $x = 4$ is $\dfrac{-2 - (-2)}{4 - 1} = 0$.

    The slope from $x = 4$ to $x = 6$ is $\dfrac{2 - (-2)}{6 - 4} = 2$.

    Note that the derivative is undefined at $x = 0$, $x = 1$,
    and $x = 4$. (The function is differentiable at $x = -4$
    and at $x = 6$ because these are endpoints of the domain
    and the one-sided derivatives exist.) The graph of the
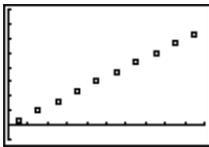    derivative is shown.
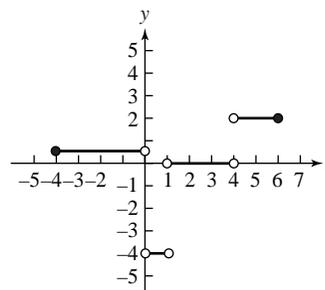
**18. continued**

(b) $x = 0, 1, 4$

**19.**

| Midpoint of Interval $(x)$ | Slope $\left(\dfrac{\Delta y}{\Delta x}\right)$ |
|---|---|
| 0.5 | $\dfrac{3.3 - 0}{1 - 0} = 3.3$ |
| 1.5 | $\dfrac{13.3 - 3.3}{2 - 1} = 10.0$ |
| 2.5 | $\dfrac{29.9 - 13.3}{3 - 2} = 16.6$ |
| 3.5 | $\dfrac{53.2 - 29.9}{4 - 3} = 23.3$ |
| 4.5 | $\dfrac{83.2 - 53.2}{5 - 4} = 30.0$ |
| 5.5 | $\dfrac{119.8 - 83.2}{6 - 5} = 36.6$ |
| 6.5 | $\dfrac{163.0 - 119.8}{7 - 6} = 43.2$ |
| 7.5 | $\dfrac{212.9 - 163.0}{8 - 7} = 49.9$ |
| 8.5 | $\dfrac{269.5 - 212.9}{9 - 8} = 56.6$ |
| 9.5 | $\dfrac{332.7 - 269.5}{10 - 9} = 63.2$ |

A graph of the derivative data is shown.



[0, 10] by[−10, 80]

(a) The derivative represents the speed of the skier.

(b) Since the distances are given in feet and the times are given in seconds, the units are feet per second.

(c) The graph appears to be approximately linear and passes through (0, 0) and (9.5, 63.2), so the slope is $\dfrac{63.2 - 0}{9.5 - 0} \approx 6.65$. The equation of the derivative is approximately $D = 6.65t$.

**20. (a)**
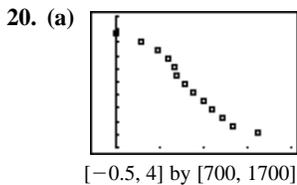


[−0.5, 4] by [700, 1700]

**(b)**

| Midpoint of Interval $(x)$ | Slope $\left(\dfrac{\Delta y}{\Delta x}\right)$ |
|---|---|
| $\dfrac{0.00 + 0.56}{2} = 0.28$ | $\dfrac{1512 - 1577}{0.56 - 0.00} \approx -116.07$ |
| $\dfrac{0.56 + 0.92}{2} = 0.74$ | $\dfrac{1448 - 1512}{0.92 - 0.56} \approx -177.78$ |
| $\dfrac{0.92 + 1.19}{2} = 1.055$ | $\dfrac{1384 - 1448}{1.19 - 0.92} \approx -237.04$ |
| $\dfrac{1.19 + 1.30}{2} = 1.245$ | $\dfrac{1319 - 1384}{1.30 - 1.19} \approx -590.91$ |
| $\dfrac{1.30 + 1.39}{2} = 1.345$ | $\dfrac{1255 - 1319}{1.39 - 1.30} \approx -711.11$ |
| $\dfrac{1.39 + 1.57}{2} = 1.48$ | $\dfrac{1191 - 1255}{1.57 - 1.39} \approx -355.56$ |
| $\dfrac{1.57 + 1.74}{2} = 1.655$ | $\dfrac{1126 - 1191}{1.74 - 1.57} \approx -382.35$ |
| $\dfrac{1.74 + 1.98}{2} = 1.86$ | $\dfrac{1062 - 1126}{1.98 - 1.74} \approx -266.67$ |
| $\dfrac{1.98 + 2.18}{2} = 2.08$ | $\dfrac{998 - 1062}{2.18 - 1.98} = -320.00$ |
| $\dfrac{2.18 + 2.41}{2} = 2.295$ | $\dfrac{933 - 998}{2.41 - 2.18} \approx -282.61$ |
| $\dfrac{2.41 + 2.64}{2} = 2.525$ | $\dfrac{869 - 933}{2.64 - 2.41} \approx -278.26$ |
| $\dfrac{2.64 + 3.24}{2} = 2.94$ | $\dfrac{805 - 869}{3.24 - 2.64} \approx -106.67$ |

A graph of the derivative data is shown.
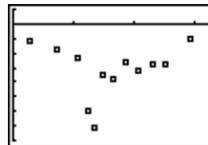


[0, 3.24] by [−800, 100]

(c) Since the elevation $y$ is given in feet and the distance $x$ down river is given in miles, the units of the gradiant are feet per mile.

(d) Since the elevation $y$ is given in feet and the distance $x$ downriver is given in miles, the units of the distance $\dfrac{dy}{dx}$ are feet per mile.

(e) Look for the steepest part of the curve. This is where the elevation is dropping most rapidly, and therefore the most likely location for significant "rapids."

(f) Look for the lowest point on the graph. This is where the elevation is dropping most rapidly, and therefore the most likely location for significant "rapids."

**21.**
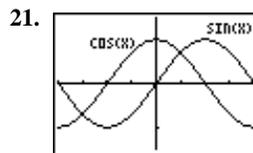


[−$\pi$, $\pi$] by [−1.5, 1.5]

The cosine function could be the derivative of the sine function. The values of the cosine are positive where the sine is increasing, zero where the sine has horizontal tangents, and negative where sine is decreasing.

**22.** We show that the right-hand derivative at 1 does not exist.

$$\lim_{h \to 0^+} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0^+} \frac{3(1 + h) - (1)^3}{h}$$

$$= \lim_{h \to 0^+} \frac{2 + 3h}{h} = \lim_{h \to 0^+} \left(\frac{2}{h} + 3\right) = \infty$$

**23.** $\lim_{h \to 0^+} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0^+} \frac{\sqrt{h} - \sqrt{0}}{h} = \lim_{h \to 0^+} \frac{\sqrt{h}}{h}$

$$= \lim_{h \to 0^+} \frac{1}{\sqrt{h}} = \infty$$

Thus, the right-hand derivative at 0 does not exist.

**24.** Two parabolas are parallel if they have the same derivative at every value of $x$. This means that their tangent lines are parallel at each value of $x$.
Two such parabolas are given by $y = x^2$ and $y = x^2 + 4$. They are graphed below.
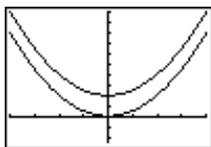


[−4, 4] by [−5, 20]

The parabolas are "everywhere equidistant," as long as the distance between them is always measured along a vertical line.

**25.** For $x > -1$, the graph of $y = f(x)$ must lie on a line of slope $-2$ that passes through $(0, -1)$: $y = -2x - 1$. Then $y(-1) = -2(-1) - 1 = 1$, so for $x < -1$, the graph of $y = f(x)$ must lie on a line of slope 1 that passes through $(-1, 1)$: $y = 1(x + 1) + 1$ or $y = x + 2$.

Thus $f(x) = \begin{cases} x + 2, & x < -1 \\ -2x - 1, & x \geq -1 \end{cases}$
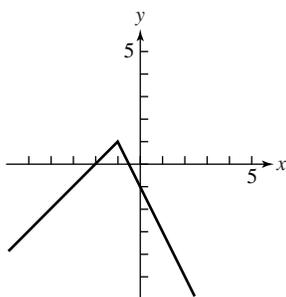


**26. (a)** $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0} (2x + h) = 2x$$

**(b)** $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{2(x + h) - 2x}{h} = \lim_{h \to 0} 2 = 2$$

**(c)** $\lim_{x \to 1^-} f'(x) = \lim_{x \to 1^-} 2x = 2(1) = 2$

**(d)** $\lim_{x \to 1^+} f'(x) = \lim_{x \to 1^+} 2 = 2$

**(e)** Yes, the one-sided limits exist and are the same, so

$$\lim_{x \to 1} f'(x) = 2.$$

**(f)** $\lim_{h \to 0^-} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0^-} \frac{(1 + h)^2 - 1^2}{h}$

$$= \lim_{h \to 0^-} \frac{2h + h^2}{h} = \lim_{h \to 0^-} (2 + h) = 2$$

**(g)** $\lim_{h \to 0^+} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0^-} \frac{2(1 + h) - 1^2}{h}$

$$= \lim_{h \to 0^-} \frac{1 + 2h}{h} = \lim_{h \to 0^-} \left(\frac{1}{h} + 2\right) = -\infty$$

The right-hand derivative does not exist.

**(h)** It does not exist because the right-hand derivative does not exist.

**27.** The $y$-intercept of the derivative is $b - a$.

**28.** Since the function must be continuous at $x = 1$, we have $\lim_{x \to 1^+} (3x + k) = f(1) = 1$, so $3 + k = 1$, or $k = -2$.

This gives $f(x) = \begin{cases} x^3, & x \leq 1 \\ 3x - 2, & x > 1. \end{cases}$

Now we confirm that $f(x)$ is differentiable at $x = 1$.

$$\lim_{h \to 0^-} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0^-} \frac{(1 + h)^3 - (1)^3}{h}$$

$$= \lim_{h \to 0} \frac{3h + 3h^2 + h^3}{h}$$

$$= \lim_{h \to 0} (3 + 3h + h^2) = 3$$

$$\lim_{h \to 0^+} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0^+} \frac{[3(1 + h) - 2] - (1)^3]}{h}$$

$$= \lim_{h \to 0^+} \frac{(1 + 3h) - 1}{h} = \lim_{h \to 0^+} 3 = 3$$

Since the right-hand derivative equals the left-hand derivative at $x = 1$, the derivative exists (and is equal to 3) when $k = -2$.

**29. (a)** $1 \cdot \dfrac{364}{365} \cdot \dfrac{363}{365} \approx 0.992$

Alternate method: $\dfrac{_{365}P_3}{365^3} \approx 0.992$

**(b)** Using the answer to part (a), the probability is about $1 - 0.992 = 0.008$.

**(c)** Let $P$ represent the answer to part (b), $P \approx 0.008$. Then the probability that three people all have different birthdays is $1 - P$. Adding a fourth person, the probability that all have different birthdays is $(1 - P)\left(\dfrac{362}{365}\right)$, so the probability of a shared birthday is $1 - (1 - P)\left(\dfrac{362}{365}\right) \approx 0.016$.

**(d)** No. Clearly February 29 is a much less likely birth date. Furthermore, census data do not support the assumption that the other 365 birth dates are equally likely. However, this simplifying assumption may still give us some insight into this problem even if the calculated probabilities aren't completely accurate.

## ■ **Section 3.2** Differentiability (pp. 105–112)

### Exploration 1   Zooming in to "See" Differentiability

1. Zooming in on the graph of *f* at the point (0, 1) always produces a graph exactly like the one shown below, provided that a square window is used. The corner shows no sign of straightening out.
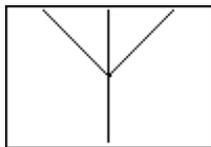


[−0.25, 0.25] by [0.836, 1.164]

2. Zooming in on the graph of *g* at the point (0, 1) begins to reveal a smooth turning point. This graph shows the result of three zooms, each by a factor of 4 horizontally and vertically, starting with the window [−4, 4] by [−1.624, 3.624].
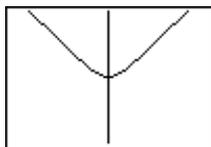


[−0.0625, 0.0625] by [0.959, 1.041]

3. On our grapher, the graph became horizontal after 8 zooms. Results can vary on different machines.

4. As we zoom in on the graphs of *f* and *g* together, the differentiable function gradually straightens out to resemble its tangent line, while the nondifferentiable function stubbornly retains its same shape.
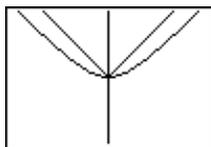


[−0.03125, 0.03125] by [0.9795, 1.0205]

### Exploration 2   Looking at the Symmetric Difference Quotient Analytically

1. $\dfrac{f(10 + h) - f(10)}{h} = \dfrac{(10.01)^2 - 10^2}{0.01} = 20.01$

   $f'(10) = 2 \cdot 10 = 20$

   The difference quotient is 0.01 away from $f'(10)$.

2. $\dfrac{f(10 + h) - f(10 - h)}{2h} = \dfrac{(10.01)^2 - (9.99)^2}{0.02} = 20$

   The symmetric difference quotient exactly equals $f'(10)$.

3. $\dfrac{f(10 + h) - f(10)}{h} = \dfrac{(10.01)^3 - 10^3}{0.01} = 300.3001$.

   $f'(10) = 3 \cdot 10^2 = 300$

   The difference quotient is 0.3001 away from $f'(10)$.

   $\dfrac{f(10 + h) - f(10 - h)}{2h} = \dfrac{(10.01)^3 - (9.99)^3}{0.02} = 300.0001$.

   The symmetric difference quotient is 0.0001 away from

   $f'(10)$.

### Quick Review 3.2

1. Yes

2. No (The $f(h)$ term in the numerator is incorrect.)

3. Yes                    4. Yes

5. No (The denominator for this expression should be 2*h)*.

6. All reals

7. $[0, \infty)$                    8. $[3, \infty)$

9. The equation is equivalent to $y = 3.2x + (3.2\pi + 5)$, so the slope is 3.2.

10. $\dfrac{f(3 + 0.001) - f(3 - 0.001)}{0.002} = \dfrac{5(3 + 0.001) - 5(3 - 0.001)}{0.002}$

$$= \dfrac{5(0.002)}{0.002} = 5$$

### Section 3.2 Exercises

1. Left-hand derivative:

$$\lim_{h \to 0^-} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0^-} \frac{h^2 - 0}{h} = \lim_{h \to 0^-} h = 0$$

Right-hand derivative:

$$\lim_{h \to 0^+} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h - 0}{h} = \lim_{h \to 0^+} 1 = 1$$

Since $0 \neq 1$, the function is not differentiable at the point $P$.

2. Left-hand derivative:

$$\lim_{h \to 0^-} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0^-} \frac{2 - 2}{h} = \lim_{h \to 0^-} 0 = 0$$

Right-hand derivative:

$$\lim_{h \to 0^+} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0^+} \frac{2(1 + h) - 2}{h} = \lim_{h \to 0^+} 2 = 2$$

Since $0 \neq 2$, the function is not differentiable at the point $P$.

3. Left-hand derivative:

$$\lim_{h \to 0^-} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0^-} \frac{\sqrt{1 + h} - 1}{h}$$

$$= \lim_{h \to 0^-} \frac{(\sqrt{1 + h} - 1)(\sqrt{1 + h} + 1)}{h(\sqrt{1 + h} + 1)}$$

$$= \lim_{h \to 0^-} \frac{(1 + h) - 1}{h(\sqrt{1 + h} + 1)}$$

$$= \lim_{h \to 0^-} \frac{1}{\sqrt{1 + h} + 1} = \frac{1}{2}$$

Right-hand derivative:

$$\lim_{h \to 0^+} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0^+} \frac{[2(1 + h) - 1] - 1}{h}$$

$$= \lim_{h \to 0^+} \frac{2h}{h}$$

$$= \lim_{h \to 0^+} 2 = 2$$

Since $\frac{1}{2} \neq 2$, the function is not differentiable at the

point $P$.

**4.** Left-hand derivative:

$$\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \frac{(1+h)-1}{h} = \lim_{h\to 0^-} 1 = 1$$

Right-hand derivative:

$$\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{\frac{1}{1+h}-1}{h}$$

$$= \lim_{h\to 0^+} \frac{1-(1+h)}{h(1+h)}$$

$$= \lim_{h\to 0^+} \frac{-h}{h(1+h)}$$

$$= \lim_{h\to 0^+} -\frac{1}{1+h} = -1$$

Since $1 \neq -1$, the function is not differentiable at the point $P$.

**5. (a)** All points in $[-3, 2]$

**(b)** None

**(c)** None

**6. (a)** All points in $[-2, 3]$

**(b)** None

**(c)** None

**7. (a)** All points in $[-3, 3]$ except $x = 0$

**(b)** None

**(c)** $x = 0$

**8. (a)** All points in $[-2, 3]$ except $x = -1, 0, 2$

**(b)** $x = -1$

**(c)** $x = 0, x = 2$

**9. (a)** All points in $[-1, 2]$ except $x = 0$

**(b)** $x = 0$

**(c)** None

**10. (a)** All points in $[-3, 3]$ except $x = -2, 2$

**(b)** $x = -2, x = 2$

**(c)** None

**11.** Since $\lim_{x\to 0} \tan^{-1} x = \tan^{-1} 0 = 0 \neq y(0)$, the problem is a discontinuity.

**12.** $\lim_{h\to 0^-} \frac{y(0+h)-y(0)}{h} = \lim_{h\to 0^-} \frac{h^{4/5}}{h} = \lim_{h\to 0^-} \frac{1}{h^{1/5}} = -\infty$

$\lim_{h\to 0^+} \frac{y(0+h)-y(0)}{h} = \lim_{h\to 0^+} \frac{h^{4/5}}{h} = \lim_{h\to 0^+} \frac{1}{h^{1/5}} = \infty$

The problem is a cusp.

**13.** Note that $y = x + \sqrt{x^2} + 2 = x + |x| + 2$

$$= \begin{cases} 2, & x \le 0 \\ 2x + 2, & x > 0. \end{cases}$$

$\lim_{h\to 0^-} \frac{y(0+h)-y(0)}{h} = \lim_{h\to 0^-} \frac{2-2}{h} = \lim_{h\to 0^-} 0 = 0$

$\lim_{h\to 0^+} \frac{y(0+h)-y(0)}{h} = \lim_{h\to 0^+} \frac{(2h+2)-2}{h} = \lim_{h\to 0^+} 2 = 2$

The problem is a corner.

**14.** $\lim_{h\to 0} \frac{y(0+h)-y(0)}{h} = \lim_{h\to 0} \frac{(3-\sqrt[3]{h})-3}{h} = \lim_{h\to 0} -\frac{\sqrt[3]{h}}{h}$

$$= \lim_{h\to 0} \left(-\frac{1}{h^{2/3}}\right) = -\infty$$

The problem is a vertical tangent.

**15.** Note that $y = 3x - 2|x| - 1 = \begin{cases} 5x - 1, & x \le 0 \\ x - 1, & x > 0 \end{cases}$

$\lim_{h\to 0^-} \frac{y(0+h)-y(0)}{h} = \lim_{h\to 0^-} \frac{(5h-1)-(-1)}{h} = \lim_{h\to 0^-} 5 = 5$

$\lim_{h\to 0^+} \frac{y(0+h)-y(0)}{h} = \lim_{h\to 0^+} \frac{(h-1)-(-1)}{h} = \lim_{h\to 0^+} 1 = 1$

The problem is a corner.

**16.** $\lim_{h\to 0^-} \frac{y(0+h)-y(0)}{h} = \lim_{h\to 0^-} \frac{\sqrt[3]{|h|}-0}{h} = \lim_{h\to 0^-} \frac{-\sqrt[3]{h}}{h}$

$$= \lim_{h\to 0^-} \left(-\frac{1}{h^{2/3}}\right) = -\infty$$

$\lim_{h\to 0^+} \frac{y(0+h)-y(0)}{h} = \lim_{h\to 0^+} \frac{\sqrt[3]{|h|}-0}{h} = \lim_{h\to 0^+} \frac{\sqrt[3]{h}}{h}$

$$= \lim_{h\to 0^+} \frac{1}{h^{2/3}} = \infty$$

The problem is a cusp.

**17.** Find the zeros of the denominator.

$$x^2 - 4x - 5 = 0$$
$$(x+1)(x-5) = 0$$
$$x = -1 \text{ or } x = 5$$

The function is a rational function, so it is differentiable for all $x$ in its domain: all reals except $x = -1, 5$.

**18.** The function is differentiable except possibly where $3x - 6 = 0$, that is, at $x = 2$. We check for differentiability at $x = 2$, using $k$ instead of the usual $h$, in order to avoid confusion with the function $h(x)$.

$$\lim_{k\to 0} \frac{h(2+k)-h(2)}{k} = \lim_{k\to 0} \frac{[\sqrt[3]{3(2+k)-6}+5]-5}{k}$$

$$= \lim_{k\to 0} \frac{\sqrt[3]{3k}}{k} = \sqrt[3]{3} \lim_{k\to 0} \frac{1}{k^{2/3}} = \infty$$

The function has a vertical tangent at $x = 2$. It is differentiable for all reals except $x = 2$.

**19.** Note that the sine function is odd, so

$$P(x) = \sin(|x|) - 1 = \begin{cases} -\sin x - 1, & x < 0 \\ \sin x - 1, & x \ge 0. \end{cases}$$

The graph of $P(x)$ has a corner at $x = 0$. The function is differentiable for all reals except $x = 0$.

**20.** Since the cosine function is even, so $Q(x) = 3\cos(|x|) = 3\cos x$. The function is differentiable for all reals.

**21.** The function is piecewise-defined in terms of polynomials, so it is differentiable everywhere except possibly at $x = 0$ and at $x = 3$. Check $x = 0$:

$$\lim_{h\to 0^-} \frac{g(0+h)-g(0)}{h} = \lim_{h\to 0^-} \frac{(h+1)^2-1}{h} = \lim_{h\to 0^-} \frac{h^2+2h}{h}$$

$$= \lim_{h\to 0^-} (h+2) = 2$$

$$\lim_{h\to 0^+} \frac{g(0+h)-g(0)}{h} = \lim_{h\to 0^+} \frac{(2h+1)-1}{h} = \lim_{h\to 0^+} 2 = 2$$

The function is differentiable at $x = 0$.

**21. continued**

Check $x = 3$:

Since $g(3) = (4 - 3)^2 = 1$ and

$\lim\limits_{x\to3^-} g(x) = \lim\limits_{x\to3^-} (2x + 1) = 2(3) + 1 = 7$, the function is

not continuous (and hence not differentiable) at $x = 3$.

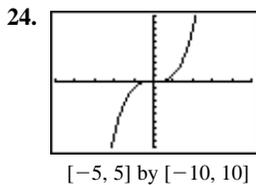The function is differentiable for all reals except $x = 3$.

**22.** Note that $C(x) = x|x| = \begin{cases} -x^2, & x < 0 \\ x^2, & x \geq 0 \end{cases}$, so it is

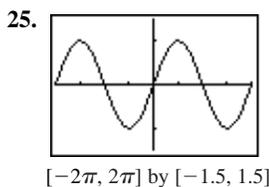differentiable for all $x$ except possibly at $x = 0$.

Check $x = 0$:

$\lim\limits_{h\to0} \dfrac{C(0 + h) - C(0)}{h} = \lim\limits_{h\to0} \dfrac{h|h| - 0}{h} = \lim\limits_{h\to0} |h| = 0$

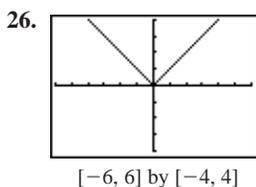The function is differentiable for all reals.

**23. (a)** $x = 0$ is not in their domains, or, they are both
discontinuous at $x = 0$.

**(b)** For $\dfrac{1}{x}$: NDER $\left(\dfrac{1}{x}, 0\right) = 1,000,000$

For $\dfrac{1}{x^2}$: NDER $\left(\dfrac{1}{x^2}, 0\right) = 0$

**(c)** It returns an incorrect response because even though

these functions are not defined at $x = 0$, they are

defined at $x = \pm0.001$. The responses differ from each

other because $\dfrac{1}{x^2}$ is even (which automatically makes

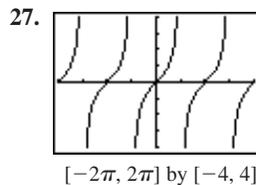NDER $\left(\dfrac{1}{x^2}, 0\right) = 0$) and $\dfrac{1}{x}$ is odd.

**24.**



$[-5, 5]$ by $[-10, 10]$

$\dfrac{dy}{dx} = x^3$

**25.**



$[-2\pi, 2\pi]$ by $[-1.5, 1.5]$

$\dfrac{dy}{dx} = \sin x$

**26.**



$[-6, 6]$ by $[-4, 4]$

$\dfrac{dy}{dx} = $ abs $(x)$ or $|x|$

**27.**



$[-2\pi, 2\pi]$ by $[-4, 4]$

$\dfrac{dy}{dx} = \tan x$

Note: Due to the way NDER is defined, the graph of

$y = $ NDER $(x)$ actually has two asymptotes for each

asymptote of $y = \tan x$. The asymptotes of $y = $ NDER $(x)$

occur at $x = \dfrac{\pi}{2} + k\pi \pm 0.001$, where $k$ is an integer. A

good window for viewing this behavior is $[1.566, 1.576]$ by

$[-1000, 1000]$.

**28.**



$[-2\pi, 2\pi]$ by $[-20, 20]$
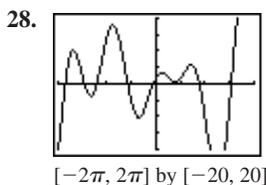
The graph of NDER $(x)$ does not look like the graph of any
basic function.

**29. (a)** $\lim\limits_{x\to1^-} f(x) = f(1)$

$\lim\limits_{x\to1^-} (3 - x) = a(1)^2 + b(1)$

$2 = a + b$

The relationship is $a + b = 2$.

**(b)** Since the function needs to be continuous, we may
assume that $a + b = 2$ and $f(1) = 2$.

$\lim\limits_{h\to0^-} \dfrac{f(1 + h) - f(1)}{h} = \lim\limits_{h\to0^-} \dfrac{3 - (1 + h) - 2}{h}$

$= \lim\limits_{h\to0^-} (-1) = -1$

$\lim\limits_{h\to0^+} \dfrac{f(1 + h) - f(1)}{h} = \lim\limits_{h\to0^-} \dfrac{a(1 + h)^2 + b(1 + h) - 2}{h}$

$= \lim\limits_{h\to0^-} \dfrac{a + 2ah + ah^2 + b + bh - 2}{h}$

$= \lim\limits_{h\to0^-} \dfrac{2ah + ah^2 + bh + (a + b - 2)}{h}$

$= \lim\limits_{h\to0^-} (2a + ah + b)$

$= 2a + b$

Therefore, $2a + b = -1$. Substituting $2 - a$ for $b$ gives
$2a + (2 - a) = -1$, so $a = -3$.
Then $b = 2 - a = 2 - (-3) = 5$. The values are
$a = -3$ and $b = 5$.

**30.** The function $f(x)$ does not have the intermediate value
property. Choose some $a$ in $(-1, 0)$ and $b$ in $(0, 1)$. Then
$f(a) = 0$ and $f(b) = 1$, but $f$ does not take on any value
between 0 and 1. Therefore, by the Intermediate Value
Theorem for Derivatives, $f$ cannot be the derivative of any
function on $[-1, 1]$.

**31. (a)** Note that $-x \le \sin \dfrac{1}{x} \le x$, for all $x$,

so $\displaystyle\lim_{x \to 0}\left(x \sin \dfrac{1}{x}\right) = 0$ by the Sandwich Theorem.

Therefore, $f$ is continuous at $x = 0$.

**(b)** $\dfrac{f(0+h) - f(0)}{h} = \dfrac{h \sin \dfrac{1}{h} - 0}{h} = \sin \dfrac{1}{h}$

**(c)** The limit does not exist because $\sin \dfrac{1}{h}$ oscillates

between $-1$ and $1$ an infinite number of times

arbitrarily close to $h = 0$ (that is, for $h$ in any open

interval containing 0).

**(d)** No, because the limit in part (c) does not exist.

**(e)** $\dfrac{g(0+h) - g(0)}{h} = \dfrac{h^2 \sin\left(\dfrac{1}{h}\right) - 0}{h} = h \sin \dfrac{1}{h}$
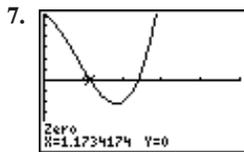
As noted in part (a), the limit of this as $x$ approaches

zero is 0, so $g'(0) = 0$.

## ■ Section 3.3 Rules for Differentiation

(pp. 112–121)

### Quick Review 3.3

**1.** $(x^2 - 2)(x^{-1} + 1) = x^2 x^{-1} + x^2 \cdot 1 - 2x^{-1} - 2 \cdot 1$
$= x + x^2 - 2x^{-1} - 2$

**2.** $\left(\dfrac{x}{x^2+1}\right)^{-1} = \dfrac{x^2+1}{x} = \dfrac{x^2}{x} + \dfrac{1}{x} = x + x^{-1}$

**3.** $3x^2 - \dfrac{2}{x} + \dfrac{5}{x^2} = 3x^2 - 2x^{-1} + 5x^{-2}$

**4.** $\dfrac{3x^4 - 2x^3 + 4}{2x^2} = \dfrac{3x^4}{2x^2} - \dfrac{2x^3}{2x^2} + \dfrac{4}{2x^2} = \dfrac{3}{2}x^2 - x + 2x^{-2}$

**5.** $(x^{-1} + 2)(x^{-2} + 1) = x^{-1}x^{-2} + x^{-1} \cdot 1 + 2x^{-2} + 2 \cdot 1$
$= x^{-3} + x^{-1} + 2x^{-2} + 2$

**6.** $\dfrac{x^{-1} + x^{-2}}{x^{-3}} = x^3(x^{-1} + x^{-2}) = x^2 + x$

**7.**



Zero
X=1.1734174  Y=0

[0, 5] by [−6, 6]

At $x \approx 1.173$, $500x^6 \approx 1305$.
At $x \approx 2.394$, $500x^6 \approx 94{,}212$
After rounding, we have:
At $x \approx 1$, $500x^6 \approx 1305$.
At $x \approx 2$, $500x^6 \approx 94{,}212$.

**8. (a)** $f(10) = 7$

**(b)** $f(0) = 7$

**(c)** $f(x + h) = 7$

**(d)** $\displaystyle\lim_{x \to a} \dfrac{f(x) - f(a)}{x - a} = \lim_{x \to a} \dfrac{7 - 7}{x - a} = \lim_{x \to a} 0 = 0$

**9.** These are all constant functions, so the graph of each
function is a horizontal line and the derivative of each
function is 0.

**10. (a)** $f'(x) = \displaystyle\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \dfrac{\dfrac{x+h}{\pi} - \dfrac{x}{\pi}}{h}$

$= \displaystyle\lim_{h \to 0} \dfrac{x+h-x}{\pi h} = \lim_{h \to 0} \dfrac{1}{\pi} = \dfrac{1}{\pi}$

**(b)** $f'(x) = \displaystyle\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \dfrac{\dfrac{\pi}{x+h} - \dfrac{\pi}{x}}{h}$

$= \displaystyle\lim_{h \to 0} \dfrac{\pi x - \pi(x+h)}{hx(x+h)} = \lim_{h \to 0} \dfrac{-\pi h}{hx(x+h)}$

$= \displaystyle\lim_{h \to 0} -\dfrac{\pi}{x(x+h)} = -\dfrac{\pi}{x^2} = -\pi x^{-2}$

## Section 3.3 Exercises

**1.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(-x^2) + \dfrac{d}{dx}(3) = -2x + 0 = -2x$

$\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}(-2x) = -2$

**2.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{1}{3}x^3\right) - \dfrac{d}{dx}(x) = x^2 - 1$

$\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}(x^2) - \dfrac{d}{dx}(1) = 2x - 0 = 2x$

**3.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(2x) + \dfrac{d}{dx}(1) = 2 + 0 = 2$

$\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}(2) = 0$

**4.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(x^2) + \dfrac{d}{dx}(x) + \dfrac{d}{dx}(1) = 2x + 1 + 0 = 2x + 1$

$\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}(2x) + \dfrac{d}{dx}(1) = 2 + 0 = 2$

**5.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{1}{3}x^3\right) + \dfrac{d}{dx}\left(\dfrac{1}{2}x^2\right) + \dfrac{d}{dx}(x)$

$= x^2 + x + 1$

$\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}(x^2) + \dfrac{d}{dx}(x) + \dfrac{d}{dx}(1) = 2x + 1 + 0 = 2x + 1$

**6.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(1) - \dfrac{d}{dx}(x) + \dfrac{d}{dx}(x^2) - \dfrac{d}{dx}(x^3)$

$= 0 - 1 + 2x - 3x^2 = -1 + 2x - 3x^2$

$\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}(-1) + \dfrac{d}{dx}(2x) - \dfrac{d}{dx}(3x^2)$

$= 0 + 2 - 6x = 2 - 6x$

**7.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(x^4) - \dfrac{d}{dx}(7x^3) + \dfrac{d}{dx}(2x^2) + \dfrac{d}{dx}(15)$

$= 4x^3 - 21x^2 + 4x + 0 = 4x^3 - 21x^2 + 4x$

$\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}(4x^3) - \dfrac{d}{dx}(21x^2) + \dfrac{d}{dx}(4x) = 12x^2 - 42x + 4$

**8.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(5x^3) - \dfrac{d}{dx}(3x^5) = 15x^2 - 15x^4$

$\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}(15x^2) - \dfrac{d}{dx}(15x^4) = 30x - 60x^3$

**9.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(4x^{-2}) - \dfrac{d}{dx}(8x) + \dfrac{d}{dx}(1)$

$= -8x^{-3} - 8 + 0 = -8x^{-3} - 8$

$\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}(-8x^{-3}) - \dfrac{d}{dx}(8) = 24x^{-4} - 0 = 24x^{-4}$

**10.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{1}{4}x^{-4}\right) - \dfrac{d}{dx}\left(\dfrac{1}{3}x^{-3}\right) + \dfrac{d}{dx}\left(\dfrac{1}{2}x^{-2}\right) - \dfrac{d}{dx}(x^{-1})$

$\quad + \dfrac{d}{dx}(3)$

$\quad = -x^{-5} + x^{-4} - x^{-3} + x^{-2} + 0$

$\quad = -x^{-5} + x^{-4} - x^{-3} + x^{-2}$

$\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}(-x^{-5}) + \dfrac{d}{dx}(x^{-4}) - \dfrac{d}{dx}(x^{-3}) + \dfrac{d}{dx}(x^{-2})$

$\quad = 5x^{-6} - 4x^{-5} + 3x^{-4} - 2x^{-3}$

**11. (a)** $\dfrac{dy}{dx} = \dfrac{d}{dx}[(x+1)(x^2+1)]$

$\quad = (x+1)\dfrac{d}{dx}(x^2+1) + (x^2+1)\dfrac{d}{dx}(x+1)$

$\quad = (x+1)(2x) + (x^2+1)(1)$

$\quad = 2x^2 + 2x + x^2 + 1$

$\quad = 3x^2 + 2x + 1$

**(b)** $\dfrac{dy}{dx} = \dfrac{d}{dx}[(x+1)(x^2+1)]$

$\quad = \dfrac{d}{dx}(x^3 + x^2 + x + 1)$

$\quad = 3x^2 + 2x + 1$

**12. (a)** $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{x^2+3}{x}\right)$

$\quad = \dfrac{x\dfrac{d}{dx}(x^2+3) - (x^2+3)\dfrac{d}{dx}(x)}{x^2}$

$\quad = \dfrac{x(2x) - (x^2+3)}{x^2}$

$\quad = \dfrac{x^2-3}{x^2}$

**(b)** $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{x^2+3}{x}\right) = \dfrac{d}{dx}(x + 3x^{-1}) = 1 - 3x^{-2}$

$\quad = 1 - \dfrac{3}{x^2}$

This is equivalent to the answer in part (a).

**13.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\dfrac{2x+5}{3x-2} = \dfrac{(3x-2)(2) - (2x+5)(3)}{(3x-2)^2} = -\dfrac{19}{(3x-2)^2}$

**14.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{x^2+5x-1}{x^2}\right) = \dfrac{d}{dx}(1 + 5x^{-1} - x^{-2})$

$\quad = 0 - 5x^{-2} + 2x^{-3} = -\dfrac{5}{x^2} + \dfrac{2}{x^3}$

**15.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{(x-1)(x^2+x+1)}{x^3}\right) = \dfrac{d}{dx}\left(\dfrac{x^3-1}{x^3}\right)$

$\quad = \dfrac{d}{dx}(1 - x^{-3}) = 0 + 3x^{-4} = \dfrac{3}{x^4}$

**16.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{1-x}{1+x^2}\right) = \dfrac{(1+x^2)(-1) - (1-x)(2x)}{(1+x^2)^2}$

$\quad = \dfrac{x^2-2x-1}{(1+x^2)^2}$

**17.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{x^2}{1-x^3}\right) = \dfrac{(1-x^3)(2x) - x^2(-3x^2)}{(1-x^3)^2} = \dfrac{x^4+2x}{(1-x^3)^2}$

**18.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{\sqrt{x}-1}{\sqrt{x}+1}\right) = \dfrac{(\sqrt{x}+1)\dfrac{1}{2\sqrt{x}} - (\sqrt{x}-1)\dfrac{1}{2\sqrt{x}}}{(\sqrt{x}+1)^2}$

$\quad = \dfrac{(\sqrt{x}+1) - (\sqrt{x}-1)}{2\sqrt{x}(\sqrt{x}+1)^2} = \dfrac{2}{2\sqrt{x}(\sqrt{x}+1)^2} = \dfrac{1}{\sqrt{x}(\sqrt{x}+1)^2}$

**19.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{(x+1)(x+2)}{(x-1)(x-2)}\right) = \dfrac{d}{dx}\left(\dfrac{x^2+3x+2}{x^2-3x+2}\right)$

$\quad = \dfrac{(x^2-3x+2)(2x+3) - (x^2+3x+2)(2x-3)}{(x^2-3x+2)^2}$

$\quad = \dfrac{(2x^3-3x^2-5x+6) - (2x^3+3x^2-5x-6)}{(x^2-3x+2)^2}$

$\quad = \dfrac{12-6x^2}{(x^2-3x+2)^2}$

**20. (a)** Let $f(x) = x$.

$\quad \dfrac{d}{dx}(x) = f'(x) = \lim\limits_{h\to0}\dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to0}\dfrac{(x+h)-x}{h}$

$\quad = \lim\limits_{h\to0}\dfrac{h}{h} = \lim\limits_{h\to0}(1) = 1$

**(b)** Note that $u = u(x)$ is a function of $x$.

$\quad \dfrac{d}{dx}(-u) = \lim\limits_{h\to0}\dfrac{-u(x+h) - [-u(x)]}{h}$

$\quad = \lim\limits_{h\to0}\left(-\dfrac{u(x+h)-u(x)}{h}\right)$

$\quad = -\lim\limits_{h\to0}\dfrac{u(x+h)-u(x)}{h} = -\dfrac{du}{dx}$

**21.** $\dfrac{d}{dx}(c \cdot f(x)) = c \cdot \dfrac{d}{dx}f(x) + f(x) \cdot \dfrac{d}{dx}(c)$

$\quad = c \cdot \dfrac{d}{dx}f(x) + 0 = c \cdot \dfrac{d}{dx}f(x)$

**22.** $\dfrac{d}{dx}\left(\dfrac{1}{f(x)}\right) = \dfrac{f(x) \cdot 0 - 1 \cdot \dfrac{d}{dx}f(x)}{[f(x)]^2} = -\dfrac{f'(x)}{[f(x)]^2}$

**23. (a)** At $x = 0$, $\dfrac{d}{dx}(uv) = u(0)v'(0) + v(0)u'(0)$

$\quad = (5)(2) + (-1)(-3) = 13$

**(b)** At $x = 0$, $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v(0)u'(0) - u(0)v'(0)}{[v(0)]^2}$

$\quad = \dfrac{(-1)(-3) - (5)(2)}{(-1)^2} = -7$

**(c)** At $x = 0$, $\dfrac{d}{dx}\left(\dfrac{v}{u}\right) = \dfrac{u(0)v'(0) - v(0)u'(0)}{[u(0)]^2}$

$\quad = \dfrac{(5)(2) - (-1)(-3)}{(5)^2} = \dfrac{7}{25}$

**(d)** At $x = 0$, $\dfrac{d}{dx}(7v - 2u) = 7v'(0) - 2u'(0)$

$\quad = 7(2) - 2(-3) = 20$

**24. (a)** At $x = 2$, $\dfrac{d}{dx}(uv) = u(2)v'(2) + v(2)u'(2)$

$= (3)(2) + (1)(-4) = 2$

**(b)** At $x = 2$, $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v(2)u'(2) - u(2)v'(2)}{[v(2)]^2}$

$= \dfrac{(1)(-4) - (3)(2)}{(1)^2} = -10$

**(c)** At $x = 2$, $\dfrac{d}{dx}\left(\dfrac{v}{u}\right) = \dfrac{u(2)v'(2) - v(2)u'(2)}{[u(2)]^2}$

$= \dfrac{(3)(2) - (1)(-4)}{(3)^2} = \dfrac{10}{9}$

**(d)** Use the result from part (a) for $\dfrac{d}{dx}(uv)$.

At $x = 2$, $\dfrac{d}{dx}(3u - 2v + 2uv)$

$= 3u'(2) - 2v'(2) + 2\dfrac{d}{dx}(uv)$

$= 3(-4) - 2(2) + 2(2)$

$= -12$

**25.** $y'(x) = 2x + 5$
$y'(3) = 2(3) + 5 = 11$
The slope is 11. (iii)

**26.** The given equation is equivalent to $y = \dfrac{3}{2}x + 6$, so the
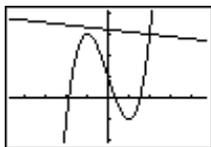
slope is $\dfrac{3}{2}$. (iii)

**27.** $y'(x) = 3x^2 - 3$
$y'(2) = 3(2)^2 - 3 = 9$

The tangent line has slope 9, so the perpendicular line has

slope $-\dfrac{1}{9}$ and passes through (2, 3).

$y = -\dfrac{1}{9}(x - 2) + 3$
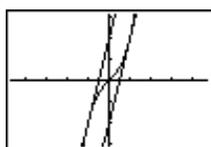
$y = -\dfrac{1}{9}x + \dfrac{29}{9}$

Graphical support:



$[-4.7, 4.7]$ by $[-2.1, 4.1]$

**28.** $y'(x) = 3x^2 + 1$
The slope is 4 when $3x^2 + 1 = 4$, at $x = \pm1$. The tangent
at $x = -1$ has slope 4 and passes through $(-1, -2)$, so its
equation is $y = 4(x + 1) - 2$, or $y = 4x + 2$. The tangent at
$x = 1$ has slope 4 and passes through (1, 2), so its equation
is $y = 4(x - 1) + 2$, or $y = 4x - 2$. The smallest slope
occurs when $3x^2 + 1$ is minimized, so the smallest slope is
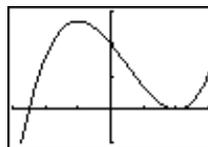1 and occurs at $x = 0$.

Graphical support:



$[-4.7, 4.7]$ by $[-3.1, 3.1]$

**29.** $y'(x) = 6x^2 - 6x - 12$
$= 6(x^2 - x - 2)$
$= 6(x + 1)(x - 2)$
The tangent is parallel to the $x$-axis when $y' = 0$, at $x = -1$
and at $x = 2$. Since $y(-1) = 27$ and $y(2) = 0$, the two
points where this occurs are $(-1, 27)$ and $(2, 0)$.

Graphical support:



$[-3, 3]$ by $[-10, 30]$

**30.** $y'(x) = 3x^2$
$y'(-2) = 12$

The tangent line has slope 12 and passes through $(-2, -8)$,

so its equation is $y = 12(x + 2) - 8$, or $y = 12x + 16$. The

$x$-intercept is $-\dfrac{4}{3}$ and the $y$-intercept is 16.

Graphical support:
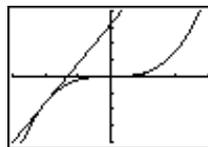


$[-3, 3]$ by $[-20, 20]$

**31.** $y'(x) = \dfrac{(x^2 + 1)(4) - 4x(2x)}{(x^2 + 1)^2} = \dfrac{-4x^2 + 4}{(x^2 + 1)^2}$

At the origin: $y'(0) = 4$

The tangent is $y = 4x$.

At (1, 2): $y'(1) = 0$

The tangent is $y = 2$.

Graphical support:
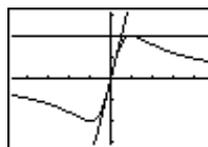


$[-4.7, 4.7]$ by $[-3.1, 3.1]$

**32.** $y'(x) = \dfrac{(4 + x^2)(0) - 8(2x)}{(4 + x^2)^2} = -\dfrac{16x}{(4 + x^2)^2}$

$y'(2) = -\dfrac{1}{2}$

The tangent has slope $-\dfrac{1}{2}$ and passes through (2, 1). Its

equation is $y = -\dfrac{1}{2}(x - 2) + 1$, or $y = -\dfrac{1}{2}x + 2$.

Graphical support:



$[-3, 5]$ by $[-2, 4]$

**33.** $\dfrac{dP}{dV} = \dfrac{d}{dV}\left(\dfrac{nRT}{V-nb} - \dfrac{an^2}{V^2}\right)$

$= \dfrac{(V-nb)\frac{d}{dV}(nRT) - (nRT)\frac{d}{dV}(V-nb)}{(V-nb)^2} - \dfrac{d}{dV}(an^2V^{-2})$
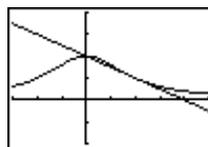
$= \dfrac{0-nRT}{(V-nb)^2} + 2an^2V^{-3}$

$= -\dfrac{nRT}{(V-nb)^2} + \dfrac{2an^2}{V^3}$

**34.** $\dfrac{ds}{dt} = \dfrac{d}{dt}(4.9t^2) = 9.8t$

$\dfrac{d^2s}{dt^2} = \dfrac{d}{dt}(9.8t) = 9.8$

**35.** $\dfrac{dR}{dM} = \dfrac{d}{dM}\left[M^2\left(\dfrac{C}{2} - \dfrac{M}{3}\right)\right]$

$= \dfrac{d}{dM}\left(\dfrac{C}{2}M^2 - \dfrac{1}{3}M^3\right)$

$= CM - M^2$

**36.** If the radius of a circle is changed by a very small amount $\Delta r$, the change in the area can be thought of as a very thin strip with length given by the circumference, $2\pi r$, and width $\Delta r$. Therefore, the change in the area can be thought of as $(2\pi r)(\Delta r)$, which means that the change in the area divided by the change in the radius is just $2\pi r$.

**37.** If the radius of a sphere is changed by a very small amount $\Delta r$, the change in the volume can be thought of as a very thin layer with an area given by the surface area, $4\pi r^2$, and a thickness given by $\Delta r$. Therefore, the change in the volume can be thought of as $(4\pi r^2)(\Delta r)$, which means that the change in the volume divided by the change in the radius is just $4\pi r^2$.

**38.** Let $t(x)$ be the number of trees and $y(x)$ be the yield per tree $x$ years from now. Then $t(0) = 156$, $y(0) = 12$, $t'(0) = 13$, and $y'(0) = 1.5$. The rate of increase of production is

$\dfrac{d}{dx}(ty) = t(0)y'(0) + y(0)t'(0) = (156)(1.5) + (12)(13)$

$= 390$ bushels of annual production per year.

**39.** Let $m(x)$ be the number of members and $c(x)$ be the pavillion cost $x$ years from now. Then $m(0) = 65$, $c(0) = 250$, $m'(0) = 6$, and $c'(0) = 10$. The rate of change of each member's share is $\dfrac{d}{dx}\left(\dfrac{c}{m}\right) = \dfrac{m(0)c'(0) - c(0)m'(0)}{[m(0)]^2}$

$= \dfrac{(65)(10) - (250)(6)}{(65)^2} \approx -0.201$ dollars per year. Each member's share of the cost is decreasing by approximately 20 cents per year.

**40. (a)** It is insignificant in the limiting case and can be treated as zero (and removed from the expression).

**(b)** It was "rejected" because it is incomparably smaller than the other terms: $v\,du$ and $u\,dv$.

**(c)** $\dfrac{d}{dx}(uv) = v\dfrac{du}{dx} + u\dfrac{dv}{dx}$. This is equivalent to the product rule given in the text.

**(d)** Because $dx$ is 'infinitely small," and this could be thought of as dividing by zero.

**(e)** $d\left(\dfrac{u}{v}\right) = \dfrac{u+du}{v+dv} - \dfrac{u}{v}$

$= \dfrac{(u+du)(v) - (u)(v+dv)}{(v+dv)(v)}$

$= \dfrac{uv + vdu - uv - udv}{v^2 + vdv}$

$= \dfrac{vdu - udv}{v^2}$

## ■ Section 3.4 Velocity and Other Rates of Change (pp. 122–133)

### Exploration 1    Growth Rings on a Tree

**1.** Figure 3.22 is a better model, as it shows rings of equal *area* as opposed to rings of equal *width*. It is not likely that a tree could sustain increased growth year after year, although climate conditions do produce some years of greater growth than others.

**2.** Rings of equal area suggest that the tree adds approximately the same amount of wood to its girth each year. With access to approximately the same raw materials from which to make the wood each year, this is how most trees actually grow.

**3.** Since change in area is constant, so also is

$\dfrac{\text{change in area}}{2\pi}$. If we denote this latter constant by $k$, we have $\dfrac{k}{\text{change in radius}} = r$, which means that $r$ varies inversely as the change in the radius. In other words, the change in radius must get smaller when $r$ gets bigger, and vice-versa.

### Exploration 2    Modeling Horizontal Motion

**1.** The particle reverses direction at about $t = 0.61$ and $t = 2.06$.
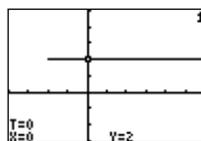


**2.** When the trace cursor is moving to the right the particle is moving to the right, and when the cursor is moving to the left the particle is moving to the left. Again we find the particle reverses direction at about $t = 0.61$ and $t = 2.06$.
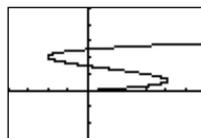
**3.** When the trace cursor is moving upward the particle is moving to the right, and when the cursor is moving downward the particle is moving to the left. Again we find the same values of $t$ for when the particle reverses direction.



**4.** We can represent the velocity by graphing the parametric equations

$x_4(t) = x_1'(t) = 12t^2 - 32t + 15$, $y_4(t) = 2$ (part 1),
$x_5(t) = x_1'(t) = 12t^2 - 32t + 15$, $y_5(t) = t$ (part 2),
$x_6(t) = t$, $y_6(t) = x_1'(t) = 12t^2 - 32t + 15$ (part 3)



$[-8, 20]$ by $[-3, 5]$
$(x_4, y_4)$



$[-8, 20]$ by $[-3, 5]$
$(x_5, y_5)$



$[-2, 5]$ by $[-10, 20]$
$(x_6, y_6)$

For $(x_4, y_4)$ and $(x_5, y_5)$, the particle is moving to the right when the $x$-coordinate of the graph (velocity) is positive, moving to the left when the $x$-coordinate of the graph (velocity) is negative, and is stopped when the $x$-coordinate of the graph (velocity) is 0. For $(x_6, y_6)$, the par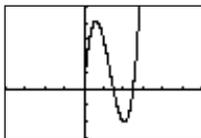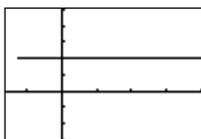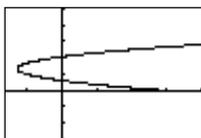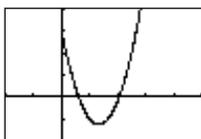ticle is moving to the right when the $y$-coordinate of the graph (velocity) is positive, moving to the left when the $y$-coordinate of the graph (velocity) is negative, and is stopped when the $y$-coordinate of the graph (velocity) is 0.

### Exploration 3   Seeing Motion on a Graphing Calculator

**1.** Let $t$Min $= 0$ and $t$Max $= 10$.

**2.** Since the rock achieves a maximum height of 400 feet, set $y$Max to be slightly greater than 400, for example $y$Max $= 420$.

**4.** The grapher proceeds with constant increments of $t$ (time), so pixels appear on the screen at regular time intervals. When the rock is moving more slowly, the pixels appear closer together. When the rock is moving faster, the pixels appear farther apart. We observe faster motion when the pixels are farther apart.

### Quick Review 3.4

**1.** The coefficient of $x^2$ is negative, so the parabola opens downward.

Graphical support:
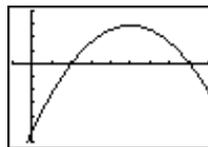


$[-1, 9]$ by $[-300, 200]$

**2.** The $y$-intercept is $f(0) = -256$.
See the solution to Exercise 1 for graphical support.

**3.** The $x$-intercepts occur when $f(x) = 0$.
$$-16x^2 + 160x - 256 = 0$$
$$-16(x^2 - 10x + 16) = 0$$
$$-16(x - 2)(x - 8) = 0$$
$$x = 2 \text{ or } x = 8$$
The $x$-intercepts are 2 and 8. See the solution to Exercise 1 for graphical support.

**4.** Since $f(x) = -16(x^2 - 10x + 16)$
$= -16(x^2 - 10x + 25 - 9) = -16(x - 5)^2 + 144$,
the range is $(-\infty, 144]$.
See the solution to Exercise 1 for graphical support.

**5.** Since $f(x) = -16(x^2 - 10x + 16)$
$= -16(x^2 - 10x + 25 - 9) = -16(x - 5)^2 + 144$,
the vertex is at $(5, 144)$. See the solution to Exercise 1 for graphical support.

**6.**
$$f(x) = 80$$
$$-16x^2 + 160x - 256 = 80$$
$$-16x^2 + 160x - 336 = 0$$
$$-16(x^2 - 10x + 21) = 0$$
$$-16(x - 3)(x - 7) = 0$$
$$x = 3 \text{ or } x = 7$$
$f(x) = 80$ at $x = 3$ and at $x = 7$.
See the solution to Exercise 1 for graphical support.

**7.**
$$\frac{dy}{dx} = 100$$
$$-32x + 160 = 100$$
$$60 = 32x$$
$$x = \frac{15}{8}$$
$\frac{dy}{dx} = 100$ at $x = \frac{15}{8}$

Graphical support: the graph of NDER $f(x)$ is shown.



$[-1, 9]$ by $[-200, 200]$

**8.**
$$\frac{dy}{dx} > 0$$
$$-32x + 160 > 0$$
$$-32x > -160$$
$$x < 5$$
$\frac{dy}{dx} > 0$ when $x < 5$.
See the solution to Exercise 7 for graphical support.

**9.** Note that $f'(x) = -32x + 160$.

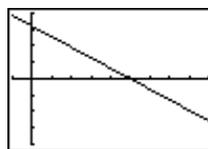$$\lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = f'(3) = -32(3) + 160 = 64$$

For graphical support, use the graph shown in the solution

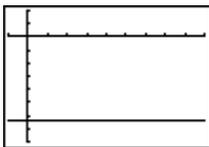to Exercise 7 and observe that NDER $(f(x), 3) \approx 64$.

**10.** $f'(x) = -32x + 160$
$f''(x) = -32$
At $x = 7$ (and, in fact, at any other of $x$),

$$\frac{d^2y}{dx^2} = -32.$$

Graphical support: the graph of NDER(NDER $f(x)$) is
shown.



$[-1, 9]$ by $[-40, 10]$

## Section 3.4 Exercises

**1.** Since $V = s^3$, the instantaneous rate of change is $\dfrac{dV}{ds} = 3s^2$.

**2. (a)** Displacement $= s(5) - s(0) = 12 - 2 = 10$ m

**(b)** Average velocity $= \dfrac{10 \text{ m}}{5 \text{ sec}} = 2$ m/sec

**(c)** Velocity $= s'(t) = 2t - 3$
At $t = 4$, velocity $= s'(4) = 2(4) - 3 = 5$ m/sec

**(d)** Acceleration $= s''(t) = 2$ m/sec$^2$

**(e)** The particle changes direction when

$$s'(t) = 2t - 3 = 0, \text{ so } t = \frac{3}{2} \text{ sec.}$$

**(f)** Since the acceleration is always positive, the position $s$

is at a minimum when the particle changes direction, at

$t = \dfrac{3}{2}$ sec. Its position at this time is $s\left(\dfrac{3}{2}\right) = -\dfrac{1}{4}$ m.

**3. (a)** Velocity: $v(t) = \dfrac{ds}{dt} = \dfrac{d}{dt}(24t - 0.8t^2) = 24 - 1.6t$ m/sec

Acceleration: $a(t) = \dfrac{dv}{dt} = \dfrac{d}{dt}(24 - 1.6t) = -1.6$ m/sec$^2$

**(b)** The rock reaches its highest point when
$v(t) = 24 - 1.6t = 0$, at $t = 15$. It took 15 seconds.

**(c)** The maximum height was $s(15) = 180$ meters.

**(d)**      $s(t) = \dfrac{1}{2}(180)$

$24t - 0.8t^2 = 90$

$0 = 0.8t^2 - 24t + 90$

$$t = \frac{24 \pm \sqrt{(-24)^2 - 4(0.8)(90)}}{2(0.8)}$$

$\approx 4.393, 25.607$

It took about 4.393 seconds to reach half its maximum

height.

**(e)**      $s(t) = 0$
$24t - 0.8t^2 = 0$
$0.8t(30 - t) = 0$
$t = 0$ or $t = 30$
The rock was aloft from $t = 0$ to $t = 30$, so it was aloft
for 30 seconds.

**4.** On Mars:

Velocity $= \dfrac{ds}{dt} = \dfrac{d}{dt}(1.86t^2) = 3.72t$

Solving $3.72t = 16.6$, the downward velocity reaches

16.6 m/sec after about 4.462 sec.

On Jupiter:

Velocity $= \dfrac{ds}{dt} = \dfrac{d}{dt}(11.44t^2) = 22.88t$

Solving $22.88t = 16.6$, the downward velocity reaches

16.6 m/sec after about 0.726 sec.

**5.** The rock reaches its maximum height when the velocity
$s'(t) = 24 - 9.8t = 0$, at $t \approx 2.449$. Its maximum height is
about $s(2.449) \approx 29.388$ meters.

**6.** Moon:
$$s(t) = 0$$
$832t - 2.6t^2 = 0$
$2.6t(320 - t) = 0$
$t = 0$ or $t = 320$
It takes 320 seconds to return.
Earth:
$$s(t) = 0$$
$832t - 16t^2 = 0$
$16t(52 - t) = 0$
$t = 0$ or $t = 52$
It takes 52 seconds to return.

**7.** The following is one way to simulate the problem situation.
For the moon:
$x_1(t) = 3(t < 160) + 3.1(t \geq 160)$
$y_1(t) = 832t - 2.6t^2$
$t$-values: 0 to 320
window: $[0, 6]$ by $[-10,000, 70,000]$
For the earth:
$x_1(t) = 3(t < 26) + 3.1(t \geq 26)$
$y_1(t) = 832t - 16t^2$
$t$-values: 0 to 52
window: $[0, 6]$ by $[-1000, 11,000]$

**8.** The growth rate is given by
$b'(t) = 10^4 - 2 \cdot 10^3 t = 10,000 - 2000t$.
At $t = 0$: $b'(0) = 10,000$ bacteria/hour
At $t = 5$: $b'(5) = 0$ bacteria/hour
At $t = 10$: $b'(10) = -10,000$ bacteria/hour

**9.** $Q(t) = 200(30 - t)^2 = 200(900 - 60t + t^2)$

$= 180{,}000 - 12{,}000t + 200t^2$

$Q'(t) = -12{,}000 + 400t$

The rate of change of the amount of water in the tank after 10 minutes is $Q'(10) = -8000$ gallons per minute.

Note that $Q'(10) < 0$, so the rate at which the water is running *out* is positive. The water is running out at the rate of 8000 gallons per minute.

The average rate for the first 10 minutes is

$\dfrac{Q(10) - Q(0)}{10 - 0} = \dfrac{80{,}000 - 180{,}000}{10} = -10{,}000$ gallons per minute.

The water is flowing out at an average rate of 10,000 gallons per minute over the first 10 min.

**10. (a)** Average cost $= \dfrac{c(100)}{100} = \dfrac{11{,}000}{100} = \$110$ per machine

**(b)** $c'(x) = 100 - 0.2x$
Marginal cost $= c'(100) = \$80$ per machine

**(c)** Actual cost of 101st machine is
$c(101) - c(100) = \$79.90$, which is very close to the marginal cost calculated in part (b).

**11. (a)**
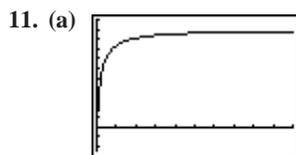


[0, 50] by [−500, 2200]

The values of $x$ which make sense are the whole numbers, $x \geq 0$.

**(b)** Marginal revenue $= r'(x) = \dfrac{d}{dx}\left[2000\left(1 - \dfrac{1}{x+1}\right)\right]$

$= \dfrac{d}{dx}\left(2000 - \dfrac{2000}{x+1}\right)$

$= 0 - \dfrac{(x+1)(0) - (2000)(1)}{(x+1)^2} = \dfrac{2000}{(x+1)^2}$

**(c)** $r'(5) = \dfrac{2000}{(5+1)^2} = \dfrac{2000}{36} \approx 55.56$

The increase in revenue is approximately \$55.56.

**(d)** The limit is 0. This means that as $x$ gets large, one reaches a point where very little extra revenue can be expected from selling more desks.

**12.** $v(t) = s'(t) = 3t^2 - 12t + 9$
$a(t) = v'(t) = 6t - 12$
Find when velocity is zero.
$3t^2 - 12t + 9 = 0$
$3(t^2 - 4t + 3) = 0$
$3(t - 1)(t - 3) = 0$
$t = 1$ or $t = 3$
At $t = 1$, the acceleration is $a(1) = -6$ m/sec$^2$
At $t = 3$, the acceleration is $a(3) = 6$ m/sec$^2$

**13.** $a(t) = v'(t) = 6t^2 - 18t + 12$
Find when acceleration is zero.
$6t^2 - 18t + 12 = 0$
$6(t^2 - 3t + 2) = 0$
$6(t - 1)(t - 2) = 0$
$t = 1$ or $t = 2$
At $t = 1$, the speed is $|v(1)| = |0| = 0$ m/sec.
At $t = 2$, the speed is $|v(2)| = |-1| = 1$ m/sec.

**14. (a)** $g'(x) = \dfrac{d}{dx}(x^3) = 3x^2$

$h'(x) = \dfrac{d}{dx}(x^3 - 2) = 3x^2$

$t'(x) = \dfrac{d}{dx}(x^3 + 3) = 3x^2$

**(b)** The graphs of NDER $g(x)$, NDER $h(x)$, and NDER $t(x)$ are all the same, as shown.
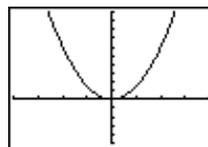


[−4, 4] by [−10, 20]

**(c)** $f(x)$ must be of the form $f(x) = x^3 + c$, where $c$ is a constant.

**(d)** Yes. $f(x) = x^3$

**(e)** Yes. $f(x) = x^3 + 3$

**15. (a)**
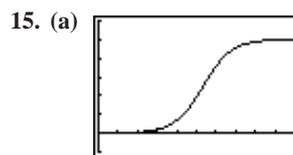


[0, 200] by [−2, 12]

**(b)** The values of $x$ which make sense are the whole numbers, $x \geq 0$.

**(c)**
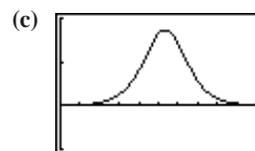


[0, 200] by [−0.1, 0.2]

$P$ is most sensitive to changes in $x$ when $|P'(x)|$ is largest. It is relatively sensitive to changes in $x$ between approximately $x = 60$ and $x = 160$.

**(d)** The marginal profit, $P'(x)$, is greatest at $x \approx 106.44$. Since $x$ must be an integer,
$P(106) \approx 4.924$ thousand dollars or \$4924.

**(e)** $P'(50) \approx 0.013$, or \$13 per package sold
$P'(100) \approx 0.165$, or \$165 per package sold
$P'(125) \approx 0.118$, or \$118 per package sold
$P'(150) \approx 0.031$, or \$31 per package sold
$P'(175) \approx 0.006$, or \$6 per package sold
$P'(300) \approx 10^{-6}$, or \$0.001 per package sold

**(f)** The limit is 10. The maximum possible profit is \$10,000 monthly.

**(g)** Yes. In order to sell more and more packages, the company might need to lower the price to a point where they won't make any additional profit.

**16. (a)** 190 ft/sec

**(b)** 2 seconds

**(c)** After 8 seconds, and its velocity was 0 ft/sec then

**(d)** After about 11 seconds, and it was falling 90 ft/sec then

**(e)** About 3 seconds (from the rocket's highest point)

**(f)** The acceleration was greatest just before the engine stopped. The acceleration was constant from $t = 2$ to $t = 11$, while the rocket was in free fall.

**17.** Note that "downward velocity" is positive when McCarthy is falling downward. His downward velocity increases steadily until the parachute opens, and then decreases to a constant downward velocity. One possible sketch:
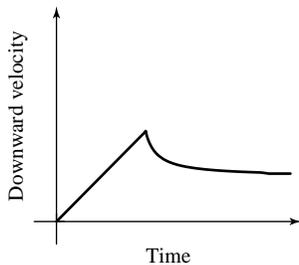


**18. (a)** We estimate the slopes at several points as follows, then connect the points to create a smooth curve.

| $t$ (days) | 0 | 10 | 20 | 30 | 40 | 50 |
|---|---|---|---|---|---|---|
| Slope (flies/day) | 0.5 | 3.0 | 13.0 | 14.0 | 3.5 | 0.5 |
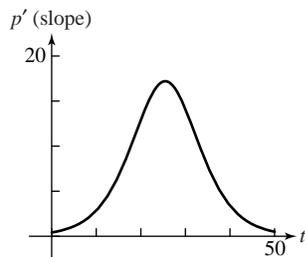


Horizontal axis: Days
Vertical axis: Flies per day

**(b)** Fastest: Around the 25th day
Slowest: Day 50 or day 0

**19.** The particle is at (5, 2) when $4t^3 - 16t^2 + 15t = 5$, which occurs at $t \approx 2.83$.

**20.** The motion can be simulated in parametric mode using $x_1(t) = 2t^3 - 13t^2 + 22t - 5$ and $y_1(t) = 2$ in $[-6, 8]$ by $[-3, 5]$.

**(a)** It begins at the point $(-5, 2)$ moving in the positive direction. After a little more than one second, it has moved a bit past (6, 2) and it turns back in the negative direction for approximately 2 seconds. At the end of that time, it is near $(-2, 2)$ and it turns back again in the positive direction. After that, it continues moving in the positive direction indefinitely, speeding up as it goes.

**(b)** The particle speeds up when its *speed* is increasing, which occurs during the approximate intervals $1.153 \leq t \leq 2.167$ and $t \geq 3.180$. It slows down during the approximate intervals $0 \leq t \leq 1.153$ and $2.167 \leq t \leq 3.180$. One way to determine the endpoints of these intervals is to use a grapher to find the minimums and maximums for the speed, $|\text{NDER } x(t)|$, using function mode in the window $[0, 5]$ by $[0, 10]$.

**(c)** The particle changes direction at $t \approx 1.153$ sec and at $t \approx 3.180$ sec.

**(d)** The particle is at rest "instantaneously" at $t \approx 1.153$ sec and at $t \approx 3.180$ sec.

**(e)** The velocity starts out positive but decreasing, it becomes negative, then starts to increase, and becomes positive again and continues to increase.
The speed is decreasing, reaches 0 at $t \approx 1.15$ sec, then increases until $t \approx 2.17$ sec, decreases until $t \approx 3.18$ sec when it is 0 again, and then increases after that.

**(f)** The particle is at (5, 2) when $2t^3 - 13t^2 + 22t - 5 = 5$ at $t \approx 0.745$ sec, $t \approx 1.626$ sec, and at $t \approx 4.129$ sec.

**21. (a)**



**(b)** $s'(1) = 18$, $s'(2.5) = 0$, $s'(3.5) = -12$

**22. (a)**
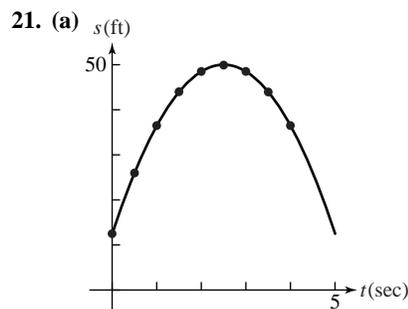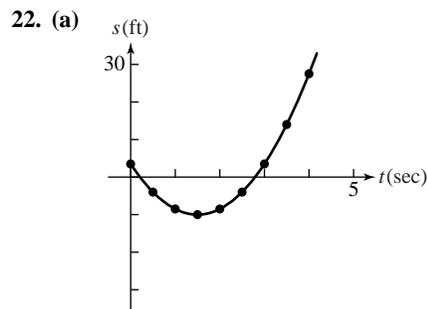


**(b)** $s'(1) = -6$, $s'(2.5) = 12$, $s'(3.5) = 24$

**23. (a)** The body reverses direction when $v$ changes sign, at $t = 2$ and at $t = 7$.

**(b)** The body is moving at a constant speed, $|v| = 3$ m/sec, between $t = 3$ and $t = 6$.

**(c)** The speed graph is obtained by reflecting the negative portion of the velocity graph, $2 < t < 7$, about the $x$-axis.

Speed(m/sec)
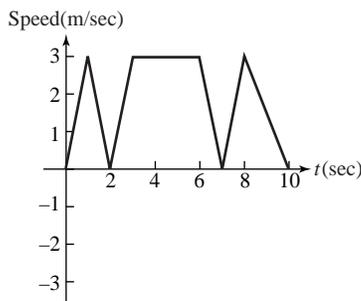


**(d)** For $0 \leq t < 1$: $a = \dfrac{3 - 0}{1 - 0} = 3$ m/sec$^2$

For $1 < t < 3$: $a = \dfrac{-3 - 3}{3 - 1} = -3$ m/sec$^2$

For $3 < t < 6$: $a = \dfrac{-3 - (-3)}{6 - 3} = 0$ m/sec$^2$

For $6 < t < 8$: $a = \dfrac{3 - (-3)}{8 - 6} = 3$ m/sec$^2$

For $8 < t \leq 10$: $a = \dfrac{0 - 3}{10 - 8} = -1.5$ m/sec$^2$

Acceleration (m/sec$^2$)
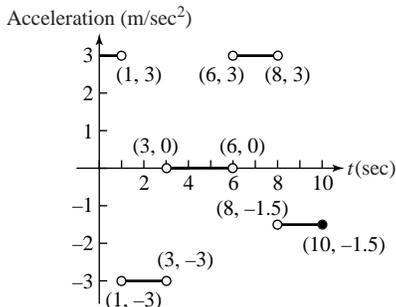


**24. (a)** The particle is moving left when the graph of $s$ has negative slope, for $2 < t < 3$ and for $5 < t \leq 6$.
The particle is moving right when the graph of $s$ has positive slope, for $0 \leq t < 1$.
The particle is standing still when the graph of $s$ is horizontal, for $1 < t < 2$ and for $3 < t < 5$.

**(b)** For $0 \leq t < 1$: $v = \dfrac{2 - 0}{1 - 0} = 2$ cm/sec

Speed $= |v| = 2$ cm/sec

For $1 < t < 2$: $v = \dfrac{2 - 2}{2 - 1} = 0$ cm/sec

Speed $= |v| = 0$ cm/sec

For $2 < t < 3$: $v = \dfrac{-2 - 2}{3 - 2} = -4$ cm/sec

Speed $= |v| = 4$ cm/sec

For $3 < t < 5$: $v = \dfrac{-2 - (-2)}{5 - 3} = 0$ cm/sec

Speed $= |v| = 0$ cm/sec

For $5 < t \leq 6$: $v = \dfrac{-4 - (-2)}{6 - 5} = -2$ cm/sec

Speed $= |v| = 2$ cm/sec

Velocity graph:

$v(t)$ (cm/sec)



Speed graph:

$|v(t)|$ (cm/sec)
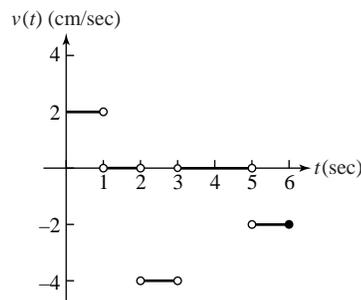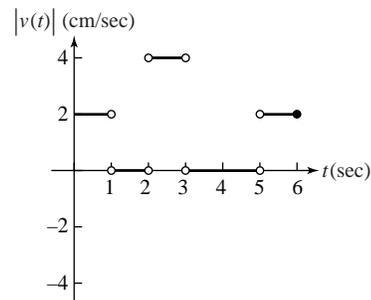


**25. (a)** The particle moves forward when $v > 0$, for $0 \leq t < 1$ and for $5 < t < 7$.
The particle moves backward when $v < 0$, for $1 < t < 5$.
The particle speeds up when $v$ is negative and decreasing, for $1 < t < 2$, and when $v$ is positive and increasing, for $5 < t < 6$.
The particle slows down when $v$ is positive and decreasing, for $0 \leq t < 1$ and for $6 < t < 7$, and when $v$ is negative and increasing, for $3 < t < 5$.

**(b)** Note that the acceleration $a = \dfrac{dv}{dt}$ is undefined at $t = 2$, $t = 3$, and $t = 6$.

The acceleration is positive when $v$ is increasing, for $3 < t < 6$.
The acceleration is negative when $v$ is decreasing, for $0 \leq t < 2$ and for $6 < t < 7$.
The acceleration is zero when $v$ is constant, for $2 < t < 3$ and for $7 < t \leq 9$.

**(c)** The particle moves at its greatest speed when $|v|$ is maximized, at $t = 0$ and for $2 < t < 3$.

**(d)** The particle stands still for more than an instant when $v$ stays at zero, for $7 < t \leq 9$.

**26. (a)** To graph the velocity, we estimate the slopes at several points as follows, then connect the points to create a smooth curve.

| $t$ (hours) | 0 | 2.5 | 5 | 7.5 | 10 | 12.5 | 15 |
|---|---|---|---|---|---|---|---|
| $v$ (km/hour) | 0 | 56 | 75 | 56 | 0 | −94 | −225 |
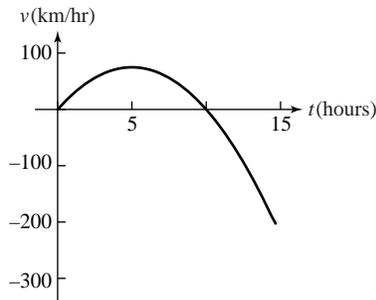


To graph the acceleration, we estimate the slope of the velocity graph at several points as follows, and then connect the points to create a smooth curve.

| $t$ (hours) | 0 | 2.5 | 5 | 7.5 | 10 | 12.5 | 15 |
|---|---|---|---|---|---|---|---|
| $v$ (km/hour$^2$) | 30 | 15 | 0 | −15 | −30 | −45 | −60 |
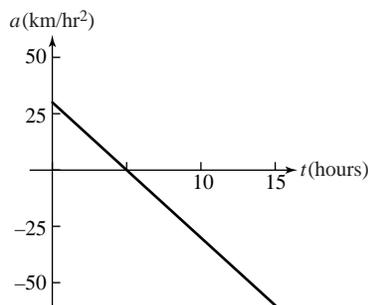


**(b)** $\dfrac{ds}{dt} = 30t - 3t^2$
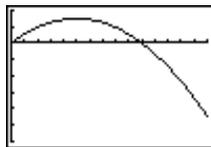


[0, 15] by [−300, 100]

$\dfrac{d^2s}{dt^2} = 30 - 6t$
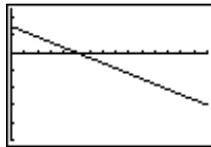


[0, 15] by [−100, 50]

The graphs are very similar.

**27. (a)** Solving $160 = 490t^2$ gives $t = \pm\dfrac{4}{7}$.

It took $\dfrac{4}{7}$ of a second. The average velocity was $\dfrac{160 \text{ cm}}{\left(\dfrac{4}{7}\right)}$ sec $= 280$ cm/sec.

**(b)** $v = s'(t) = 980t$

$a = s''(t) = 980$

The velocity was $s'\left(\dfrac{4}{7}\right) = 560$ cm/sec.

The acceleration was $s''\left(\dfrac{4}{7}\right) = 980$ cm/sec$^2$.

**(c)** Since there were about 16 flashes during $\dfrac{4}{7}$ of a second, the light was flashing at a rate of about 28 flashes per second.

**28.** Graph C is position, graph A is velocity, and graph B is acceleration.
A is the derivative of C because it is positive, negative, and zero where C is increasing, decreasing, and has horizontal tangents, respectively. The relationship between B and A is similar.

**29.** Graph C is position, graph B is velocity, and graph A is acceleration.
B is the derivative of C because it is negative and zero where C is decreasing and has horizontal tangents, respectively.
A is the derivative of B because it is positive, negative, and zero where B is increasing, decreasing, and has horizontal tangents, respectively.

**30. (a)** $\dfrac{dy}{dt} = \dfrac{d}{dt}\left[6\left(1 - \dfrac{t}{12}\right)^2\right]$

$= \dfrac{d}{dt}\left[6\left(1 - \dfrac{t}{6} + \dfrac{t^2}{144}\right)\right]$

$= \dfrac{d}{dt}\left(6 - t + \dfrac{1}{24}t^2\right)$

$= 0 - 1 + \dfrac{t}{12} = \dfrac{t}{12} - 1$

**(b)** The fluid level is falling fastest when $\dfrac{dy}{dt}$ is the most negative, at $t = 0$, when $\dfrac{dy}{dt} = -1$. The fluid level is falling slowest at $t = 12$, when $\dfrac{dy}{dt} = 0$.

**(c)**
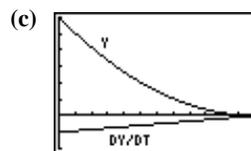


[0, 12] by [−2, 6]

$y$ is decreasing and $\dfrac{dy}{dt}$ is negative over the entire interval. $y$ decreases more rapidly early in the interval, and the magnitude of $\dfrac{dy}{dt}$ is larger then. $\dfrac{dy}{dt}$ is 0 at $t = 12$, where the graph of $y$ seems to have a horizontal tangent.

**31. (a)** $\dfrac{dV}{dr} = \dfrac{d}{dr}\left(\dfrac{4}{3}\pi r^3\right) = 4\pi r^2$

When $r = 2$, $\dfrac{dV}{dr} = 4\pi(2)^2 = 16\pi$ cubic feet of

volume per foot of radius.

**(b)** The increase in the volume is

$\dfrac{4}{3}\pi(2.2)^3 - \dfrac{4}{3}\pi(2)^3 \approx 11.092$ cubic feet.

**32.** For $t > 0$, the speed of the aircraft in meters per second

after $t$ seconds is $\dfrac{20}{9}t$. Multiplying by $\dfrac{3600 \text{ sec}}{1 \text{ h}} \cdot \dfrac{1 \text{ km}}{1000 \text{ m}}$,

we find that this is equivalent to $8t$ kilometers per hour.

Solving $8t = 200$ gives $t = 25$ seconds. The aircraft takes

25 seconds to become airborne, and the distance it travels

during this time is $D(25) \approx 694.444$ meters.

**33.** Let $v_0$ be the exit velocity of a particle of lava. Then

$s(t) = v_0 t - 16t^2$ feet, so the velocity is $\dfrac{ds}{dt} = v_0 - 32t$.

Solving $\dfrac{ds}{dt} = 0$ gives $t = \dfrac{v_0}{32}$. Then the maximum height, in

feet, is $s\left(\dfrac{v_0}{32}\right) = v_0\left(\dfrac{v_0}{32}\right) - 16\left(\dfrac{v_0}{32}\right)^2 = \dfrac{v_0^2}{64}$. Solving

$\dfrac{v_0^2}{64} = 1900$ gives $v_0 \approx \pm 348.712$. The exit velocity was

about 348.712 ft/sec. Multiplying by $\dfrac{3600 \text{ sec}}{1 \text{ h}} \cdot \dfrac{1 \text{ mi}}{5280 \text{ ft}}$, we

find that this is equivalent to about 237.758 mi/h.

**34.** By estimating the slope of the velocity graph at that point.

**35.** Since profit = revenue − cost, the Sum and Difference

Rule gives $\dfrac{d}{dx}(\text{profit}) = \dfrac{d}{dx}(\text{revenue}) - \dfrac{d}{dx}(\text{cost})$, where $x$ is

the number of units produced. This means that marginal

profit = marginal revenue − marginal cost.

**36. (a)** It takes 135 seconds.

**(b)** Average speed $= \dfrac{\Delta F}{\Delta t} = \dfrac{5 - 0}{73 - 0} = \dfrac{5}{73}$

$\approx 0.068$ furlongs/sec.

**(c)** Using a symmetric difference quotient, the horse's

speed is approximately

$\dfrac{\Delta F}{\Delta t} = \dfrac{4 - 2}{59 - 33} = \dfrac{2}{26} = \dfrac{1}{13} \approx 0.077$ furlongs/sec.

**(d)** The horse is running the fastest during the last furlong
(between 9th and 10th furlong markers). This furlong
takes only 11 seconds to run, which is the least amount
of time for a furlong.

**(e)** The horse accelerates the fastest during the first furlong
(between markers 0 and 1).

**37. (a)** Assume that $f$ is even. Then,

$\begin{aligned}
f'(-x) &= \lim_{h \to 0} \dfrac{f(-x + h) - f(-x)}{h} \\
&= \lim_{h \to 0} \dfrac{f(x - h) - f(x)}{h}, \text{ and substituting } k = -h, \\
&= \lim_{k \to 0} \dfrac{f(x + k) - f(x)}{-k} \\
&= -\lim_{k \to 0} \dfrac{f(x + k) - f(x)}{k} = -f'(x)
\end{aligned}$

So, $f'$ is an odd function.

**(b)** Assume that $f$ is odd. Then,

$\begin{aligned}
f'(-x) &= \lim_{h \to 0} \dfrac{f(-x + h) - f(-x)}{h} \\
&= \lim_{h \to 0} \dfrac{-f(x - h) + f(x)}{h},
\end{aligned}$

and substituting $k = -h$,

$\begin{aligned}
&= \lim_{k \to 0} \dfrac{-f(x + k) + f(x)}{-k} \\
&= \lim_{k \to 0} \dfrac{f(x + k) - f(x)}{k} = f'(x)
\end{aligned}$

So, $f'$ is an even function.

**38.** $\dfrac{d}{dx}(fgh) = \dfrac{d}{dx}[f(gh)] = f \cdot \dfrac{d}{dx}(gh) + gh \cdot \dfrac{d}{dx}(f)$

$= f\left(g \cdot \dfrac{dh}{dx} + h \cdot \dfrac{dg}{dx}\right) + gh \cdot \dfrac{df}{dx}$

$= \left(\dfrac{df}{dx}\right)gh + f\left(\dfrac{dg}{dx}\right)h + fg\left(\dfrac{dh}{dx}\right)$

# ■ Section 3.5 Derivatives of Trigonometric Functions (pp. 134–141)

### Exploration 1   Making a Conjecture with NDER

**1.** When the graph of sin $x$ is increasing, the graph of
NDER (sin $x$) is positive (above the $x$-axis).

**2.** When the graph of sin $x$ is decreasing, the graph of
NDER (sin $x$) is negative (below the $x$-axis).

**3.** When the graph of sin $x$ stops increasing and starts
decreasing, the graph of NDER (sin $x$) crosses the $x$-axis
from above to below.

**4.** The slope of the graph of sin $x$ matches the value of
NDER (sin $x$) at these points.

**5.** We conjecture that NDER (sin $x$) = cos $x$. The graphs
coincide, supporting our conjecture.



$[-2\pi, 2\pi]$ by $[-4, 4]$

**6.** When the graph of cos $x$ is increasing, the graph of NDER (cos $x$) is positive (above the $x$-axis).
When the graph of cos $x$ is decreasing, the graph of NDER (cos $x$) is negative (below the $x$-axis).
When the graph of cos $x$ stops increasing and starts decreasing, the graph of NDER (cos $x$) crosses the $x$-axis from above to below.
The slope of the graph of cos $x$ matches the value of NDER (cos $x$) at these points.
We conjecture that NDER (cos $x$) = $-\sin x$. The graphs coincide, supporting our conjecture.
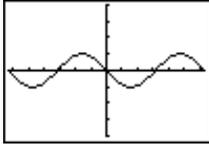


$[-2\pi, 2\pi]$ by $[-4, 4]$

## Quick Review 3.5

**1.** $135° \cdot \dfrac{\pi}{180°} = \dfrac{3\pi}{4} \approx 2.356$

**2.** $1.7 \cdot \dfrac{180°}{\pi} = \left(\dfrac{306}{\pi}\right)° \approx 97.403°$

**3.** $\sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$

**4.** Domain: All reals
Range: $[-1, 1]$

**5.** Domain: $x \neq \dfrac{k\pi}{2}$ for odd integers $k$

Range: All reals

**6.** $\cos a = \pm\sqrt{1 - \sin^2 a} = \pm\sqrt{1 - (-1)^2} = \pm\sqrt{0} = 0$

**7.** If $\tan a = -1$, then $a = \dfrac{3\pi}{4} + k\pi$ for some integer $k$, so

$\sin a = \pm\dfrac{1}{\sqrt{2}}$.

**8.** $\dfrac{1 - \cos h}{h} = \dfrac{(1 - \cos h)(1 + \cos h)}{h(1 + \cos h)} = \dfrac{1 - \cos^2 h}{h(1 + \cos h)}$

$= \dfrac{\sin^2 h}{h(1 + \cos h)}$

**9.** $y'(x) = 6x^2 - 14x$
$y'(3) = 12$
The tangent line has slope 12 and passes through (3, 1), so its equation is $y = 12(x - 3) + 1$, or $y = 12x - 35$.

**10.** $a(t) = v'(t) = 6t^2 - 14t$
$a(3) = 12$

## Section 3.5 Exercises

**1.** $\dfrac{d}{dx}(1 + x - \cos x) = 0 + 1 -(-\sin x) = 1 + \sin x$

**2.** $\dfrac{d}{dx}(2 \sin x - \tan x) = 2 \cos x - \sec^2 x$

**3.** $\dfrac{d}{dx}\left(\dfrac{1}{x} + 5 \sin x\right) = -\dfrac{1}{x^2} + 5 \cos x$

**4.** $\dfrac{d}{dx}(x \sec x) = x\dfrac{d}{dx}(\sec x) + \sec x\dfrac{d}{dx}(x)$

$= x \sec x \tan x + \sec x$

**5.** $\dfrac{d}{dx}(4 - x^2 \sin x) = \dfrac{d}{dx}(4) - \left[x^2\dfrac{d}{dx}(\sin x) + (\sin x)\dfrac{d}{dx}(x^2)\right]$

$= 0 - [x^2 \cos x + (\sin x)(2x)]$

$= -x^2 \cos x - 2x \sin x$

**6.** $\dfrac{d}{dx}(3x + x \tan x) = \dfrac{d}{dx}(3x) + \left[x\dfrac{d}{dx}(\tan x) + (\tan x)\dfrac{d}{dx}(x)\right]$

$= 3 + x \sec^2 x + \tan x$

**7.** $\dfrac{d}{dx}\left(\dfrac{4}{\cos x}\right) = \dfrac{d}{dx}(4 \sec x) = 4 \sec x \tan x$

**8.** $\dfrac{d}{dx}\dfrac{x}{1 + \cos x} = \dfrac{(1 + \cos x)\dfrac{d}{dx}(x) - x\dfrac{d}{dx}(1 + \cos x)}{(1 + \cos x)^2}$

$= \dfrac{1 + \cos x + x \sin x}{(1 + \cos x)^2}$

**9.** $\dfrac{d}{dx}\dfrac{\cot x}{1 + \cot x} = \dfrac{(1 + \cot x)\dfrac{d}{dx}(\cot x) - (\cot x)\dfrac{d}{dx}(1 + \cot x)}{(1 + \cot x)^2}$

$= \dfrac{(1 + \cot x)(-\csc^2 x) - (\cot x)(-\csc^2 x)}{(1 + \cot x)^2}$

$= -\dfrac{\csc^2 x}{(1 + \cot x)^2} = -\dfrac{\csc^2 x \sin^2 x}{(1 + \cot x)^2 \sin^2 x} = -\dfrac{1}{(\sin x + \cos x)^2}$

**10.** $\dfrac{d}{dx}\dfrac{\cos x}{1 + \sin x} = \dfrac{(1 + \sin x)\dfrac{d}{dx}(\cos x) - (\cos x)\dfrac{d}{dx}(1 + \sin x)}{(1 + \sin x)^2}$

$= \dfrac{(1 + \sin x)(-\sin x) - (\cos x)(\cos x)}{(1 + \sin x)^2}$

$= \dfrac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2}$

$= \dfrac{-(1 + \sin x)}{(1 + \sin x)^2}$

$= -\dfrac{1}{1 + \sin x}$

**11.** $y'(x) = \dfrac{d}{dx}(\sin x + 3) = \cos x$

$y'(\pi) = \cos \pi = -1$

The tangent line has slope $-1$ and passes through

$(\pi, \sin \pi + 3) = (\pi, 3)$.

Its equation is $y = -1(x - \pi) + 3$, or $y = -x + \pi + 3$.

**12.** $y'(x) = \dfrac{d}{dx}\dfrac{\tan x}{x} = \dfrac{x\frac{d}{dx}(\tan x) - \tan x\frac{d}{dx}(x)}{x^2}$

$= \dfrac{x\sec^2 x - \tan x}{x^2}$

$y'\left(\dfrac{\pi}{4}\right) = \dfrac{\frac{\pi}{4}\cdot(\sqrt{2})^2 - 1}{\left(\frac{\pi}{4}\right)^2} = \dfrac{8\pi - 16}{\pi^2}$

The normal line has slope $-\dfrac{\pi^2}{8\pi - 16} = \dfrac{\pi^2}{16 - 8\pi}$

and passes through $\left(\dfrac{\pi}{4}, \dfrac{\tan\left(\frac{\pi}{4}\right)}{\frac{\pi}{4}}\right) = \left(\dfrac{\pi}{4}, \dfrac{4}{\pi}\right)$.

Its equation is $y = \dfrac{\pi^2}{16 - 8\pi}\left(x - \dfrac{\pi}{4}\right) + \dfrac{4}{\pi}$, or

$y = \dfrac{\pi^2}{16 - 8\pi}x - \dfrac{\pi^3}{64 - 32\pi} + \dfrac{4}{\pi}$.

Using decimals, this equation is approximately

$y = -1.081x + 2.122$.

**13.** $y'(x) = \dfrac{d}{dx}(x^2 \sin x) = x^2\dfrac{d}{dx}(\sin x) + (\sin x)\dfrac{d}{dx}(x^2)$

$= x^2 \cos x + 2x \sin x$

$y'(3) = 9\cos 3 + 6\sin 3$

The tangent line has slope $9\cos 3 + 6\sin 3$ and passes
through $(3, 9\sin 3)$. Its equation is
$y = (9\cos 3 + 6\sin 3)(x - 3) + 9\sin 3$, or
$y = (9\cos 3 + 6\sin 3)x - 27\cos 3 - 9\sin 3$. Using
decimals, this equation is approximately
$y = -8.063x + 25.460$.

**14.** $\lim\limits_{h \to 0} \dfrac{\cos(x+h) - \cos x}{h}$

$= \lim\limits_{h \to 0} \dfrac{(\cos x \cos h - \sin x \sin h) - \cos x}{h}$

$= \lim\limits_{h \to 0} \dfrac{\cos x(\cos h - 1) - \sin x \sin h}{h}$

$= \lim\limits_{h \to 0}\left((\cos x)\dfrac{\cos h - 1}{h} - (\sin x)\dfrac{\sin h}{h}\right)$

$= (\cos x)\left(\lim\limits_{h \to 0}\dfrac{\cos h - 1}{h}\right) - (\sin x)\left(\lim\limits_{h \to 0}\dfrac{\sin h}{h}\right)$

$= (\cos x)(0) - (\sin x)(1) = -\sin x$

**15. (a)** $\dfrac{d}{dx}\tan x = \dfrac{d}{dx}\dfrac{\sin x}{\cos x} = \dfrac{(\cos x)\frac{d}{dx}(\sin x) - (\sin x)\frac{d}{dx}(\cos x)}{(\cos x)^2}$

$= \dfrac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x}$

$= \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x} = \sec^2 x$

**(b)** $\dfrac{d}{dx}\sec x = \dfrac{d}{dx}\dfrac{1}{\cos x} = \dfrac{(\cos x)\frac{d}{dx}(1) - (1)\frac{d}{dx}(\cos x)}{(\cos x)^2}$

$= \dfrac{(\cos x)(0) - (1)(-\sin x)}{\cos^2 x}$

$= \dfrac{\sin x}{\cos^2 x} = \sec x \tan x$

**16. (a)** $\dfrac{d}{dx}\cot x = \dfrac{d}{dx}\dfrac{\cos x}{\sin x}$

$= \dfrac{(\sin x)\frac{d}{dx}(\cos x) - (\cos x)\frac{d}{dx}(\sin x)}{(\sin x)^2}$

$= \dfrac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x}$

$= \dfrac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$

$= -\dfrac{1}{\sin^2 x} = -\csc^2 x$

**(b)** $\dfrac{d}{dx}\csc x = \dfrac{d}{dx}\dfrac{1}{\sin x}$

$= \dfrac{(\sin x)\frac{d}{dx}(1) - (1)\frac{d}{dx}(\sin x)}{(\sin x)^2}$

$= \dfrac{(\sin x)(0) - (1)(\cos x)}{\sin^2 x}$

$= -\dfrac{\cos x}{\sin^2 x} = -\csc x \cot x$

**17.** $\dfrac{d}{dx}\sec x = \sec x \tan x$ which is 0 at $x = 0$, so the slope of
the tangent line is 0. $\dfrac{d}{dx}\cos x = -\sin x$ which is 0 at $x = 0$,
so the slope of the tangent line is 0.

**18.** $\dfrac{d}{dx}\tan x = \sec^2 x = \dfrac{1}{\cos^2 x}$, which is never 0.
$\dfrac{d}{dx}\cot x = -\csc^2 x = -\dfrac{1}{\sin^2 x}$, which is never 0.

**19.** $y'(x) = \dfrac{d}{dx}(\sqrt{2}\cos x) = -\sqrt{2}\sin x$

$y'\left(\dfrac{\pi}{4}\right) = -\sqrt{2}\sin\dfrac{\pi}{4} = -\sqrt{2}\left(\dfrac{1}{\sqrt{2}}\right) = -1$

The tangent line has slope $-1$ and passes through

$\left(\dfrac{\pi}{4}, \sqrt{2}\cos\dfrac{\pi}{4}\right) = \left(\dfrac{\pi}{4}, 1\right)$, so its equation is

$y = -1\left(x - \dfrac{\pi}{4}\right) + 1$, or $y = -x + \dfrac{\pi}{4} + 1$.

The normal line has slope 1 and passes through $\left(\dfrac{\pi}{4}, 1\right)$, so

its equation is $y = 1\left(x - \dfrac{\pi}{4}\right) + 1$, or $y = x + 1 - \dfrac{\pi}{4}$.

**20.** $y'(x) = \dfrac{d}{dx}\tan x = \sec^2 x$

$y'(x) = \dfrac{d}{dx}(2x) = 2$

$\sec^2 x = 2$
$\sec x = \pm\sqrt{2}$

$\cos x = \pm\dfrac{1}{\sqrt{2}}$

On $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$, the solutions are $x = \pm\dfrac{\pi}{4}$. The points on the

curve are $\left(-\dfrac{\pi}{4}, -1\right)$ and $\left(\dfrac{\pi}{4}, 1\right)$.

**21.** $y'(x) = \dfrac{d}{dx}(4 + \cot x - 2\csc x)$

$\qquad = 0 - \csc^2 x + 2\csc x \cot x$

$\qquad = -\csc^2 x + 2\csc x \cot x$

**(a)** $y'\left(\dfrac{\pi}{2}\right) = -\csc^2 \dfrac{\pi}{2} + 2\csc \dfrac{\pi}{2} \cot \dfrac{\pi}{2}$

$\qquad = -1^2 + 2(1)(0) = -1$

The tangent line has slope $-1$ and passes through

$P\left(\dfrac{\pi}{2}, 2\right)$. Its equation is $y = -1\left(x - \dfrac{\pi}{2}\right) + 2$, or

$y = -x + \dfrac{\pi}{2} + 2$.

**(b)** $\qquad\qquad\qquad y'(x) = 0$

$-\csc^2 x + 2\csc x \cot x = 0$

$-\dfrac{1}{\sin^2 x} + \dfrac{2\cos x}{\sin^2 x} = 0$

$\dfrac{1}{\sin^2 x}(2\cos x - 1) = 0$

$\cos x = \dfrac{1}{2}$

$x = \dfrac{\pi}{3}$ at point $Q$

$y\left(\dfrac{\pi}{3}\right) = 4 + \cot \dfrac{\pi}{3} - 2\csc \dfrac{\pi}{3}$

$\qquad = 4 + \dfrac{1}{\sqrt{3}} - 2\left(\dfrac{2}{\sqrt{3}}\right)$

$\qquad = 4 - \dfrac{3}{\sqrt{3}} = 4 - \sqrt{3}$

The coordinates of $Q$ are $\left(\dfrac{\pi}{3}, 4 - \sqrt{3}\right)$.

The equation of the horizontal line is $y = 4 - \sqrt{3}$.

**22.** $y'(x) = \dfrac{d}{dx}(1 + \sqrt{2}\csc x + \cot x)$

$\qquad = 0 + \sqrt{2}(-\csc x \cot x) + (-\csc^2 x)$

$\qquad = -\sqrt{2}\csc x \cot x - \csc^2 x$

**(a)** $y'\left(\dfrac{\pi}{4}\right) = -\sqrt{2}\csc \dfrac{\pi}{4} \cot \dfrac{\pi}{4} - \csc^2 \dfrac{\pi}{4}$

$\qquad = -\sqrt{2}(\sqrt{2})(1) - (\sqrt{2})^2$

$\qquad = -2 - 2 = -4$

The tangent line has slope $-4$ and passes through

$P\left(\dfrac{\pi}{4}, 4\right)$. Its equation is $y = -4\left(x - \dfrac{\pi}{4}\right) + 4$, or

$y = -4x + \pi + 4$.

**(b)** $\qquad\qquad\qquad y'(x) = 0$

$-\sqrt{2}\csc x \cot x - \csc^2 x = 0$

$-\dfrac{\sqrt{2}\cos x}{\sin^2 x} - \dfrac{1}{\sin^2 x} = 0$

$-\dfrac{1}{\sin^2 x}(\sqrt{2}\cos x + 1) = 0$

$\cos x = -\dfrac{1}{\sqrt{2}}$

$x = \dfrac{3\pi}{4}$ at point $Q$

$y\left(\dfrac{3\pi}{4}\right) = 1 + \sqrt{2}\csc \dfrac{3\pi}{4} + \cot \dfrac{3\pi}{4}$

$\qquad = 1 + \sqrt{2}(\sqrt{2}) + (-1)$

$\qquad = 2$

The coordinates of $Q$ are $\left(\dfrac{3\pi}{4}, 2\right)$.

The equation of the horizontal line is $y = 2$.

**23. (a)** Velocity: $s'(t) = -2\cos t$ m/sec
Speed: $|s'(t)| = |2\cos t|$ m/sec
Acceleration: $s''(t) = 2\sin t$ m/sec$^2$
Jerk: $s'''(t) = 2\cos t$ m/sec$^3$

**(b)** Velocity: $-2\cos \dfrac{\pi}{4} = -\sqrt{2}$ m/sec

Speed: $|-\sqrt{2}| = \sqrt{2}$ m/sec

Acceleration: $2\sin \dfrac{\pi}{4} = \sqrt{2}$ m/sec$^2$

Jerk: $2\cos \dfrac{\pi}{4} = \sqrt{2}$ m/sec$^3$

**(c)** The body starts at 2, goes to 0 and then oscillates between 0 and 4.

Speed:

*Greatest* when $\cos t = \pm 1$ (or $t = k\pi$), at the center of

the interval of motion.

*Zero* when $\cos t = 0$ (or $t = \dfrac{k\pi}{2}$, $k$ odd), at the

endpoints of the interval of motion.

Acceleration:

*Greatest* (in magnitude) when $\sin t = \pm 1$

(or $t = \dfrac{k\pi}{2}$, $k$ odd)

*Zero* when $\sin t = 0$ (or $t = k\pi$)

Jerk:

*Greatest* (in magnitude) when $\cos t = \pm 1$ (or $t = k\pi$)

*Zero* when $\cos t = 0$ $\left(\text{or } t = \dfrac{k\pi}{2}, k \text{ odd}\right)$

**24. (a)** Velocity: $s'(t) = \cos t - \sin t$ m/sec
Speed: $|s'(t)| = |\cos t - \sin t|$ m/sec
Acceleration: $s''(t) = -\sin t - \cos t$ m/sec$^2$
Jerk: $s'''(t) = -\cos t + \sin t$ m/sec$^3$

**(b)** Velocity: $\cos\dfrac{\pi}{4} - \sin\dfrac{\pi}{4} = 0$ m/sec

Speed: $|0| = 0$ m/sec

Acceleration: $-\sin\dfrac{\pi}{4} - \cos\dfrac{\pi}{4} = -\sqrt{2}$ m/sec$^2$

Jerk: $-\cos\dfrac{\pi}{4} + \sin\dfrac{\pi}{4} = 0$ m/sec$^3$

**(c)** The body starts at 1, goes to $\sqrt{2}$ and then oscillates between $\pm\sqrt{2}$.

Speed:

*Greatest* when $t = \dfrac{3\pi}{4} + k\pi$

*Zero* when $t = \dfrac{\pi}{4} + k\pi$

Acceleration:

*Greatest* (in magnitude) when $t = \dfrac{\pi}{4} + k\pi$

*Zero* when $t = \dfrac{3\pi}{4} + k\pi$

Jerk:

*Greatest* (in magnitude) when $t = \dfrac{3\pi}{4} + k\pi$

*Zero* when $t = \dfrac{\pi}{4} + k\pi$

**25. (a)**
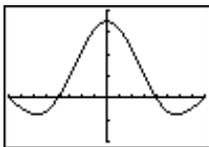


$[-360, 360]$ by $[-0.01, 0.02]$

The limit is $\dfrac{\pi}{180}$ because this is the conversion factor for changing from degrees to radians.

**(b)**
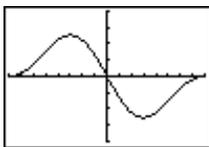


$[-360, 360]$ by $[-0.02, 0.02]$

This limit is still 0.

**(c)** $\dfrac{d}{dx}\sin x = \lim_{h\to 0}\dfrac{\sin(x+h) - \sin x}{h}$

$= \lim_{h\to 0}\dfrac{\sin x\cos h + \cos x\sin h - \sin x}{h}$

$= \lim_{h\to 0}\dfrac{\sin x(\cos h - 1) + \cos x\sin h}{h}$

$= \left(\lim_{h\to 0}\sin x\right)\left(\lim_{h\to 0}\dfrac{\cos h - 1}{h}\right) + \left(\lim_{h\to 0}\cos x\right)\left(\lim_{h\to 0}\dfrac{\sin h}{h}\right)$

$= (\sin x)(0) + (\cos x)\left(\dfrac{\pi}{180}\right)$

$= \dfrac{\pi}{180}\cos x$

**(d)** $\dfrac{d}{dx}\cos x = \lim_{h\to 0}\dfrac{\cos(x+h) - \cos x}{h}$

$= \lim_{h\to 0}\dfrac{\cos x\cos h - \sin x\sin h - \cos x}{h}$

$= \lim_{h\to 0}\dfrac{(\cos x)(\cos h - 1) - \sin x\sin h}{h}$

$= \left(\lim_{h\to 0}\cos x\right)\left(\lim_{h\to 0}\dfrac{\cos h - 1}{h}\right) - \left(\lim_{h\to 0}\sin x\right)\left(\lim_{h\to 0}\dfrac{\sin h}{h}\right)$

$= (\cos x)(0) - (\sin x)\left(\dfrac{\pi}{180}\right)$

$= -\dfrac{\pi}{180}\sin x$

**(e)** $\dfrac{d^2}{dx^2}\sin x = \dfrac{d}{dx}\dfrac{\pi}{180}\cos x = \dfrac{\pi}{180}\left(-\dfrac{\pi}{180}\sin x\right)$

$= -\dfrac{\pi^2}{180^2}\sin x$

$\dfrac{d^3}{dx^3}\sin x = \dfrac{d}{dx}\left(-\dfrac{\pi^2}{180^2}\sin x\right) = -\dfrac{\pi^2}{180^2}\left(\dfrac{\pi}{180}\cos x\right)$

$= -\dfrac{\pi^3}{180^3}\cos x$

$\dfrac{d^2}{dx^2}\cos x = \dfrac{d}{dx}\left(-\dfrac{\pi}{180}\sin x\right) = -\dfrac{\pi}{180}\left(\dfrac{\pi}{180}\cos x\right)$

$= -\dfrac{\pi^2}{180^2}\cos x$

$\dfrac{d^3}{dx^3}\cos x = \dfrac{d}{dx}\left(-\dfrac{\pi^2}{180^2}\cos x\right) = -\dfrac{\pi^2}{180^2}\left(-\dfrac{\pi}{180}\sin x\right)$

$= \dfrac{\pi^3}{180^3}\sin x$

**26.** $y' = \dfrac{d}{dx}\csc x = -\csc x\cot x$

$y'' = \dfrac{d}{dx}(-\csc x\cot x)$

$= -(\csc x)\dfrac{d}{dx}(\cot x) - (\cot x)\dfrac{d}{dx}(\csc x)$

$= -(\csc x)(-\csc^2 x) - (\cot x)(-\csc x\cot x)$

$= \csc^3 x + \csc x\cot^2 x$

**27.** $y' = \dfrac{d}{d\theta}(\theta\tan\theta)$

$= \theta\dfrac{d}{d\theta}(\tan\theta) + (\tan\theta)\dfrac{d}{d\theta}(\theta)$

$= \theta\sec^2\theta + \tan\theta$

$y'' = \dfrac{d}{d\theta}(\theta\sec^2\theta + \tan\theta)$

$= \theta\dfrac{d}{d\theta}[(\sec\theta)(\sec\theta)] + (\sec^2\theta)\dfrac{d}{d\theta}(\theta) + \dfrac{d}{d\theta}(\tan\theta)$

$= \theta\left[(\sec\theta)\dfrac{d}{d\theta}(\sec\theta) + (\sec\theta)\dfrac{d}{d\theta}(\sec\theta)\right] + \sec^2\theta + \sec^2\theta$

$= 2\theta\sec^2\theta\tan\theta + 2\sec^2\theta$

$= (2\theta\tan\theta + 2)(\sec^2\theta)$

or, writing in terms of sines and cosines,

$= \dfrac{2 + 2\theta\tan\theta}{\cos^2\theta}$

$= \dfrac{2\cos\theta + 2\theta\sin\theta}{\cos^3\theta}$

**28.** Continuous:

Note that $g(0) = \lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} \cos x = \cos(0) = 1$, and

$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} (x + b) = b$. We require $\lim_{x \to 0^-} g(x) = g(0)$,

so $b = 1$. The function is continuous if $b = 1$.

Differentiable:
For $b = 1$, the left-hand derivative is 1 and the right-hand derivative is $-\sin(0) = 0$, so the function is not differentiable. For other values of $b$, the function is discontinuous at $x = 0$ and there is no left-hand derivative. So, there is no value of $b$ that will make the function differentiable at $x = 0$.

**29.** Observe the pattern:

$\dfrac{d}{dx} \cos x = -\sin x$ $\qquad$ $\dfrac{d^5}{dx^5} \cos x = -\sin x$

$\dfrac{d^2}{dx^2} \cos x = -\cos x$ $\qquad$ $\dfrac{d^6}{dx^6} \cos x = -\cos x$

$\dfrac{d^3}{dx^3} \cos x = \sin x$ $\qquad$ $\dfrac{d^7}{dx^7} \cos x = \sin x$

$\dfrac{d^4}{dx^4} \cos x = \cos x$ $\qquad$ $\dfrac{d^8}{dx^8} \cos x = \cos x$

Continuing the pattern, we see that

$\dfrac{d^n}{dx^n} \cos x = \sin x$ when $n = 4k + 3$ for any whole number $k$.

Since $999 = 4(249) + 3$, $\dfrac{d^{999}}{dx^{999}} \cos x = \sin x$.

**30.** Observe the pattern:

$\dfrac{d}{dx} \sin x = \cos x$ $\qquad$ $\dfrac{d^5}{dx^5} \sin x = \cos x$

$\dfrac{d^2}{dx^2} \sin x = -\sin x$ $\qquad$ $\dfrac{d^6}{dx^6} \sin x = -\sin x$

$\dfrac{d^3}{dx^3} \sin x = -\cos x$ $\qquad$ $\dfrac{d^7}{dx^7} \sin x = -\cos x$

$\dfrac{d^4}{dx^4} \sin x = \sin x$ $\qquad$ $\dfrac{d^8}{dx^8} \sin x = \sin x$

Continuing the pattern, we see that

$\dfrac{d^n}{dx^n} \sin x = \cos x$ when $n = 4k + 1$ for any whole number $k$.

Since $725 = 4(181) + 1$, $\dfrac{d^{725}}{dx^{725}} \sin x = \cos x$.

**31.** The line is tangent to the graph of $y = \sin x$ at $(0, 0)$. Since $y'(0) = \cos(0) = 1$, the line has slope 1 and its equation is $y = x$.

**32.** (a) Using $y = x$, $\sin(0.12) \approx 0.12$.

(b) $\sin(0.12) \approx 0.1197122$; The approximation is within 0.0003 of the actual value.

**33.** $\dfrac{d}{dx} \sin 2x = \dfrac{d}{dx}(2 \sin x \cos x)$

$= 2\dfrac{d}{dx}(\sin x \cos x)$

$= 2\left[(\sin x)\dfrac{d}{dx}(\cos x) + (\cos x)\dfrac{d}{dx}(\sin x)\right]$

$= 2[(\sin x)(-\sin x) + (\cos x)(\cos x)]$

$= 2(\cos^2 x - \sin^2 x)$

$= 2 \cos 2x$

**34.** $\dfrac{d}{dx} \cos 2x = \dfrac{d}{dx}[(\cos x)(\cos x) - (\sin x)(\sin x)]$

$= \left[(\cos x)\dfrac{d}{dx}(\cos x) + (\cos x)\dfrac{d}{dx}(\cos x)\right] -$

$\qquad \left[(\sin x)\dfrac{d}{dx}(\sin x) + (\sin x)\dfrac{d}{dx}(\sin x)\right]$

$= 2(\cos x)(-\sin x) - 2(\sin x)(\cos x)$

$= -4 \sin x \cos x$

$= -2(2 \sin x \cos x)$

$= -2 \sin 2x$

**35.** $\lim_{h \to 0} \dfrac{(\cos h - 1)}{h} = \lim_{h \to 0} \dfrac{(\cos h - 1)(\cos h + 1)}{h(\cos h + 1)}$

$= \lim_{h \to 0} \dfrac{\cos^2 h - 1}{h(\cos h + 1)}$

$= \lim_{h \to 0} \dfrac{-\sin^2 h}{h(\cos h + 1)}$

$= -\left(\lim_{h \to 0} \dfrac{\sin h}{h}\right)\left(\lim_{h \to 0} \dfrac{\sin h}{\cos h + 1}\right)$

$= -(1)\left(\dfrac{0}{2}\right) = 0$

**36.** $y' = \dfrac{d}{dx}(A \sin x + B \cos x) = A \cos x - B \sin x$

$y'' = \dfrac{d}{dx}(A \cos x - B \sin x) = -A \sin x - B \cos x$

Solve: $\qquad\qquad\qquad\qquad\qquad\qquad y'' - y = \sin x$

$(-A \sin x - B \cos x) - (A \sin x + B \cos x) = \sin x$

$\qquad\qquad\qquad\qquad -2A \sin x - 2B \cos x = \sin x$

At $x = \dfrac{\pi}{2}$, this gives $-2A = 1$, so $A = -\dfrac{1}{2}$.

At $x = 0$, we have $-2B = 0$, so $B = 0$.

Thus, $A = -\dfrac{1}{2}$ and $B = 0$.

# ■ Section 3.6 Chain Rule (pp. 141–149)

## Quick Review 3.6

**1.** $f(g(x)) = f(x^2 + 1) = \sin(x^2 + 1)$

**2.** $f(g(h(x))) = f(g(7x)) = f((7x)^2 + 1)$
$= \sin[(7x)^2 + 1] = \sin(49x^2 + 1)$

**3.** $(g \circ h)(x) = g(h(x)) = g(7x) = (7x)^2 + 1 = 49x^2 + 1$

**4.** $(h \circ g)(x) = h(g(x)) = h(x^2 + 1) = 7(x^2 + 1) = 7x^2 + 7$

**5.** $f\left(\dfrac{g(x)}{h(x)}\right) = f\left(\dfrac{x^2 + 1}{7x}\right) = \sin \dfrac{x^2 + 1}{7x}$

**6.** $\sqrt{\cos x + 2} = g(\cos x) = g(f(x))$

**7.** $\sqrt{3 \cos^2 x + 2} = g(3 \cos^2 x) = g(h(\cos x)) = g(h(f(x)))$

**8.** $3 \cos x + 6 = 3(\cos x + 2) = 3(\sqrt{\cos x + 2})^2$

$= h(\sqrt{\cos x + 2}) = h(g(\cos x)) = h(g(f(x)))$

**9.** $\cos 27x^4 = f(27x^4) = f(3(3x^2)^2) = f(h(3x^2)) = f(h(h(x)))$

**10.** $\cos \sqrt{2 + 3x^2} = \cos \sqrt{3x^2 + 2} = f(\sqrt{3x^2 + 2})$

$= f(g(3x^2)) = f(g(h(x)))$

## Section 3.6 Exercises

**1.** $\dfrac{dy}{dx} = \dfrac{d}{dx} \sin (3x + 1) = [\cos (3x + 1)]\dfrac{d}{dx}(3x + 1)$

$= [\cos (3x + 1)](3) = 3 \cos (3x + 1)$

**2.** $\dfrac{dy}{dx} = \dfrac{d}{dx} \sin (7 - 5x) = [\cos (7 - 5x)]\dfrac{d}{dx}(7 - 5x)$

$= [\cos (7 - 5x)](-5) = -5 \cos (7 - 5x)$

**3.** $\dfrac{dy}{dx} = \dfrac{d}{dx} \cos (\sqrt{3}x) = [-\sin (\sqrt{3}x)]\dfrac{d}{dx}(\sqrt{3}x)$

$= [-\sin (\sqrt{3}x)](\sqrt{3}) = -\sqrt{3} \sin (\sqrt{3}x)$

**4.** $\dfrac{dy}{dx} = \dfrac{d}{dx} \tan (2x - x^3) = [\sec^2 (2x - x^3)]\dfrac{d}{dx}(2x - x^3)$

$= [\sec^2 (2x - x^3)](2 - 3x^2) = (2 - 3x^2) \sec^2 (2x - x^3)$

**5.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\left[5 \cot \left(\dfrac{2}{x}\right)\right] = \left[-5 \csc^2 \left(\dfrac{2}{x}\right)\right]\dfrac{d}{dx}(2x^{-1})$

$= \left[-5 \csc^2 \left(\dfrac{2}{x}\right)\right](-2x^{-2}) = \dfrac{10}{x^2} \csc^2 \left(\dfrac{2}{x}\right)$

**6.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{\sin x}{1 + \cos x}\right)^2 = 2\left(\dfrac{\sin x}{1 + \cos x}\right)\dfrac{d}{dx}\left(\dfrac{\sin x}{1 + \cos x}\right)$

$= 2\left(\dfrac{\sin x}{1 + \cos x}\right)\left(\dfrac{(1 + \cos x)\dfrac{d}{dx} \sin x - \sin x \dfrac{d}{dx}(1 + \cos x)}{(1 + \cos x)^2}\right)$

$= 2\left(\dfrac{\sin x}{1 + \cos x}\right)\left(\dfrac{(1 + \cos x)(\cos x) - (\sin x)(-\sin x)}{(1 + \cos x)^2}\right)$

$= 2\left(\dfrac{\sin x}{1 + \cos x}\right)\left(\dfrac{\cos x + \cos^2 x + \sin^2 x}{(1 + \cos x)^2}\right)$

$= 2\left(\dfrac{\sin x}{1 + \cos x}\right)\left(\dfrac{1 + \cos x}{(1 + \cos x)^2}\right)$

$= 2\left(\dfrac{\sin x}{1 + \cos x}\right)\left(\dfrac{1}{1 + \cos x}\right)$

$= \dfrac{2 \sin x}{(1 + \cos x)^2}$

**7.** $\dfrac{dy}{dx} = \dfrac{d}{dx} \cos (\sin x) = [-\sin (\sin x)]\dfrac{d}{dx} (\sin x)$

$= -\sin (\sin x) \cos x$

**8.** $\dfrac{dy}{dx} = \dfrac{d}{dx} \sec (\tan x) = \sec (\tan x) \tan (\tan x)\dfrac{d}{dx}(\tan x)$

$= \sec (\tan x) \tan (\tan x) \sec^2 x$

**9.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(x + \sqrt{x})^{-2} = -2(x + \sqrt{x})^{-3}\dfrac{d}{dx}(x + \sqrt{x})$

$= -2(x + \sqrt{x})^{-3}\left(1 + \dfrac{1}{2\sqrt{x}}\right)$

**10.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(\csc x + \cot x)^{-1}$

$= -(\csc x + \cot x)^{-2}\dfrac{d}{dx}(\csc x + \cot x)$

$= -\dfrac{1}{(\csc x + \cot x)^2}(-\cot x \csc x - \csc^2 x)$

$= \dfrac{(\csc x)(\cot x + \csc x)}{(\csc x + \cot x)^2} = \dfrac{\csc x}{\csc x + \cot x}$

**11.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(\sin^{-5} x - \cos^3 x)$

$= (-5 \sin^{-6} x)\dfrac{d}{dx}(\sin x) - (3 \cos^2 x)\dfrac{d}{dx}(\cos x)$

$= -5 \sin^{-6} x \cos x + 3 \cos^2 x \sin x$

**12.** $\dfrac{dy}{dx} = \dfrac{d}{dx}[x^3 (2x - 5)^4]$

$= (x^3)\dfrac{d}{dx}(2x - 5)^4 + (2x - 5)^4\dfrac{d}{dx}(x^3)$

$= (x^3)(4)(2x - 5)^3\dfrac{d}{dx}(2x - 5) + (2x - 5)^4(3x^2)$

$= (x^3)(4)(2x - 5)^3(2) + 3x^2(2x - 5)^4$

$= 8x^3(2x - 5)^3 + 3x^2(2x - 5)^4$

$= x^2(2x - 5)^3[8x + 3(2x - 5)]$

$= x^2(2x - 5)^3(14x - 15)$

**13.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(\sin^3 x \tan 4x)$

$= (\sin^3 x)\dfrac{d}{dx}(\tan 4x) + (\tan 4x)\dfrac{d}{dx}(\sin^3 x)$

$= (\sin^3 x)(\sec^2 4x)\dfrac{d}{dx}(4x) + (\tan 4x)(3 \sin^2 x)\dfrac{d}{dx} (\sin x)$

$= (\sin^3 x)(\sec^2 4x)(4) + (\tan 4x)(3 \sin^2 x)(\cos x)$

$= 4 \sin^3 x \sec^2 4x + 3 \sin^2 x \cos x \tan 4x$

**14.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(4\sqrt{\sec x + \tan x})$

$= 4 \cdot \dfrac{1}{2\sqrt{\sec x + \tan x}}\dfrac{d}{dx}(\sec x + \tan x)$

$= \dfrac{2}{\sqrt{\sec x + \tan x}}(\sec x \tan x + \sec^2 x)$

$= 2 \sec x\dfrac{\sec x + \tan x}{\sqrt{\sec x + \tan x}}$

$= 2 \sec x\sqrt{\sec x + \tan x}$

**15.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{3}{\sqrt{2x+1}}\right)$

$\quad = \dfrac{(\sqrt{2x+1})\frac{d}{dx}(3) - 3\frac{d}{dx}(\sqrt{2x+1})}{(\sqrt{2x+1})^2}$

$\quad = \dfrac{(\sqrt{2x+1})(0) - 3\left(\frac{1}{2\sqrt{2x+1}}\right)\frac{d}{dx}(2x+1)}{2x+1}$

$\quad = \dfrac{-3\left(\frac{1}{2\sqrt{2x+1}}\right)(2)}{2x+1}$

$\quad = -\dfrac{3}{(2x+1)\sqrt{2x+1}}$

$\quad = -3(2x+1)^{-3/2}$

**16.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\dfrac{x}{\sqrt{1+x^2}}$

$\quad = \dfrac{(\sqrt{1+x^2})\frac{d}{dx}(x) - x\frac{d}{dx}(\sqrt{1+x^2})}{(\sqrt{1+x^2})^2}$

$\quad = \dfrac{(\sqrt{1+x^2})(1) - x\left(\frac{1}{2\sqrt{1+x^2}}\right)\frac{d}{dx}(1+x^2)}{1+x^2}$

$\quad = \dfrac{\sqrt{1+x^2} - x\left(\frac{1}{2\sqrt{1+x^2}}\right)(2x)}{1+x^2}$

$\quad = \dfrac{(1+x^2) - x^2}{(1+x^2)(\sqrt{1+x^2})}$

$\quad = (1+x^2)^{-3/2}$

**17.** The last step here uses the identity $2\sin a \cos a = \sin 2a$.

$\dfrac{dy}{dx} = \dfrac{d}{dx}\sin^2(3x-2)$

$\quad = 2\sin(3x-2)\dfrac{d}{dx}\sin(3x-2)$

$\quad = 2\sin(3x-2)\cos(3x-2)\dfrac{d}{dx}(3x-2)$

$\quad = 2\sin(3x-2)\cos(3x-2)(3)$

$\quad = 6\sin(3x-2)\cos(3x-2)$

$\quad = 3\sin(6x-4)$

**18.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(1+\cos 2x)^2 = 2(1+\cos 2x)\dfrac{d}{dx}(1+\cos 2x)$

$\quad = 2(1+\cos 2x)(-\sin 2x)\dfrac{d}{dx}(2x)$

$\quad = 2(1+\cos 2x)(-\sin 2x)(2)$

$\quad = -4(1+\cos 2x)(\sin 2x)$

**19.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(1+\cos^2 7x)^3$

$\quad = 3(1+\cos^2 7x)^2\dfrac{d}{dx}(1+\cos^2 7x)$

$\quad = 3(1+\cos^2 7x)^2(2\cos 7x)\dfrac{d}{dx}(\cos 7x)$

$\quad = 3(1+\cos^2 7x)^2(2\cos 7x)(-\sin 7x)\dfrac{d}{dx}(7x)$

$\quad = 3(1+\cos^2 7x)^2(2\cos 7x)(-\sin 7x)(7)$

$\quad = -42(1+\cos^2 7x)^2\cos 7x \sin 7x$

**20.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(\sqrt{\tan 5x}) = \dfrac{1}{2\sqrt{\tan 5x}}\dfrac{d}{dx}\tan 5x$

$\quad = \dfrac{1}{2\sqrt{\tan 5x}}(\sec^2 5x)\dfrac{d}{dx}(5x)$

$\quad = \dfrac{1}{2\sqrt{\tan 5x}}(\sec^2 5x)(5)$

$\quad = \dfrac{5\sec^2 5x}{2\sqrt{\tan 5x}}$ or $\dfrac{5}{2}(\tan 5x)^{-1/2}\sec^2 5x$

**21.** $\dfrac{ds}{dt} = \dfrac{d}{dt}\cos\left(\dfrac{\pi}{2} - 3t\right)$

$\quad = \left[-\sin\left(\dfrac{\pi}{2} - 3t\right)\right]\dfrac{d}{dt}\left(\dfrac{\pi}{2} - 3t\right)$

$\quad = \left[-\sin\left(\dfrac{\pi}{2} - 3t\right)\right](-3)$

$\quad = 3\sin\left(\dfrac{\pi}{2} - 3t\right)$

**22.** $\dfrac{ds}{dt} = \dfrac{d}{dt}[t\cos(\pi-4t)]$

$\quad = (t)\dfrac{d}{dt}[\cos(\pi-4t)] + \cos(\pi-4t)\dfrac{d}{dt}(t)$

$\quad = t[-\sin(\pi-4t)]\dfrac{d}{dt}(\pi-4t) + \cos(\pi-4t)(1)$

$\quad = t[-\sin(\pi-4t)](-4) + \cos(\pi-4t)$

$\quad = 4t\sin(\pi-4t) + \cos(\pi-4t)$

**23.** $\dfrac{ds}{dt} = \dfrac{d}{dt}\left(\dfrac{4}{3\pi}\sin 3t + \dfrac{4}{5\pi}\cos 5t\right)$

$\quad = \dfrac{4}{3\pi}(\cos 3t)\dfrac{d}{dt}(3t) + \dfrac{4}{5\pi}(-\sin 5t)\dfrac{d}{dt}(5t)$

$\quad = \dfrac{4}{3\pi}(\cos 3t)(3) + \dfrac{4}{5\pi}(-\sin 5t)(5)$

$\quad = \dfrac{4}{\pi}\cos 3t - \dfrac{4}{\pi}\sin 5t$

**24.** $\dfrac{ds}{dt} = \dfrac{d}{dt}\left[\sin\left(\dfrac{3\pi}{2}t\right) + \cos\left(\dfrac{7\pi}{4}t\right)\right]$

$\quad = \cos\left(\dfrac{3\pi}{2}t\right)\dfrac{d}{dt}\left(\dfrac{3\pi}{2}t\right) - \sin\left(\dfrac{7\pi}{4}t\right)\dfrac{d}{dt}\left(\dfrac{7\pi}{4}t\right)$

$\quad = \dfrac{3\pi}{2}\cos\left(\dfrac{3\pi}{2}t\right) - \dfrac{7\pi}{4}\sin\left(\dfrac{7\pi}{4}t\right)$

**25.** $\dfrac{dr}{d\theta} = \dfrac{d}{d\theta}\tan(2-\theta) = \sec^2(2-\theta)\dfrac{d}{d\theta}(2-\theta)$

$\quad = \sec^2(2-\theta)(-1) = -\sec^2(2-\theta)$

**26.** $\dfrac{dr}{d\theta} = \dfrac{d}{d\theta}(\sec 2\theta \tan 2\theta)$

$\quad = (\sec 2\theta)\dfrac{d}{d\theta}(\tan 2\theta) + (\tan 2\theta)\dfrac{d}{d\theta}(\sec 2\theta)$

$\quad = (\sec 2\theta)(\sec^2 2\theta)\dfrac{d}{d\theta}(2\theta) + (\tan 2\theta)(\sec 2\theta \tan 2\theta)\dfrac{d}{d\theta}(2\theta)$

$\quad = 2\sec^3 2\theta + 2\sec 2\theta \tan^2 2\theta$

**27.** $\dfrac{dr}{d\theta} = \dfrac{d}{d\theta}\sqrt{\theta \sin \theta} = \dfrac{1}{2\sqrt{\theta \sin \theta}}\dfrac{d}{d\theta}(\theta \sin \theta)$

$\qquad = \dfrac{1}{2\sqrt{\theta \sin \theta}}\left[\theta \dfrac{d}{d\theta}(\sin \theta) + (\sin \theta)\dfrac{d}{d\theta}(\theta)\right]$

$\qquad = \dfrac{1}{2\sqrt{\theta \sin \theta}}(\theta \cos \theta + \sin \theta)$

$\qquad = \dfrac{\theta \cos \theta + \sin \theta}{2\sqrt{\theta \sin \theta}}$

**28.** $\dfrac{dr}{d\theta} = \dfrac{d}{d\theta}(2\theta\sqrt{\sec \theta})$

$\qquad = (2\theta)\dfrac{d}{d\theta}(\sqrt{\sec \theta}) + (\sqrt{\sec \theta})\dfrac{d}{d\theta}(2\theta)$

$\qquad = (2\theta)\left(\dfrac{1}{2\sqrt{\sec \theta}}\right)\dfrac{d}{d\theta}(\sec \theta) + 2\sqrt{\sec \theta}$

$\qquad = (2\theta)\left(\dfrac{1}{2\sqrt{\sec \theta}}\right)(\sec \theta \tan \theta) + 2\sqrt{\sec \theta}$

$\qquad = \theta(\sqrt{\sec \theta})(\tan \theta) + 2\sqrt{\sec \theta}$

$\qquad = \sqrt{\sec \theta}(\theta \tan \theta + 2)$

**29.** $y' = \dfrac{d}{dx}\tan x = \sec^2 x$

$\qquad y'' = \dfrac{d}{dx}\sec^2 x = (2 \sec x)\dfrac{d}{dx}(\sec x)$

$\qquad\qquad = (2 \sec x)(\sec x \tan x)$

$\qquad\qquad = 2 \sec^2 x \tan x$

**30.** $y' = \dfrac{d}{dx}\cot x = -\csc^2 x$

$\qquad y'' = \dfrac{d}{dx}(-\csc^2 x) = (-2 \csc x)\dfrac{d}{dx}(\csc x)$

$\qquad\qquad = (-2 \csc x)(-\csc x \cot x)$

$\qquad\qquad = 2 \csc^2 x \cot x$

**31.** $y' = \dfrac{d}{dx}\cot (3x - 1) = -\csc^2 (3x - 1)\dfrac{d}{dx}(3x - 1)$

$\qquad = -3 \csc^2 (3x - 1)$

$\qquad y'' = \dfrac{d}{dx}[-3 \csc^2(3x - 1)]$

$\qquad = -3[2 \csc (3x - 1)]\dfrac{d}{dx}\csc (3x - 1)$

$\qquad = -3[2 \csc (3x - 1)] \cdot$

$\qquad\quad [-\csc (3x - 1) \cot (3x - 1)]\dfrac{d}{dx}(3x - 1)$

$\qquad = -3[2 \csc (3x - 1)][-\csc (3x - 1) \cot (3x - 1)](3)$

$\qquad = 18 \csc^2 (3x - 1) \cot (3x - 1)$

**32.** $y' = \dfrac{d}{dx}\left[9 \tan \left(\dfrac{x}{3}\right)\right] = 9 \sec^2 \left(\dfrac{x}{3}\right)\dfrac{d}{dx}\left(\dfrac{x}{3}\right)$

$\qquad = 3 \sec^2 \left(\dfrac{x}{3}\right)$

$\qquad y'' = \dfrac{d}{dx}\left[3 \sec^2 \left(\dfrac{x}{3}\right)\right] = 3\left[2 \sec \left(\dfrac{x}{3}\right)\right]\dfrac{d}{dx}\sec \left(\dfrac{x}{3}\right)$

$\qquad = 6\left[\sec \left(\dfrac{x}{3}\right)\right]\left[\sec \left(\dfrac{x}{3}\right)\tan \left(\dfrac{x}{3}\right)\right]\dfrac{d}{dx}\left(\dfrac{x}{3}\right)$

$\qquad = 2 \sec^2 \left(\dfrac{x}{3}\right)\tan \left(\dfrac{x}{3}\right)$

**33.** $f'(u) = \dfrac{d}{du}(u^5 + 1) = 5u^4$

$\qquad g'(x) = \dfrac{d}{dx}(\sqrt{x}) = \dfrac{1}{2\sqrt{x}}$

$\qquad (f \circ g)'(1) = f'(g(1))g'(1) = f'(1)g'(1) = (5)\left(\dfrac{1}{2}\right) = \dfrac{5}{2}$

**34.** $f'(u) = \dfrac{d}{du}(1 - u^{-1}) = u^{-2} = \dfrac{1}{u^2}$

$\qquad g'(x) = \dfrac{d}{dx}(1 - x)^{-1} = -(1 - x)^{-2}\dfrac{d}{dx}(1 - x)$

$\qquad = -(1 - x)^{-2}(-1) = \dfrac{1}{(1 - x)^2}$

$\qquad (f \circ g)'(-1) = f'(g(-1))g'(-1) = f'\left(\dfrac{1}{2}\right)g'(-1)$

$\qquad = (4)\left(\dfrac{1}{4}\right) = 1$

**35.** $f'(u) = \dfrac{d}{du}\left(\cot \dfrac{\pi u}{10}\right) = -\csc^2 \left(\dfrac{\pi u}{10}\right)\dfrac{d}{du}\left(\dfrac{\pi u}{10}\right)$

$\qquad = -\dfrac{\pi}{10}\csc^2 \left(\dfrac{\pi u}{10}\right)$

$\qquad g'(x) = \dfrac{d}{dx}(5\sqrt{x}) = \dfrac{5}{2\sqrt{x}}$

$\qquad (f \circ g)'(1) = f'(g(1))g'(1) = f'(5)g'(1)$

$\qquad = -\dfrac{\pi}{10}\left[\csc^2 \left(\dfrac{\pi}{2}\right)\right]\left(\dfrac{5}{2}\right)$

$\qquad = -\dfrac{\pi}{10}(1)\left(\dfrac{5}{2}\right) = -\dfrac{\pi}{4}$

**36.** $f'(u) = \dfrac{d}{du}[u + (\cos u)^{-2}]$

$\qquad = 1 - 2(\cos u)^{-3}\dfrac{d}{du}\cos u$

$\qquad = 1 + \dfrac{2 \sin u}{\cos^3 u}$

$\qquad g'(x) = \dfrac{d}{dx}(\pi x) = \pi$

$\qquad (f \circ g)'\left(\dfrac{1}{4}\right) = f'(g(\tfrac{1}{4}))g'\left(\dfrac{1}{4}\right)$

$\qquad\qquad = f'\left(\dfrac{\pi}{4}\right)g'\left(\dfrac{1}{4}\right)$

$\qquad\qquad = \left(1 + \dfrac{\frac{2}{\sqrt{2}}}{\left(\frac{1}{\sqrt{2}}\right)^3}\right)(\pi)$

$\qquad\qquad = 5\pi$

**37.** $f'(u) = \dfrac{d}{du}\dfrac{2u}{u^2 + 1} = \dfrac{(u^2 + 1)\frac{d}{du}(2u) - (2u)\frac{d}{du}(u^2 + 1)}{(u^2 + 1)^2}$

$\qquad = \dfrac{(u^2 + 1)(2) - (2u)(2u)}{(u^2 + 1)^2} = \dfrac{-2u^2 + 2}{(u^2 + 1)^2}$

$\qquad g'(x) = \dfrac{d}{dx}(10x^2 + x + 1) = 20x + 1$

$\qquad (f \circ g)'(0) = f'(g(0))g'(0) = f'(1)g'(0) = (0)(1) = 0$

**38.** $f'(u) = \dfrac{d}{du}\left(\dfrac{u-1}{u+1}\right)^2 = 2\left(\dfrac{u-1}{u+1}\right)\dfrac{d}{du}\left(\dfrac{u-1}{u+1}\right)$

$= 2\left(\dfrac{u-1}{u+1}\right)\dfrac{(u+1)\dfrac{d}{du}(u-1) - (u-1)\dfrac{d}{du}(u+1)}{(u+1)^2}$

$= 2\left(\dfrac{u-1}{u+1}\right)\dfrac{(u+1) - (u-1)}{(u+1)^2} = \dfrac{4(u-1)}{(u+1)^3}$

$g'(x) = \dfrac{d}{dx}(x^{-2} - 1) = -2x^{-3}$

$(f \circ g)'(-1) = f'(g(-1))g'(-1)$

$\qquad = f'(0)g'(-1)$

$\qquad = (-4)(2) = -8$

**39. (a)** $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

$= \dfrac{d}{du}(\cos u)\dfrac{d}{dx}(6x+2)$

$= (-\sin u)(6)$

$= -6 \sin u$

$= -6 \sin (6x+2)$

**(b)** $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

$= \dfrac{d}{du}(\cos 2u)\dfrac{d}{dx}(3x+1)$

$= (-\sin 2u)(2) \cdot (3)$

$= -6 \sin 2u$

$= -6 \sin (6x+2)$

**40. (a)** $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

$= \dfrac{d}{du}\sin (u+1)\dfrac{d}{dx}(x^2)$

$= \cos (u+1)(1) \cdot 2x$

$= 2x \cos (u+1)$

$= 2x \cos (x^2+1)$

**(b)** $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

$= \dfrac{d}{du}(\sin u)\dfrac{d}{dx}(x^2+1)$

$= (\cos u)(2x)$

$= 2x \cos u$

$= 2x \cos (x^2+1)$

**41.** $\dfrac{dx}{dt} = \dfrac{d}{dt}(2 \cos t) = -2 \sin t$

$\dfrac{dy}{dt} = \dfrac{d}{dt}(2 \sin t) = 2 \cos t$

$\dfrac{dy}{dx} = \dfrac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \dfrac{2 \cos t}{-2 \sin t} = -\cot t$

The line passes through $\left(2 \cos \dfrac{\pi}{4}, 2 \sin \dfrac{\pi}{4}\right) = (\sqrt{2}, \sqrt{2})$

and has slope $-\cot \dfrac{\pi}{4} = -1$. Its equation is

$y = -(x - \sqrt{2}) + \sqrt{2}$, or $y = -x + 2\sqrt{2}$.

**42.** $\dfrac{dx}{dt} = \dfrac{d}{dt}(\sin 2\pi t) = (\cos 2\pi t)\dfrac{d}{dt}(2\pi t) = 2\pi \cos 2\pi t$

$\dfrac{dy}{dt} = \dfrac{d}{dt}(\cos 2\pi t) = (-\sin 2\pi t)\dfrac{d}{dt}(2\pi t) = -2\pi \sin 2\pi t$

$\dfrac{dy}{dx} = \dfrac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \dfrac{-2\pi \sin 2\pi t}{2\pi \cos 2\pi t} = -\tan 2\pi t$

The line passes through $\left(\sin \dfrac{2\pi}{-6}, \cos \dfrac{2\pi}{-6}\right) = \left(-\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right)$

and has slope $-\tan \dfrac{2\pi}{-6} = \sqrt{3}$. Its equation is

$y = \sqrt{3}\left(x + \dfrac{\sqrt{3}}{2}\right) + \dfrac{1}{2}$, or $y = \sqrt{3}x + 2$.

**43.** $\dfrac{dx}{dt} = \dfrac{d}{dt}(\sec^2 t - 1) = (2 \sec t)\dfrac{d}{dt}(\sec t)$

$\qquad\qquad = (2 \sec t)(\sec t \tan t)$

$\qquad\qquad = 2 \sec^2 t \tan t$

$\dfrac{dy}{dt} = \dfrac{d}{dt}\tan t = \sec^2 t$

$\dfrac{dy}{dx} = \dfrac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \dfrac{\sec^2 t}{2 \sec^2 t \tan t} = \dfrac{1}{2} \cot t.$

The line passes through

$\left(\sec^2\left(-\dfrac{\pi}{4}\right) - 1, \tan\left(-\dfrac{\pi}{4}\right)\right) = (1, -1)$ and has

slope $\dfrac{1}{2}\cot\left(-\dfrac{\pi}{4}\right) = -\dfrac{1}{2}$. Its equation is

$y = -\dfrac{1}{2}(x-1) - 1$, or $y = -\dfrac{1}{2}x - \dfrac{1}{2}$.

**44.** $\dfrac{dx}{dt} = \dfrac{d}{dt}\sec t = \sec t \tan t$

$\dfrac{dy}{dt} = \dfrac{d}{dt}\tan t = \sec^2 t$

$\dfrac{dy}{dx} = \dfrac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \dfrac{\sec^2 t}{\sec t \tan t} = \dfrac{\sec t}{\tan t} = \dfrac{1}{\sin t} = \csc t$

The line passes through $\left(\sec \dfrac{\pi}{6}, \tan \dfrac{\pi}{6}\right) = \left(\dfrac{2}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$ and

has slope $\csc \dfrac{\pi}{6} = 2$. Its equation is $y = 2\left(x - \dfrac{2}{\sqrt{3}}\right) + \dfrac{1}{\sqrt{3}}$,

or $y = 2x - \sqrt{3}$.

**45.** $\dfrac{dx}{dt} = \dfrac{d}{dt}t = 1$

$\dfrac{dy}{dt} = \dfrac{d}{dt}\sqrt{t} = \dfrac{1}{2\sqrt{t}}$

$\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{1/(2\sqrt{t})}{1} = \dfrac{1}{2\sqrt{t}}$

The line passes through $\left(\dfrac{1}{4}, \sqrt{\dfrac{1}{4}}\right) = \left(\dfrac{1}{4}, \dfrac{1}{2}\right)$ and has slope

$\dfrac{1}{2\sqrt{\frac{1}{4}}} = 1$. Its equation is $y = 1\left(x - \dfrac{1}{4}\right) + \dfrac{1}{2}$, or

$y = x + \dfrac{1}{4}$.

**46.** $\dfrac{dx}{dt} = \dfrac{d}{dt}(2t^2 + 3) = 4t$

$\dfrac{dy}{dt} = \dfrac{d}{dt}(t^4) = 4t^3$

$\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{4t^3}{4t} = t^2$

The line passes through $(2(-1)^2 + 3, (-1)^4) = (5, 1)$ and has slope $(-1)^2 = 1$. Its equation is $y = 1(x - 5) + 1$, or $y = x - 4$.

**47.** $\dfrac{dx}{dt} = \dfrac{d}{dt}(t - \sin t) = 1 - \cos t$

$\dfrac{dy}{dt} = \dfrac{d}{dt}(1 - \cos t) = \sin t$

$\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{\sin t}{1 - \cos t}$

The line passes through

$\left(\dfrac{\pi}{3} - \sin\dfrac{\pi}{3}, 1 - \cos\dfrac{\pi}{3}\right) = \left(\dfrac{\pi}{3} - \dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right)$ and has slope

$\dfrac{\sin\left(\frac{\pi}{3}\right)}{1 - \cos\left(\frac{\pi}{3}\right)} = \sqrt{3}$. Its equation is

$y = \sqrt{3}\left(x - \dfrac{\pi}{3} + \dfrac{\sqrt{3}}{2}\right) + \dfrac{1}{2}$, or

$y = \sqrt{3}x + 2 - \dfrac{\pi}{\sqrt{3}}$.

**48.** $\dfrac{dx}{dt} = \dfrac{d}{dt}\cos t = -\sin t$

$\dfrac{dy}{dt} = \dfrac{d}{dt}(1 + \sin t) = \cos t$

$\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{\cos t}{-\sin t} = -\cot t$

The line passes through $\left(\cos\dfrac{\pi}{2}, 1 + \sin\dfrac{\pi}{2}\right) = (0, 2)$ and

has slope $-\cot\left(\dfrac{\pi}{2}\right) = 0$. Its equation is $y = 2$.

**49. (a)** $\dfrac{dx}{dt} = \dfrac{d}{dt}(t^2 + t) = 2t + 1$

$\dfrac{dy}{dt} = \dfrac{d}{dt}\sin t = \cos t$

$\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{\cos t}{2t + 1}$

**(b)** $\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dt}\dfrac{\cos t}{2t + 1}$

$= \dfrac{(2t + 1)\frac{d}{dt}(\cos t) - (\cos t)\frac{d}{dt}(2t + 1)}{(2t + 1)^2}$

$= \dfrac{(2t + 1)(-\sin t) - (\cos t)(2)}{(2t + 1)^2}$

$= -\dfrac{(2t + 1)(\sin t) + 2\cos t}{(2t + 1)^2}$

**(c)** Let $u = \dfrac{dy}{dx}$.

Then $\dfrac{du}{dt} = \dfrac{du}{dx}\dfrac{dx}{dt}$, so $\dfrac{du}{dx} = \dfrac{du}{dt} \div \dfrac{dx}{dt}$. Therefore,

$\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) \div \dfrac{dx}{dt}$

$= -\dfrac{(2t + 1)(\sin t) + 2\cos t}{(2t + 1)^2} \div (2t + 1)$

$= -\dfrac{(2t + 1)(\sin t) + 2\cos t}{(2t + 1)^3}$

**(d)** The expression in part (c).

**50.** Since the radius passes through $(0, 0)$ and $(2\cos t, 2\sin t)$,

it has slope given by $\tan t$. But the slope of the tangent is

$\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{-2\cos t}{2\sin t} = -\cot t$, which is the negative

reciprocal of $\tan t$. This means that the radius and the

tangent are perpendicular. (The preceding argument breaks

down when $t = \dfrac{k\pi}{2}$, where $k$ is an integer. At these values,

either the radius is horizontal and the tangent is vertical or

the radius is vertical and the tangent is horizontal, so the

result still holds.)

**51.** $\dfrac{ds}{dt} = \dfrac{ds}{d\theta}\dfrac{d\theta}{dt} = \dfrac{d}{d\theta}(\cos\theta)\dfrac{d\theta}{dt}$

$= (-\sin\theta)\left(\dfrac{d\theta}{dt}\right)$

When $\theta = \dfrac{3\pi}{2}$ and $\dfrac{d\theta}{dt} = 5$, $\dfrac{ds}{dt} = \left(-\sin\dfrac{3\pi}{2}\right)(5) = 5$.

**52.** $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt} = \dfrac{d}{dx}(x^2 + 7x - 5)\dfrac{dx}{dt}$

$= (2x + 7)\left(\dfrac{dx}{dt}\right)$

When $x = 1$ and $\dfrac{dy}{dt} = \dfrac{1}{3}$, $\dfrac{dy}{dt} = [2(1) + 7]\left(\dfrac{1}{3}\right) = 3$.

**53.** $\dfrac{dy}{dx} = \dfrac{d}{dx} \sin \dfrac{x}{2} = \left(\cos \dfrac{x}{2}\right)\dfrac{d}{dx}\left(\dfrac{x}{2}\right) = \dfrac{1}{2}\cos\dfrac{x}{2}$

Since the range of the function $f(x) = \dfrac{1}{2}\cos\dfrac{x}{2}$ is $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$,

the largest possible value of $\dfrac{dy}{dx}$ is $\dfrac{1}{2}$.
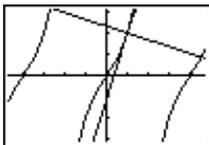
**54.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(\sin mx) = (\cos mx)\dfrac{d}{dx}(mx) = m\cos mx$

The desired line has slope $y'(0) = m\cos 0 = m$ and passes

through $(0, 0)$, so its equation is $y = mx$.

**55.** $\dfrac{dy}{dx} = \dfrac{d}{dx}2\tan\dfrac{\pi x}{4} = \left(2\sec^2\dfrac{\pi x}{4}\right)\dfrac{d}{dx}\left(\dfrac{\pi x}{4}\right)$

$= \dfrac{\pi}{2}\sec^2\left(\dfrac{\pi x}{4}\right)$

$y'(1) = \dfrac{\pi}{2}\sec^2\left(\dfrac{\pi}{4}\right) = \dfrac{\pi}{2}(\sqrt{2})^2 = \pi.$

The tangent line has slope $\pi$ and passes through

$\left(1, 2\tan\dfrac{\pi}{4}\right) = (1, 2)$. Its equation is $y = \pi(x-1) + 2$, or

$y = \pi x - \pi + 2.$

The normal line has slope $-\dfrac{1}{\pi}$ and passes through $(1, 2)$. Its

equation is $y = -\dfrac{1}{\pi}(x-1) + 2$, or $y = -\dfrac{1}{\pi}x + \dfrac{1}{\pi} + 2.$

Graphical support:



$[-4.7, 4.7]$ by $[-3.1, 3.1]$

**56. (a)** $\dfrac{d}{dx}[2f(x)] = 2f'(x)$

At $x = 2$, the derivative is $2f'(2) = 2\left(\dfrac{1}{3}\right) = \dfrac{2}{3}.$

**(b)** $\dfrac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$

At $x = 3$, the derivative is $f'(3) + g'(3) = 2\pi + 5.$

**(c)** $\dfrac{d}{dx}[f(x) \cdot g(x)] = f(x)g'(x) + g(x)f'(x)$

At $x = 3$, the derivative is

$f(3)g'(3) + g(3)f'(3) = (3)(5) + (-4)(2\pi)$

$= 15 - 8\pi.$

**(d)** $\dfrac{d}{dx}\dfrac{f(x)}{g(x)} = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

At $x = 2$, the derivative is

$\dfrac{g(2)f'(2) - f(2)g'(2)}{[g(2)]^2} = \dfrac{(2)\left(\dfrac{1}{3}\right) - (8)(-3)}{(2)^2}$

$= \dfrac{\dfrac{74}{3}}{4} = \dfrac{37}{6}.$

**(e)** $\dfrac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

At $x = 2$, the derivative is

$f'(g(2))g'(2) = f'(2)g'(2) = \left(\dfrac{1}{3}\right)(-3) = -1.$

**(f)** $\dfrac{d}{dx}\sqrt{f(x)} = \dfrac{1}{2\sqrt{f(x)}}\dfrac{d}{dx}f(x) = \dfrac{f'(x)}{2\sqrt{f(x)}}$

At $x = 2$, the derivative is

$\dfrac{f'(2)}{2\sqrt{f(2)}} = \dfrac{\dfrac{1}{3}}{2\sqrt{8}} = \dfrac{1}{6(2\sqrt{2})} = \dfrac{1}{12\sqrt{2}}.$

**(g)** $\dfrac{d}{dx}\dfrac{1}{g^2(x)} = \dfrac{d}{dx}[g(x)]^{-2} = -2[g(x)]^{-3}\dfrac{d}{dx}g(x) = -\dfrac{2g'(x)}{[g(x)]^3}$

At $x = 3$, the derivative is

$-\dfrac{2g'(3)}{[g(3)]^3} = -\dfrac{2(5)}{(-4)^3} = -\dfrac{10}{-64} = \dfrac{5}{32}.$

**(h)** $\dfrac{d}{dx}\sqrt{f^2(x) + g^2(x)} = \dfrac{1}{2\sqrt{f^2(x) + g^2(x)}}\dfrac{d}{dx}[f^2(x) + g^2(x)]$

$= \dfrac{1}{2\sqrt{f^2(x) + g^2(x)}}[2f(x)\dfrac{d}{dx}f(x) + 2g(x)\dfrac{d}{dx}g(x)]$

$= \dfrac{f(x)f'(x) + g(x)g'(x)}{\sqrt{f^2(x) + g^2(x)}}$

At $x = 2$, the derivative is

$\dfrac{f(2)f'(2) + g(2)g'(2)}{\sqrt{f^2(2) + g^2(2)}} = \dfrac{(8)\left(\dfrac{1}{3}\right) + (2)(-3)}{\sqrt{8^2 + 2^2}}$

$= \dfrac{-\dfrac{10}{3}}{\sqrt{68}} = -\dfrac{\dfrac{10}{3}}{2\sqrt{17}} = -\dfrac{5}{3\sqrt{17}}$

**57. (a)** $\dfrac{d}{dx}[5f(x) - g(x)] = 5f'(x) - g'(x)$

At $x = 1$, the derivative is

$5f'(1) - g'(1) = 5\left(-\dfrac{1}{3}\right) - \left(-\dfrac{8}{3}\right) = 1.$

**(b)** $\dfrac{d}{dx}f(x)g^3(x) = f(x)\dfrac{d}{dx}g^3(x) + g^3(x)\dfrac{d}{dx}f(x)$

$= f(x)[3g^2(x)]\dfrac{d}{dx}g(x) + g^3(x)f'(x)$

$= 3f(x)g^2(x)g'(x) + g^3(x)f'(x)$

At $x = 0$, the derivative is $3f(0)g^2(0)g'(0) + g^3(0)f'(0)$

$= 3(1)(1)^2\left(\dfrac{1}{3}\right) + (1)^3(5) = 6.$

**(c)** $\dfrac{d}{dx}\dfrac{f(x)}{g(x)+1} = \dfrac{[g(x)+1]\frac{d}{dx}f(x) - f(x)\frac{d}{dx}[g(x)+1]}{[g(x)+1]^2}$

$\qquad\qquad = \dfrac{[g(x)+1]f'(x) - f(x)g'(x)}{[g(x)+1]^2}$

At $x = 1$, the derivative is

$\dfrac{[g(1)+1]f'(1) - f(1)g'(1)}{[g(1)+1]^2} = \dfrac{(-4+1)\left(-\frac{1}{3}\right) - (3)\left(-\frac{8}{3}\right)}{(-4+1)^2}$

$\qquad\qquad = \dfrac{9}{9} = 1.$

**(d)** $\dfrac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

At $x = 0$, the derivative is

$f'(g(0))g'(0) = f'(1)g'(0) = \left(-\dfrac{1}{3}\right)\left(\dfrac{1}{3}\right) = -\dfrac{1}{9}.$

**(e)** $\dfrac{d}{dx}g(f(x)) = g'(f(x))f'(x)$

At $x = 0$, the derivative is

$g'(f(0))f'(0) = g'(1)f'(0) = \left(-\dfrac{8}{3}\right)(5) = -\dfrac{40}{3}$

**(f)** $\dfrac{d}{dx}[g(x)+f(x)]^{-2} = -2[g(x)+f(x)]^{-3}\dfrac{d}{dx}[g(x)+f(x)]$

$\qquad\qquad = -\dfrac{2[g'(x)+f'(x)]}{[g(x)+f(x)]^3}$

At $x = 1$, the derivative is

$-\dfrac{2[g'(1)+f'(1)]}{[g(1)+f(1)]^3} = -\dfrac{2\left(-\frac{8}{3}-\frac{1}{3}\right)}{(-4+3)^3} = -\dfrac{-6}{-1} = -6.$

**(g)** $\dfrac{d}{dx}[f(x+g(x))] = f'(x+g(x))\dfrac{d}{dx}[x+g(x)]$

$\qquad = f'(x+g(x))(1+g'(x))$

At $x = 0$, the derivative is

$f'(0+g(0))(1+g'(0)) = f'(0+1)\left(1+\dfrac{1}{3}\right)$

$\qquad\qquad = f'(1)\left(\dfrac{4}{3}\right)$

$\qquad\qquad = \left(-\dfrac{1}{3}\right)\left(\dfrac{4}{3}\right) = -\dfrac{4}{9}.$

**58.** For $y = \sin 2x$, $y' = (\cos 2x)\dfrac{d}{dx}(2x) = 2\cos 2x$ and the

slope at the origin is 2.

For $y = -\sin\dfrac{x}{2}$, $y' = \left(-\cos\dfrac{x}{2}\right)\dfrac{d}{dx}\left(\dfrac{x}{2}\right) = -\dfrac{1}{2}\cos\dfrac{x}{2}$ and the

slope at the origin is $-\dfrac{1}{2}$. Since the slopes of the two

tangent lines are 2 and $-\dfrac{1}{2}$, the lines are perpendicular and

the curves are orthogonal.

A graph of the two curves along with the tangents $y = 2x$

and $y = -\dfrac{1}{2}x$ is shown.
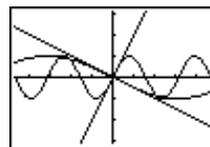


$[-4.7, 4.7]$ by $[-3.1, 3.1]$

**59.** Because the symbols $\dfrac{dy}{dx}$, $\dfrac{dy}{du}$, and $\dfrac{du}{dx}$ are not fractions. The

individual symbols $dy$, $dx$, and $du$ do not have numerical

values.

**60.** Velocity: $s'(t) = -2\pi bA \sin(2\pi bt)$
acceleration: $s''(t) = -4\pi^2 b^2 A \cos(2\pi bt)$
jerk: $s'''(t) = 8\pi^3 b^3 A \sin(2\pi bt)$
The velocity, amplitude, and jerk are proportional to $b$, $b^2$,
and $b^3$, respectively. If the frequency $b$ is doubled, then the
amplitude of the velocity is doubled, the amplitude of the
acceleration is quadrupled, and the amplitude of the jerk is
multiplied by 8.

**61. (a)** $y'(t) = \dfrac{d}{dt}37\sin\left[\dfrac{2\pi}{365}(x-101)\right] + \dfrac{d}{dt}(25)$

$\qquad = 37\cos\left[\dfrac{2\pi}{365}(x-101)\right]\cdot\dfrac{d}{dx}\left[\dfrac{2\pi}{365}(x-101)\right] + 0$

$\qquad = 37\cos\left[\dfrac{2\pi}{365}(x-101)\right]\cdot\dfrac{2\pi}{365}$

$\qquad = \dfrac{74\pi}{365}\cos\left[\dfrac{2\pi}{365}(x-101)\right]$

Since $\cos u$ is greatest when $u = 0, \pm 2\pi$, and so on,

$y'(t)$ is greatest when $\dfrac{2\pi}{365}(x-101) = 0$, or

$x = 101$. The temperature is increasing the fastest on

day 101 (April 11).

**(b)** The rate of increase is

$y'(101) = \dfrac{74\pi}{365} \approx 0.637$ degrees per day.

**62.** Velocity: $s'(t) = \dfrac{d}{dt}\sqrt{1+4t} = \dfrac{1}{2\sqrt{1+4t}}\dfrac{d}{dt}(1+4t)$

$$= \dfrac{4}{2\sqrt{1+4t}} = \dfrac{2}{\sqrt{1+4t}}$$

At $t = 6$, the velocity is $\dfrac{2}{\sqrt{1+4(6)}} = \dfrac{2}{5}$ m/sec

Acceleration: $s''(t) = \dfrac{d}{dt}\dfrac{2}{\sqrt{1+4t}}$

$$= \dfrac{(\sqrt{1+4t})\dfrac{d}{dt}(2) - 2\dfrac{d}{dt}\sqrt{1+4t}}{(\sqrt{1+4t})^2}$$

$$= \dfrac{-2\left(\dfrac{1}{2\sqrt{1+4t}}\right)\dfrac{d}{dt}(1+4t)}{1+4t}$$

$$= \dfrac{\dfrac{-4}{\sqrt{1+4t}}}{1+4t} = -\dfrac{4}{(1+4t)^{3/2}}$$

At $t = 6$, the acceleration is $-\dfrac{4}{[1+4(6)]^{3/2}} = -\dfrac{4}{125}$ m/sec$^2$

**63.** Acceleration $= \dfrac{dv}{dt} = \dfrac{dv}{ds}\dfrac{ds}{dt} = \left(\dfrac{dv}{ds}\right)(v) = \left[\dfrac{d}{ds}(k\sqrt{s})\right](k\sqrt{s})$

$= \left(\dfrac{k}{2\sqrt{s}}\right)(k\sqrt{s}) = \dfrac{k^2}{2}$, a constant.

**64.** Note that this Exercise concerns itself with the slowing

down caused by the earth's atmosphere, *not* the acceleration

caused by gravity.

Given: $v = \dfrac{k}{\sqrt{s}}$

Acceleration $= \dfrac{dv}{dt} = \dfrac{dv}{ds}\dfrac{ds}{dt} = \left(\dfrac{dv}{ds}\right)(v) = (v)\left(\dfrac{dv}{ds}\right)$

$$= \left(\dfrac{k}{\sqrt{s}}\right)\dfrac{d}{ds}\dfrac{k}{\sqrt{s}}$$

$$= \left(\dfrac{k}{\sqrt{s}}\right)\left(\dfrac{\sqrt{s}\dfrac{d}{ds}(k) - k\dfrac{d}{ds}\sqrt{s}}{(\sqrt{s})^2}\right)$$

$$= \left(\dfrac{k}{\sqrt{s}}\right)\left(\dfrac{\dfrac{-k}{(2\sqrt{s})}}{s}\right)$$

$$= -\dfrac{k^2}{2s^2}, \; s \geq 0$$

Thus, the acceleration is inversely proportional to $s^2$.

**65.** Acceleration $= \dfrac{dv}{dt} = \dfrac{df(x)}{dt} = \dfrac{df(x)}{dx}\dfrac{dx}{dt} = f'(x)f(x)$

**66.** $\dfrac{dT}{du} = \dfrac{dT}{dL}\dfrac{dL}{du} = \left(\dfrac{d}{dL}2\pi\sqrt{\dfrac{L}{g}}\right)(kL)$

$$= \left(2\pi\dfrac{1}{2\sqrt{\dfrac{L}{g}}}\right)\left(\dfrac{d}{dL}\dfrac{L}{g}\right)(kL)$$

$$= \left(\dfrac{\pi}{\sqrt{\dfrac{L}{g}}}\right)\left(\dfrac{1}{g}\right)(kL) = k\pi\sqrt{\dfrac{L}{g}} = \dfrac{kT}{2}$$

**67.** No, this does not contradict the Chain Rule. The Chain Rule states that if two functions are differentiable at the appropriate points, then their composite must also be differentiable. It does not say: If a composite is differentiable, then the functions which make up the composite must all be differentiable.

**68.** Yes. Note that $\dfrac{d}{dx}f(g(x)) = f'(g(x))g'(x)$. If the graph of $y = f(g(x))$ has a horizontal tangent at $x = 1$, then $f'(g(1))g'(1) = 0$, so either $g'(1) = 0$ or $f'(g(1)) = 0$. This means that either the graph of $y = g(x)$ has a horizontal tangent at $x = 1$, or the graph of $y = f(u)$ has a horizontal tangent at $u = g(1)$.

**69.** For $h = 1$:
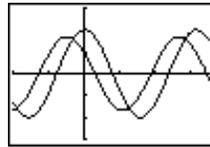


$[-2, 3.5]$ by $[-3, 3]$

For $h = 0.5$:
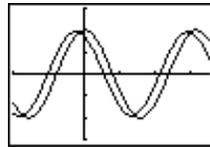


$[-2, 3.5]$ by $[-3, 3]$

For $h = 0.2$:
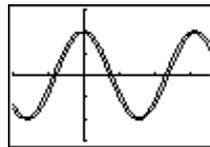


$[-2, 3.5]$ by $[-3, 3]$

As $h \to 0$, the second curve (the difference quotient) approaches the first ($y = 2\cos 2x$). This is because $2\cos 2x$ is the derivative of $\sin 2x$, and the second curve is the difference quotient used to define the derivative of $\sin 2x$. As $h \to 0$, the difference quotient expression should be approaching the derivative.

**70.** For $h = 1$:
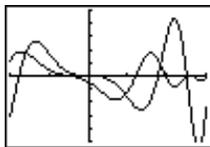


[−2, 3] by [−5, 5]

For $h = 0.7$:
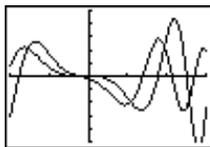


[−2, 3] by [−5, 5]

For $h = 0.3$:
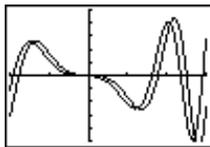


[−2, 3] by [−5, 5]

As $h \to 0$, the second curve (the difference quotient) approaches the first ($y = -2x \sin(x^2)$). This is because $-2x \sin(x^2)$ is the derivative of $\cos(x^2)$, and the second curve is the difference quotient used to define the derivative of $\cos(x^2)$. As $h \to 0$, the difference quotient expression should be approaching the derivative.

**71. (a)** Let $f(x) = |x|$.

Then

$$\frac{d}{dx}|u| = \frac{d}{dx} f(u) = f'(u) \frac{du}{dx} = \left(\frac{d}{du}|u|\right)\left(\frac{du}{dx}\right) = \frac{u}{|u|} u'.$$

The derivative of the absolute value function is $+1$ for

positive values, $-1$ for negative values, and undefined

at 0. So $f'(u) = \begin{cases} -1, & u < 0 \\ 1, & u > 0. \end{cases}$

But this is exactly how the expression $\frac{u}{|u|}$ evaluates.

**(b)** $f'(x) = \left[\frac{d}{dx}(x^2 - 9)\right] \cdot \frac{x^2 - 9}{|x^2 - 9|} = \frac{(2x)(x^2 - 9)}{|x^2 - 9|}$

$g'(x) = \frac{d}{dx}(|x| \sin x)$

$= |x| \frac{d}{dx}(\sin x) + (\sin x) \frac{d}{dx}|x|$

$= |x| \cos x + \frac{x \sin x}{|x|}$

Note: The expression for $g'(x)$ above is undefined at

$x = 0$, but actually

$g'(0) = \lim_{h \to 0} \frac{g(0 + h) - g(0)}{h} = \lim_{h \to 0} \frac{|h| \sin h}{h} = 0.$

Therefore, we may express the derivative as

$g'(x) = \begin{cases} |x| \cos x + \frac{x \sin x}{|x|}, & x \neq 0 \\ 0, & x = 0. \end{cases}$

**72.** $\dfrac{dG}{dx} = \dfrac{d}{dx} \sqrt{uv} = \dfrac{d}{dx} \sqrt{x(x + c)} = \dfrac{d}{dx} \sqrt{x^2 + cx}$

$= \dfrac{1}{2\sqrt{x^2 + cx}} \dfrac{d}{dx}(x^2 + cx) = \dfrac{2x + c}{2\sqrt{x^2 + cx}} = \dfrac{x + (x + c)}{2\sqrt{x(x + c)}}$

$= \dfrac{u + v}{2\sqrt{uv}} = \dfrac{A}{G}$

# ■ Section 3.7 Implicit Differentiation
(pp. 149–157)

### Exploration 1    An Unexpected Derivative

**1.** $2x - 2y - 2xy' + 2yy' = 0$. Solving for $y'$, we find that $\frac{dy}{dx} = 1$ (provided $y \neq x$).

**2.** With a constant derivative of 1, the graph would seem to be a line with slope 1.

**3.** Letting $x = 0$ in the original equation, we find that $y = \pm 2$. This would seem to indicate that this equation defines two lines implicitly, both with slope 1. The two lines are $y = x + 2$ and $y = x - 2$.

**4.** Factoring the original equation, we have
$[(x - y) - 2][(x - y) + 2] = 0$
$\therefore x - y - 2 = 0$ or $x - y + 2 = 0$
$\therefore y = x - 2$ or $y = x + 2$.
The graph is shown below.
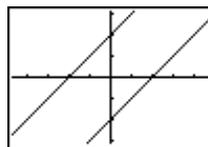


[−4.7, 4.7] by [−3.1, 3.1]

**5.** At each point $(x, y)$ on either line, $\frac{dy}{dx} = 1$. The condition $y \neq x$ is true because both lines are parallel to the line $y = x$. The derivative is surprising because it does not depend on $x$ or $y$, but there are no inconsistencies.

### Quick Review 3.7

**1.** $x - y^2 = 0$
$x = y^2$
$\pm\sqrt{x} = y$
$y_1 = \sqrt{x}, y_2 = -\sqrt{x}$
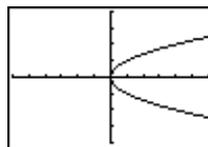


[−6, 6] by [−4, 4]

**2.** $4x^2 + 9y^2 = 36$

$$9y^2 = 36 - 4x^2$$

$$y^2 = \frac{36 - 4x^2}{9} = \frac{4}{9}(9 - x^2)$$

$$y = \pm\frac{2}{3}\sqrt{9 - x^2}$$

$$y_1 = \frac{2}{3}\sqrt{9 - x^2}, \, y_2 = -\frac{2}{3}\sqrt{9 - x^2}$$
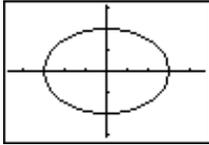


[−4.7, 4.7] by [−3.1, 3.1]

**3.** $x^2 - 4y^2 = 0$

$$(x + 2y)(x - 2y) = 0$$

$$y = \pm\frac{x}{2}$$

$$y_1 = \frac{x}{2}, \, y_2 = -\frac{x}{2}$$
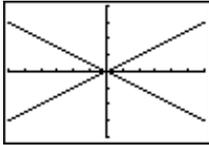


[−6, 6] by [−4, 4]

**4.** $x^2 + y^2 = 9$

$$y^2 = 9 - x^2$$

$$y = \pm\sqrt{9 - x^2}$$

$$y_1 = \sqrt{9 - x^2}, \, y_2 = -\sqrt{9 - y^2}$$
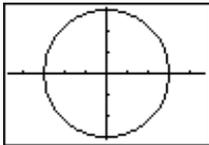


[−4.7, 4.7] by [−3.1, 3.1]

**5.** $x^2 + y^2 = 2x + 3$

$$y^2 = 2x + 3 - x^2$$

$$y = \pm\sqrt{2x + 3 - x^2}$$

$$y_1 = \sqrt{2x + 3 - x^2}, \, y_2 = -\sqrt{2x + 3 - x^2}$$
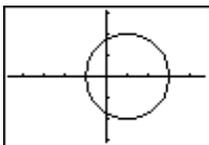


[−4.7, 4.7] by [−3.1, 3.1]

**6.** $x^2 y' - 2xy = 4x - y$

$$x^2 y' = 4x - y + 2xy$$

$$y' = \frac{4x - y + 2xy}{x^2}$$

**7.** $y' \sin x - x \cos x = xy' + y$

$$y' \sin x - xy' = y + x \cos x$$

$$(\sin x - x)y' = y + x \cos x$$

$$y' = \frac{y + x \cos x}{\sin x - x}$$

**8.** $x(y^2 - y') = y'(x^2 - y)$

$$xy^2 = y'(x^2 - y + x)$$

$$y' = \frac{xy^2}{x^2 - y + x}$$

**9.** $\sqrt{x}(x - \sqrt[3]{x}) = x^{1/2}(x - x^{1/3})$

$$= x^{1/2}x - x^{1/2}x^{1/3}$$

$$= x^{3/2} - x^{5/6}$$

**10.** $\dfrac{x + \sqrt[3]{x^2}}{\sqrt{x^3}} = \dfrac{x + x^{2/3}}{x^{3/2}}$

$$= \frac{x}{x^{3/2}} + \frac{x^{2/3}}{x^{3/2}}$$

$$= x^{-1/2} + x^{-5/6}$$

## Section 3.7 Exercises

**1.** $\dfrac{dy}{dx} = \dfrac{d}{dx}x^{9/4} = \dfrac{9}{4}x^{(9/4)-1} = \dfrac{9}{4}x^{5/4}$

**2.** $\dfrac{dy}{dx} = \dfrac{d}{dx}x^{-3/5} = -\dfrac{3}{5}x^{(-3/5)-1} = -\dfrac{3}{5}x^{-8/5}$

**3.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\sqrt[3]{x} = \dfrac{d}{dx}x^{1/3} = \dfrac{1}{3}x^{(1/3)-1} = \dfrac{1}{3}x^{-2/3}$

**4.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\sqrt[4]{x} = \dfrac{d}{dx}x^{1/4} = \dfrac{1}{4}x^{(1/4)-1} = \dfrac{1}{4}x^{-3/4}$

**5.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(2x + 5)^{-1/2} = -\dfrac{1}{2}(2x + 5)^{(-1/2)-1}\dfrac{d}{dx}(2x + 5)$

$$= -\frac{1}{2}(2x + 5)^{-3/2}(2) = -(2x + 5)^{-3/2}$$

**6.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(1 - 6x)^{2/3}$

$$= \frac{2}{3}(1 - 6x)^{(2/3)-1}\frac{d}{dx}(1 - 6x)$$

$$= \frac{2}{3}(1 - 6x)^{-1/3}(-6)$$

$$= -4(1 - 6x)^{-1/3}$$

**7.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(x\sqrt{x^2 + 1})$

$$= x\frac{d}{dx}\sqrt{x^2 + 1} + \sqrt{x^2 + 1}\frac{d}{dx}(x)$$

$$= x\frac{d}{dx}(x^2 + 1)^{1/2} + (x^2 + 1)^{1/2}$$

$$= x \cdot \frac{1}{2}(x^2 + 1)^{-1/2}(2x) + (x^2 + 1)^{1/2}$$

$$= x^2(x^2 + 1)^{-1/2} + (x^2 + 1)^{1/2}$$

Note: This answer is equivalent to $\dfrac{2x^2 + 1}{\sqrt{x^2 + 1}}$.

**8.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\dfrac{x}{\sqrt{x^2+1}} = \dfrac{(x^2+1)^{1/2}\frac{d}{dx}x - x\frac{d}{dx}(x^2+1)^{1/2}}{x^2+1}$

$= \dfrac{(x^2+1)^{1/2} - x \cdot \frac{1}{2}(x^2+1)^{-1/2}(2x)}{x^2+1}$

$= \dfrac{x^2+1-x^2}{(x^2+1)(x^2+1)^{1/2}}$

$= \dfrac{1}{(x^2+1)^{3/2}}$

$= (x^2+1)^{-3/2}$

**9.**                     $x^2y + xy^2 = 6$

$\dfrac{d}{dx}(x^2y) + \dfrac{d}{dx}(xy^2) = \dfrac{d}{dx}(6)$

$x^2\dfrac{dy}{dx} + y(2x) + x(2y)\dfrac{dy}{dx} + y^2(1) = 0$

$x^2\dfrac{dy}{dx} + 2xy\dfrac{dy}{dx} = -(2xy + y^2)$

$(2xy + x^2)\dfrac{dy}{dx} = -(2xy + y^2)$

$\dfrac{dy}{dx} = -\dfrac{2xy + y^2}{2xy + x^2}$

**10.**        $x^3 + y^3 = 18xy$

$\dfrac{d}{dx}(x^3) + \dfrac{d}{dx}(y^3) = \dfrac{d}{dx}(18xy)$

$3x^2 + 3y^2\dfrac{dy}{dx} = 18x\dfrac{dy}{dx} + 18y\,(1)$

$3y^2\dfrac{dy}{dx} - 18x\dfrac{dy}{dx} = 18y - 3x^2$

$(3y^2 - 18x)\dfrac{dy}{dx} = 18y - 3x^2$

$\dfrac{dy}{dx} = \dfrac{18y - 3x^2}{3y^2 - 18x}$

$\dfrac{dy}{dx} = \dfrac{6y - x^2}{y^2 - 6x}$

**11.**   $y^2 = \dfrac{x-1}{x+1}$

$\dfrac{d}{dx}y^2 = \dfrac{d}{dx}\dfrac{x-1}{x+1}$

$2y\dfrac{dy}{dx} = \dfrac{(x+1)(1) - (x-1)(1)}{(x+1)^2}$

$2y\dfrac{dy}{dx} = \dfrac{2}{(x+1)^2}$

$\dfrac{dy}{dx} = \dfrac{1}{y(x+1)^2}$

**12.**        $x^2 = \dfrac{x-y}{x+y}$

$\dfrac{d}{dx}(x^2) = \dfrac{d}{dx}\dfrac{x-y}{x+y}$

$2x = \dfrac{(x+y)\left(1 - \frac{dy}{dx}\right) - (x-y)\left(1 + \frac{dy}{dx}\right)}{(x+y)^2}$

$2x = \dfrac{\left[x - x\frac{dy}{dx} + y - y\frac{dy}{dx}\right] - \left[x + x\frac{dy}{dx} - y - y\frac{dy}{dx}\right]}{(x+y)^2}$

$2x = \dfrac{2y - 2x\frac{dy}{dx}}{(x+y)^2}$

$x(x+y)^2 = y - x\dfrac{dy}{dx}$

$x\dfrac{dy}{dx} = y - x(x+y)^2$

$\dfrac{dy}{dx} = \dfrac{y - x(x+y)^2}{x} = \dfrac{y}{x} - (x+y)^2$

Alternate solution:

$x^2 = \dfrac{x-y}{x+y}$

$x^2(x+y) = x - y$

$x^3 + x^2y = x - y$

$\dfrac{d}{dx}(x^3) + \dfrac{d}{dx}(x^2y) = \dfrac{d}{dx}(x) - \dfrac{d}{dx}(y)$

$3x^2 + x^2\dfrac{dy}{dx} + y(2x) = 1 - \dfrac{dy}{dx}$

$(x^2+1)\dfrac{dy}{dx} = 1 - 3x^2 - 2xy$

$\dfrac{dy}{dx} = \dfrac{1 - 3x^2 - 2xy}{x^2+1}$

**13.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(1 - x^{1/2})^{1/2}$

$= \dfrac{1}{2}(1 - x^{1/2})^{-1/2}\dfrac{d}{dx}(1 - x^{1/2})$

$= \dfrac{1}{2}(1 - x^{1/2})^{-1/2}\left(-\dfrac{1}{2}x^{-1/2}\right)$

$= -\dfrac{1}{4}(1 - x^{1/2})^{-1/2}x^{-1/2}$

**14.** $\dfrac{dy}{dx} = \dfrac{d}{dx}3(2x^{-1/2} + 1)^{-1/3}$

$= -(2x^{-1/2} + 1)^{-4/3}\dfrac{d}{dx}(2x^{-1/2} + 1)$

$= -(2x^{-1/2} + 1)^{-4/3}(-x^{-3/2})$

$= x^{-3/2}(2x^{-1/2} + 1)^{-4/3}$

**15.** $\dfrac{dy}{dx} = \dfrac{d}{dx}3(\csc x)^{3/2}$

$= \dfrac{9}{2}(\csc x)^{1/2}\dfrac{d}{dx}(\csc x)$

$= \dfrac{9}{2}(\csc x)^{1/2}(-\csc x \cot x)$

$= -\dfrac{9}{2}(\csc x)^{3/2}\cot x$

**16.** $\dfrac{dy}{dx} = \dfrac{d}{dx}[\sin{(x+5)}]^{5/4}$

$\quad = \dfrac{5}{4}[\sin{(x+5)}]^{1/4}\dfrac{d}{dx}\sin{(x+5)}$

$\quad = \dfrac{5}{4}[\sin{(x+5)}]^{1/4}\cos{(x+5)}$

**17.** $\qquad x = \tan y$

$\dfrac{d}{dx}(x) = \dfrac{d}{dx}(\tan y)$

$\quad 1 = \sec^2 y\dfrac{dy}{dx}$

$\quad \dfrac{dy}{dx} = \dfrac{1}{\sec^2 y} = \cos^2 y$

**18.** $\qquad x = \sin y$

$\dfrac{d}{dx}(x) = \dfrac{d}{dx}(\sin y)$

$\quad 1 = \cos y\dfrac{dy}{dx}$

$\quad \dfrac{dy}{dx} = \dfrac{1}{\cos y} = \sec y$

**19.** $\qquad\qquad x + \tan xy = 0$

$\dfrac{d}{dx}(x) + \dfrac{d}{dx}(\tan xy) = 0$

$1 + \sec^2(xy)\dfrac{d}{dx}(xy) = 0$

$1 + (\sec^2 xy)[x\dfrac{dy}{dx} + (y)(1)] = 0$

$(\sec^2 xy)(x)\dfrac{dy}{dx} = -1 - (\sec^2 xy)(y)$

$\qquad \dfrac{dy}{dx} = \dfrac{-1 - y\sec^2 xy}{x\sec^2 xy}$

$\qquad \dfrac{dy}{dx} = -\dfrac{1}{x}\cos^2 xy - \dfrac{y}{x}$

**20.** $\qquad\quad x + \sin y = xy$

$\dfrac{d}{dx}(x) + \dfrac{d}{dx}(\sin y) = \dfrac{d}{dx}(xy)$

$\quad 1 + (\cos y)\dfrac{dy}{dx} = x\dfrac{dy}{dx} + (y)(1)$

$\quad (\cos y - x)\dfrac{dy}{dx} = -1 + y$

$\qquad \dfrac{dy}{dx} = \dfrac{-1+y}{\cos y - x} = \dfrac{1-y}{x - \cos y}$

**21. (a)** If $f(x) = \dfrac{3}{2}x^{2/3} - 3$, then

$f'(x) = x^{-1/3}$ and $f''(x) = -\dfrac{1}{3}x^{-4/3}$

which contradicts the given equation $f''(x) = x^{-1/3}$.

**(b)** If $f(x) = \dfrac{9}{10}x^{5/3} - 7$, then

$f'(x) = \dfrac{3}{2}x^{2/3}$ and $f''(x) = x^{-1/3}$,

which matches the given equation.

**(c)** Differentiating both sides of the given equation

$f''(x) = x^{-1/3}$ gives $f'''(x) = -\dfrac{1}{3}x^{-4/3}$, so it *must* be true

that $f'''(x) = -\dfrac{1}{3}x^{-4/3}$.

**(d)** If $f'(x) = \dfrac{3}{2}x^{2/3} + 6$, then $f''(x) = x^{-1/3}$, which matches

the given equation.

Conclusion: (b), (c), and (d) could be true.

**22. (a)** If $g'(t) = 4\sqrt[4]{t} - 4$, then

$g''(t) = \dfrac{d}{dx}(4t^{1/4} - 4) = t^{-3/4} = \dfrac{1}{t^{3/4}}$, which matches the

given equation.

**(b)** Differentiating both sides of the given equation

$g''(t) = \dfrac{1}{t^{3/4}} = t^{-3/4}$ gives $g'''(t) = -\dfrac{3}{4}t^{-7/4}$, which is not

consistent with $g'''(t) = -\dfrac{4}{\sqrt[4]{t}}$.

**(c)** If $g(t) = t - 7 + \dfrac{16}{5}t^{5/4}$, then $g'(t) = 1 + 4t^{1/4}$ and

$g''(t) = t^{-3/4} = \dfrac{1}{t^{3/4}}$, which matches the given

equation.

**(d)** If $g'(t) = \dfrac{1}{4}t^{1/4}$, then $g''(t) = \dfrac{1}{16}t^{-3/4}$, which

contradicts the given equation.

Conclusion: (a) and (c) could be true.

**23.** $\qquad\qquad x^2 + y^2 = 1$

$\dfrac{d}{dx}(x^2) + \dfrac{d}{dx}(y^2) = \dfrac{d}{dx}(1)$

$\qquad 2x + 2yy' = 0$

$\qquad\quad 2yy' = -2x$

$\qquad\qquad y' = -\dfrac{x}{y}$

$\qquad\qquad y'' = \dfrac{d}{dx}\left(-\dfrac{x}{y}\right)$

$\qquad\qquad\quad = -\dfrac{(y)(1) - (x)(y')}{y^2}$

$\qquad\qquad\quad = -\dfrac{y - x\left(-\dfrac{x}{y}\right)}{y^2}$

$\qquad\qquad\quad = -\dfrac{x^2 + y^2}{y^3}$

Since our original equation was $x^2 + y^2 = 1$, we may

substitute 1 for $x^2 + y^2$, giving $y'' = -\dfrac{1}{y^3}$.

**24.**     $x^{2/3} + y^{2/3} = 1$

$$\frac{d}{dx}(x^{2/3}) + \frac{d}{dx}(y^{2/3}) = \frac{d}{dx}(1)$$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$$

$$y' = -\frac{x^{-1/3}}{y^{-1/3}} = -\left(\frac{y}{x}\right)^{1/3}$$

$$y'' = \frac{d}{dx}\left[-\left(\frac{y}{x}\right)^{1/3}\right]$$

$$= -\frac{1}{3}\left(\frac{y}{x}\right)^{-2/3}\frac{d}{dx}\left(\frac{y}{x}\right)$$

$$= -\frac{1}{3}\left(\frac{y}{x}\right)^{-2/3}\frac{xy' - (y)(1)}{x^2}$$

$$= -\frac{1}{3}\frac{-(x)\left(\frac{y}{x}\right)^{1/3} - y}{x^{4/3}y^{2/3}}$$

$$= \frac{1}{3}\frac{x^{2/3}y^{1/3} + y}{x^{4/3}y^{2/3}}$$

$$= \frac{x^{2/3} + y^{2/3}}{3x^{4/3}y^{1/3}}$$

Since our original equation was $x^{2/3} + y^{2/3} = 1$, we may

substitute 1 for $x^{2/3} + y^{2/3}$, giving $y'' = \frac{1}{3x^{4/3}y^{1/3}}$.

**25.**     $y^2 = x^2 + 2x$

$$\frac{d}{dy}(y^2) = \frac{d}{dy}(x^2) + \frac{d}{dy}(2x)$$

$$2yy' = 2x + 2$$

$$y' = \frac{2x + 2}{2y} = \frac{x + 1}{y}$$

$$y'' = \frac{d}{dx}\frac{x + 1}{y}$$

$$= \frac{(y)(1) - (x + 1)y'}{y^2}$$

$$= \frac{y - (x + 1)\left(\frac{x + 1}{y}\right)}{y^2}$$

$$= \frac{y^2 - (x + 1)^2}{y^3}$$

Since our original equation was $y^2 = x^2 + 2x$, we may
write $y^2 - (x + 1)^2 = (x^2 + 2x) - (x^2 + 2x + 1) = -1$,
which gives $y'' = -\frac{1}{y^3}$.

**26.** $y^2 + 2y = 2x + 1$

$$\frac{d}{dx}(y^2 + 2y) = \frac{d}{dx}(2x + 1)$$

$$(2y + 2)y' = 2$$

$$y' = \frac{1}{y + 1}$$

$$y'' = \frac{d}{dx}\frac{1}{y + 1}$$

$$= -(y + 1)^{-2}y'$$

$$= -(y + 1)^{-2}\left(\frac{1}{y + 1}\right)$$

$$= -\frac{1}{(y + 1)^3}$$

**27.**     $x^2 + xy - y^2 = 1$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(xy) - \frac{d}{dx}(y^2) = \frac{d}{dx}(1)$$

$$2x + x\frac{dy}{dx} + (y)(1) - 2y\frac{dy}{dx} = 0$$

$$(x - 2y)\frac{dy}{dx} = -2x - y$$

$$\frac{dy}{dx} = \frac{-2x - y}{x - 2y} = \frac{2x + y}{2y - x}$$

Slope at (2, 3): $\frac{2(2) + 3}{2(3) - 2} = \frac{7}{4}$

**(a)** Tangent: $y = \frac{7}{4}(x - 2) + 3$ or $y = \frac{7}{4}x - \frac{1}{2}$

**(b)** Normal: $y = -\frac{4}{7}(x - 2) + 3$ or $y = -\frac{4}{7}x + \frac{29}{7}$

**28.**     $x^2 + y^2 = 25$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Slope at (3, −4): $-\frac{3}{-4} = \frac{3}{4}$

**(a)** Tangent: $y = \frac{3}{4}(x - 3) + (-4)$ or $y = \frac{3}{4}x - \frac{25}{4}$

**(b)** Normal: $y = -\frac{4}{3}(x - 3) + (-4)$ or $y = -\frac{4}{3}x$

**29.**     $x^2y^2 = 9$

$$\frac{d}{dx}(x^2y^2) = \frac{d}{dx}(9)$$

$$(x^2)(2y)\frac{dy}{dx} + (y^2)(2x) = 0$$

$$2x^2y\frac{dy}{dx} = -2xy^2$$

$$\frac{dy}{dx} = -\frac{2xy^2}{2x^2y} = -\frac{y}{x}$$

Slope at (−1, 3): $-\frac{3}{-1} = 3$

**(a)** Tangent: $y = 3(x + 1) + 3$ or $y = 3x + 6$

**(b)** Normal: $y = -\frac{1}{3}(x + 1) + 3$ or $y = -\frac{1}{3}x + \frac{8}{3}$

**30.**     $y^2 - 2x - 4y - 1 = 0$

$$\frac{d}{dx}(y^2) - \frac{d}{dx}(2x) - \frac{d}{dx}(4y) - \frac{d}{dx}(1) = \frac{d}{dx}(0)$$

$$2y\frac{dy}{dx} - 2 - 4\frac{dy}{dx} - 0 = 0$$

$$(2y - 4)\frac{dy}{dx} = 2$$

$$\frac{dy}{dx} = \frac{1}{y - 2}$$

Slope at (−2, 1): $\frac{1}{1 - 2} = -1$

**(a)** Tangent: $y = -(x + 2) + 1$ or $y = -x - 1$

**(b)** Normal: $y = 1(x + 2) + 1$ or $y = x + 3$

**31.**
$$6x^2 + 3xy + 2y^2 + 17y - 6 = 0$$

$$\frac{d}{dx}(6x^2) + \frac{d}{dx}(3xy) + \frac{d}{dx}(2y^2) + \frac{d}{dx}(17y) - \frac{d}{dx}(6) = \frac{d}{dx}(0)$$

$$12x + 3x\frac{dy}{dx} + (3y)(1) + 4y\frac{dy}{dx} + 17\frac{dy}{dx} - 0 = 0$$

$$3x\frac{dy}{dx} + 4y\frac{dy}{dx} + 17\frac{dy}{dx} = -12x - 3y$$

$$(3x + 4y + 17)\frac{dy}{dx} = -12x - 3y$$

$$\frac{dy}{dx} = \frac{-12x - 3y}{3x + 4y + 17}$$

Slope at $(-1, 0)$: $\dfrac{-12(-1) - 3(0)}{3(-1) + 4(0) + 17} = \dfrac{12}{14} = \dfrac{6}{7}$

**(a)** Tangent: $y = \dfrac{6}{7}(x + 1) + 0$ or $y = \dfrac{6}{7}x + \dfrac{6}{7}$

**(b)** Normal: $y = -\dfrac{7}{6}(x + 1) + 0$ or $y = -\dfrac{7}{6}x - \dfrac{7}{6}$

**32.**
$$x^2 - \sqrt{3}xy + 2y^2 = 5$$

$$\frac{d}{dx}(x^2) - \sqrt{3}\frac{d}{dx}(xy) + 2\frac{d}{dx}(y^2) = \frac{d}{dx}(5)$$

$$2x - \sqrt{3}(x)\frac{dy}{dx} - \sqrt{3}(y)(1) + 4y\frac{dy}{dx} = 0$$

$$(-x\sqrt{3} + 4y)\frac{dy}{dx} = y\sqrt{3} - 2x$$

$$\frac{dy}{dx} = \frac{y\sqrt{3} - 2x}{-x\sqrt{3} + 4y}$$

Slope at $(\sqrt{3}, 2)$: $\dfrac{2\sqrt{3} - 2\sqrt{3}}{-\sqrt{3}\sqrt{3} + 4(2)} = 0$

**(a)** Tangent: $y = 2$

**(b)** Normal: $x = \sqrt{3}$

**33.**
$$2xy + \pi \sin y = 2\pi$$

$$2\frac{d}{dx}(xy) + \pi\frac{d}{dx}(\sin y) = \frac{d}{dx}(2\pi)$$

$$2x\frac{dy}{dx} + 2y(1) + \pi \cos y \frac{dy}{dx} = 0$$

$$(2x + \pi \cos y)\frac{dy}{dx} = -2y$$

$$\frac{dy}{dx} = -\frac{2y}{2x + \pi \cos y}$$

Slope at $\left(1, \dfrac{\pi}{2}\right)$: $-\dfrac{2(\pi/2)}{2(1) + \pi \cos(\pi/2)} = -\dfrac{\pi}{2}$

**(a)** Tangent: $y = -\dfrac{\pi}{2}(x - 1) + \dfrac{\pi}{2}$ or $y = -\dfrac{\pi}{2}x + \pi$

**(b)** Normal: $y = \dfrac{2}{\pi}(x - 1) + \dfrac{\pi}{2}$ or $y = \dfrac{2}{\pi}x - \dfrac{2}{\pi} + \dfrac{\pi}{2}$

**34.** $x \sin 2y = y \cos 2x$

$$\frac{d}{dx}(x \sin 2y) = \frac{d}{dx}(y \cos 2x)$$

$$(x)(\cos 2y)(2)\frac{dy}{dx} + (\sin 2y)(1) =$$

$$(y)(-\sin 2x)(2) + (\cos 2x)\left(\frac{dy}{dx}\right)$$

$$(2x \cos 2y)\frac{dy}{dx} - (\cos 2x)\frac{dy}{dx} = -2y \sin 2x - \sin 2y$$

$$\frac{dy}{dx} = -\frac{2y \sin 2x + \sin 2y}{2x \cos 2y - \cos 2x}$$

Slope at $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$: $-\dfrac{2\left(\dfrac{\pi}{2}\right)\sin\left(\dfrac{\pi}{2}\right) + \sin(\pi)}{2\left(\dfrac{\pi}{4}\right)\cos(\pi) - \cos\left(\dfrac{\pi}{2}\right)}$

$$= -\frac{(\pi)(1) + 0}{\left(\dfrac{\pi}{2}\right)(-1) - 0} = 2$$

**(a)** Tangent: $y = 2\left(x - \dfrac{\pi}{4}\right) + \dfrac{\pi}{2}$ or $y = 2x$

**(b)** Normal: $y = -\dfrac{1}{2}\left(x - \dfrac{\pi}{4}\right) + \dfrac{\pi}{2}$ or $y = -\dfrac{1}{2}x + \dfrac{5\pi}{8}$

**35.**
$$y = 2 \sin(\pi x - y)$$

$$\frac{dy}{dx} = \frac{d}{dx} 2 \sin(\pi x - y)$$

$$\frac{dy}{dx} = 2 \cos(\pi x - y)\left(\pi - \frac{dy}{dx}\right)$$

$$[1 + 2 \cos(\pi x - y)]\frac{dy}{dx} = 2\pi \cos(\pi x - y)$$

$$\frac{dy}{dx} = \frac{2\pi \cos(\pi x - y)}{1 + 2 \cos(\pi x - y)}$$

Slope at $(1, 0)$: $\dfrac{2\pi \cos \pi}{1 + 2 \cos \pi} = \dfrac{2\pi(-1)}{1 + 2(-1)} = 2\pi$

**(a)** Tangent: $y = 2\pi(x - 1) + 0$ or $y = 2\pi x - 2\pi$

**(b)** Normal: $y = -\dfrac{1}{2\pi}(x - 1) + 0$ or $y = -\dfrac{x}{2\pi} + \dfrac{1}{2\pi}$

**36.** $x^2 \cos^2 y - \sin y = 0$

$$\frac{d}{dx}(x^2 \cos^2 y) - \frac{d}{dx}(\sin y) = \frac{d}{dx}(0)$$

$$(x^2)(2 \cos y)(-\sin y)\left(\frac{dy}{dx}\right) + (\cos^2 y)(2x) - (\cos y)\frac{dy}{dx} = 0$$

$$-(2x^2 \cos y \sin y + \cos y)\frac{dy}{dx} = -2x \cos^2 y$$

$$\frac{dy}{dx} = \frac{2x \cos^2 y}{\cos y + 2x^2 \cos y \sin y} = \frac{2x \cos y}{1 + 2x^2 \sin y}$$

Slope at $(0, \pi)$: $\dfrac{2(0) \cos \pi}{1 + 2(0)^2 \sin \pi} = 0$

**(a)** Tangent: $y = \pi$

**(b)** Normal: $x = 0$

**37. (a)**
$$y^4 = y^2 - x^2$$

$$\frac{d}{dx}(y^4) = \frac{d}{dx}(y^2) - \frac{d}{dx}x^2$$

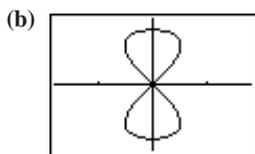$$4y^3\frac{dy}{dx} = 2y\frac{dy}{dx} - 2x$$

$$(4y^3 - 2y)\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{4y^3 - 2y} = \frac{x}{y - 2y^3}$$

At $\left(\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{2}\right)$:

$$\text{Slope} = \frac{\dfrac{\sqrt{3}}{4}}{\dfrac{\sqrt{3}}{2} - 2\left(\dfrac{\sqrt{3}}{2}\right)^3}$$

$$= \frac{\dfrac{\sqrt{3}}{4}}{\dfrac{\sqrt{3}}{2} - \dfrac{3\sqrt{3}}{4}} \cdot \frac{\dfrac{4}{\sqrt{3}}}{\dfrac{4}{\sqrt{3}}} = \frac{1}{2 - 3} = -1$$

At $\left(\frac{\sqrt{3}}{4}, \frac{1}{2}\right)$:

$$\text{Slope} = \frac{\dfrac{\sqrt{3}}{4}}{\dfrac{1}{2} - 2\left(\dfrac{1}{2}\right)^3} = \frac{\dfrac{\sqrt{3}}{4}}{\dfrac{1}{2} - \dfrac{1}{4}} \cdot \frac{4}{4} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

**(b)**



[−1.8, 1.8] by [−1.2, 1.2]

Parameter interval: $-1 \le t \le 1$

**38. (a)**
$$y^2(2 - x) = x^3$$

$$\frac{d}{dx}[y^2(2 - x)] = \frac{d}{dx}(x^3)$$

$$(y^2)(-1) + (2 - x)(2y)\frac{dy}{dx} = 3x^2$$

$$2y(2 - x)\frac{dy}{dx} = 3x^2 + y^2$$

$$\frac{dy}{dx} = \frac{3x^2 + y^2}{2y(2 - x)}$$

Slope at $(1, 1)$: $\dfrac{3(1)^2 + (1)^2}{2(1)(2 - 1)} = \dfrac{4}{2} = 2$

Tangent: $y = 2(x - 1) + 1$ or $y = 2x - 1$

Normal: $y = -\dfrac{1}{2}(x - 1) + 1$ or $y = -\dfrac{1}{2}x + \dfrac{3}{2}$

**(b)** One way is to graph the equations $y = \pm\sqrt{\dfrac{x^3}{2 - x}}$.

**39. (a)** $(-1)^3(1)^2 = \cos(\pi)$ is true since both sides equal $-1$.

**(b)**
$$x^3y^2 = \cos(\pi y)$$

$$\frac{d}{dx}(x^3y^2) = \frac{d}{dx}\cos(\pi y)$$

$$(x^3)(2y)\frac{dy}{dx} + (y^2)(3x^2) = (-\sin \pi y)(\pi)\frac{dy}{dx}$$

$$(2x^3y + \pi \sin \pi y)\frac{dy}{dx} = -3x^2y^2$$

$$\frac{dy}{dx} = -\frac{3x^2y^2}{2x^3y + \pi \sin \pi y}$$

Slope at $(-1, 1)$: $-\dfrac{3(-1)^2(1)}{2(-1)^3(1) + \pi \sin \pi} = \dfrac{-3}{-2} = \dfrac{3}{2}$

The slope of the tangent line is $\dfrac{3}{2}$.

**40. (a)** When $x = 2$, we have $y^3 - 2y = -1$, or

$y^3 - 2y + 1 = 0$. Clearly, $y = 1$ is one solution, and

we may factor $y^3 - 2y + 1$ as $(y - 1)(y^2 + y - 1)$. The

solutions of $y^2 + y - 1 = 0$ are

$$y = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(-1)}}{2(1)} = \frac{-1 \pm \sqrt{5}}{2}. \text{ Hence,}$$

there are three possible $y$-values: $1, \dfrac{-1 - \sqrt{5}}{2}$, and

$\dfrac{-1 + \sqrt{5}}{2}$.

**(b)**
$$y^3 - xy = -1$$

$$\frac{d}{dx}(y^3) - \frac{d}{dx}(xy) = \frac{d}{dx}(-1)$$

$$3y^2y' - xy' - (y)(1) = 0$$

$$(3y^2 - x)y' = y$$

$$y' = \frac{y}{3y^2 - x}$$

$$y'' = \frac{d}{dx}\frac{y}{3y^2 - x}$$

$$= \frac{(3y^2 - x)(y') - (y)(6y\,y' - 1)}{(3y^2 - x)^2}$$

$$= \frac{y - xy' - 3y^2\,y'}{(3y^2 - x)^2}$$

Since we are working with numerical information,
there is no need to write a general expression for $y''$ in
terms of $x$ and $y$.

To evaluate $f'(2)$, evaluate the expression for $y'$ using

$x = 2$ and $y = 1$:

$$f'(2) = \frac{1}{3(1)^2 - 2} = \frac{1}{1} = 1$$

To evaluate $f''(2)$, evaluate the expression for $y''$ using

$x = 2$, $y = 1$, and $y' = 1$:

$$fz''(2) = \frac{(1) - 2(1) - 3(1)^2(1)}{[3(1)^2 - 2]^2} = \frac{-4}{1} = -4$$

**41.** Find the two points:

The curve crosses the $x$-axis when $y = 0$, so the equation becomes $x^2 + 0x + 0 = 7$, or $x^2 = 7$. The solutions are $x = \pm\sqrt{7}$, so the points are $(\pm\sqrt{7}, 0)$.

Show tangents are parallel:

$$x^2 + xy + y^2 = 7$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(7)$$

$$2x + x\frac{dy}{dx} + (y)(1) + 2y\frac{dy}{dx} = 0$$

$$(x + 2y)\frac{dy}{dx} = -(2x + y)$$

$$\frac{dy}{dx} = -\frac{2x + y}{x + 2y}$$

Slope at $(\sqrt{7}, 0)$: $-\dfrac{2\sqrt{7} + 0}{\sqrt{7} + 2(0)} = -2$

Slope at $(-\sqrt{7}, 0)$: $-\dfrac{2(-\sqrt{7}) + 0}{-\sqrt{7} + 2(0)} = -2$

The tangents at these points are parallel because they have the same slope. The common slope is $-2$.

**42.**
$$x^2 + xy + y^2 = 7$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(7)$$

$$2x + x\frac{dy}{dx} + (y)(1) + 2y\frac{dy}{dx} = 0$$

$$(x + 2y)\frac{dy}{dx} = -(2x + y)$$

$$\frac{dy}{dx} = -\frac{2x + y}{x + 2y}$$

**(a)** The tangent is parallel to the $x$-axis when

$$\frac{dy}{dx} = -\frac{2x + y}{x + 2y} = 0, \text{ or } y = -2x.$$

Substituting $-2x$ for $y$ in the original equation, we have

$$x^2 + xy + y^2 = 7$$
$$x^2 + (x)(-2x) + (-2x)^2 = 7$$
$$x^2 - 2x^2 + 4x^2 = 7$$
$$3x^2 = 7$$
$$x = \pm\sqrt{\frac{7}{3}}$$

The points are $\left(-\sqrt{\dfrac{7}{3}}, 2\sqrt{\dfrac{7}{3}}\right)$ and $\left(\sqrt{\dfrac{7}{3}}, -2\sqrt{\dfrac{7}{3}}\right)$.

**(b)** Since $x$ and $y$ are interchangeable in the original equation, $\dfrac{dx}{dy}$ can be obtained by interchanging $x$ and $y$ in the expression for $\dfrac{dy}{dx}$. That is, $\dfrac{dx}{dy} = -\dfrac{2y + x}{x + 2y}$. The tangent is parallel to the $y$-axis when $\dfrac{dx}{dy} = 0$, or $x = -2y$. Substituting $-2y$ for $x$ in the original equation, we have:

$$x^2 + xy + y^2 = 7$$
$$(-2y)^2 + (-2y)(y) + y^2 = 7$$
$$4y^2 - 2y^2 + y^2 = 7$$
$$3y^2 = 7$$
$$y = \pm\sqrt{\frac{7}{3}}$$

The points are $\left(-2\sqrt{\dfrac{7}{3}}, \sqrt{\dfrac{7}{3}}\right)$ and $\left(2\sqrt{\dfrac{7}{3}}, -\sqrt{\dfrac{7}{3}}\right)$.

Note that these are the same points that would be obtained by interchanging $x$ and $y$ in the solution to part (a).

**43.** First curve:

$$2x^2 + 3y^2 = 5$$

$$\frac{d}{dx}(2x^2) + \frac{d}{dx}(3y^2) = \frac{d}{dx}(5)$$

$$4x + 6y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{4x}{6y} = -\frac{2x}{3y}$$

Second curve:

$$y^2 = x^3$$

$$\frac{d}{dx}y^2 = \frac{d}{dx}x^3$$

$$2y\frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2}{2y}$$

At $(1, 1)$, the slopes are $-\dfrac{2}{3}$ and $\dfrac{3}{2}$ respectively. At $(1, -1)$, the slopes are $\dfrac{2}{3}$ and $-\dfrac{3}{2}$ respectively. In both cases, the tangents are perpendicular. To graph the curves and normal lines, we may use the following parametric equations for $-\pi \le t \le \pi$:

First curve: $x = \sqrt{\dfrac{5}{2}} \cos t, \ y = \sqrt{\dfrac{5}{3}} \sin t$
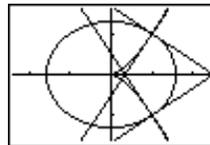
Second curve: $x = \sqrt[3]{t^2}, \ y = t$

Tangents at $(1, 1)$:     $x = 1 + 3t, \ y = 1 - 2t$
                    $x = 1 + 2t, \ y = 1 + 3t$

Tangents at $(1, -1)$:     $x = 1 + 3t, \ y = -1 + 2t$
                    $x = 1 + 2t, \ y = -1 - 3t$



$[-2.4, 2.4]$ by $[-1.6, 1.6]$

**44.** $v(t) = s'(t) = \dfrac{d}{dt}(4 + 6t)^{3/2} = \dfrac{3}{2}(4 + 6t)^{1/2}(6)$

$\qquad = 9(4 + 6t)^{1/2}$

$\quad a(t) = v'(t) = \dfrac{d}{dt}[9(4 + 6t)^{1/2}]$

$\qquad = \dfrac{9}{2}(4 + 6t)^{-1/2}(6) = 27(4 + 6t)^{-1/2}$

At $t = 2$, the velocity is $v(2) = 36$ m/sec and the

acceleration is $a(2) = \dfrac{27}{4}$ m/sec$^2$.

**45.** Acceleration $= \dfrac{dv}{dt} = \dfrac{d}{dt}[8(s - t)^{1/2} + 1]$

$\qquad = 4(s - t)^{-1/2}\left(\dfrac{ds}{dt} - 1\right)$

$\qquad = 4(s - t)^{-1/2}(v - 1)$

$\qquad = 4(s - t)^{-1/2}[(8(s - t)^{1/2} + 1) - 1]$

$\qquad = 32(s - t)^{-1/2}(s - t)^{1/2}$

$\qquad = 32$ ft/sec$^2$

**46.** $\qquad\qquad y^4 - 4y^2 = x^4 - 9x^2$

$\dfrac{d}{dx}(y^4) - \dfrac{d}{dx}(4y^2) = \dfrac{d}{dx}(x^4) - \dfrac{d}{dx}(9x^2)$

$4y^3\dfrac{dy}{dx} - 8y\dfrac{dy}{dx} = 4x^3 - 18x$

$\qquad \dfrac{dy}{dx} = \dfrac{4x^3 - 18x}{4y^3 - 8y} = \dfrac{2x^3 - 9x}{2y^3 - 4y}$

Slope at (3, 2): $\dfrac{2(3)^3 - 9(3)}{2(2)^3 - 4(2)} = \dfrac{27}{8}$

Slope at (−3, 2): $\dfrac{2(-3)^3 - 9(-3)}{2(2)^3 - 4(2)} = -\dfrac{27}{8}$

Slope at (−3, −2): $\dfrac{2(-3)^3 - 9(-3)}{2(-2)^3 - 4(-2)} = \dfrac{27}{8}$

Slope at (3, −2): $\dfrac{2(3)^3 - 9(3)}{2(-2)^3 - 4(-2)} = -\dfrac{27}{8}$

**47. (a)** $\qquad\qquad x^3 + y^3 - 9xy = 0$

$\dfrac{d}{dx}(x^3) + \dfrac{d}{dx}(y^3) - 9\dfrac{d}{dx}(xy) = \dfrac{d}{dx}(0)$

$3x^2 + 3y^2\dfrac{dy}{dx} - 9x\dfrac{dy}{dx} - 9(y)(1) = 0$

$\qquad (3y^2 - 9x)\dfrac{dy}{dx} = 9y - 3x^2$

$\qquad\qquad \dfrac{dy}{dx} = \dfrac{9y - 3x^2}{3y^2 - 9x} = \dfrac{3y - x^2}{y^2 - 3x}$

Slope at (4, 2): $\dfrac{3(2) - (4)^2}{(2)^2 - 3(4)} = \dfrac{-10}{-8} = \dfrac{5}{4}$

Slope at (2, 4): $\dfrac{3(4) - (2)^2}{(4)^2 - 3(2)} = \dfrac{8}{10} = \dfrac{4}{5}$

**(b)** The tangent is horizontal when

$\dfrac{dy}{dx} = \dfrac{3y - x^2}{y^2 - 3x} = 0$, or $y = \dfrac{x^2}{3}$.

Substituting $\dfrac{x^2}{3}$ for $y$ in the original equation, we have:

$x^3 + y^3 - 9xy = 0$

$x^3 + \left(\dfrac{x^2}{3}\right)^3 - 9x\left(\dfrac{x^2}{3}\right) = 0$

$x^3 + \dfrac{x^6}{27} - 3x^3 = 0$

$\dfrac{x^3}{27}(x^3 - 54) = 0$

$x = 0$ or $x = \sqrt[3]{54} = 3\sqrt[3]{2}$

At $x = 0$, we have $y = \dfrac{0^2}{3} = 0$, which gives the

point (0, 0), which is the origin. At $x = 3\sqrt[3]{2}$, we have

$y = \dfrac{1}{3}(3\sqrt[3]{2})^2 = \dfrac{1}{3}(9\sqrt[3]{4}) = 3\sqrt[3]{4}$, so the point other

than the origin is $(3\sqrt[3]{2}, 3\sqrt[3]{4})$ or approximately

(3.780, 4.762).

**(c)** The equation $x^3 + y^3 - 9xy$ is not affected by

interchanging $x$ and $y$, so its graph is symmetric about

the line $y = x$ and we may find the desired point by

interchanging the $x$-value and the $y$-value in the answer

to part (b). The desired point is $(3\sqrt[3]{4}, 3\sqrt[3]{2})$ or

approximately (4.762, 3.780).

**48.** $\qquad\qquad x^2 + 2xy - 3y^2 = 0$

$\dfrac{d}{dx}(x^2) + 2\dfrac{d}{dx}(xy) - \dfrac{d}{dx}(3y^2) = \dfrac{d}{dx}(0)$

$2x + 2x\dfrac{dy}{dx} + 2(y)(1) - 6y\dfrac{dy}{dx} = 0$

$\qquad (2x - 6y)\dfrac{dy}{dx} = -2x - 2y$

$\qquad\qquad \dfrac{dy}{dx} = \dfrac{-2x - 2y}{2x - 6y} = \dfrac{x + y}{3y - x}$

At (1, 1) the curve has slope $\dfrac{1 + 1}{3(1) - 1} = \dfrac{2}{2} = 1$, so the

normal line is $y = -1(x - 1) + 1$ or $y = -x + 2$.

Substituting $-x + 2$ for $y$ in the original equation, we have:

$x^2 + 2xy - 3y^2 = 0$

$x^2 + 2x(-x + 2) - 3(-x + 2)^2 = 0$

$x^2 - 2x^2 + 4x - 3(x^2 - 4x + 4) = 0$

$-4x^2 + 16x - 12 = 0$

$-4(x - 1)(x - 3) = 0$

$x = 1$ or $x = 3$

Since the given point (1, 1) had $x = 1$, we choose $x = 3$

and so $y = -(3) + 2 = -1$. The desired point is (3, −1).

**49.**
$$xy + 2x - y = 0$$

$$\frac{d}{dx}(xy) + \frac{d}{dx}(2x) - \frac{d}{dx}(y) = 0$$

$$x\frac{dy}{dx} + (y)(1) + 2 - \frac{dy}{dx} = 0$$

$$(x-1)\frac{dy}{dx} = -2 - y$$

$$\frac{dy}{dx} = \frac{-2-y}{x-1} = \frac{2+y}{1-x}$$

Since the slope of the line $2x + y = 0$ is $-2$, we wish to find points where the normal has slope $-2$, that is, where the tangent has slope $\frac{1}{2}$. Thus, we have

$$\frac{2+y}{1-x} = \frac{1}{2}$$

$$2(2+y) = 1 - x$$

$$4 + 2y = 1 - x$$

$$x = -2y - 3$$

Substituting $-2y - 3$ in the original equation, we have:

$$xy + 2x - y = 0$$
$$(-2y - 3)y + 2(-2y - 3) - y = 0$$
$$-2y^2 - 8y - 6 = 0$$
$$-2(y+1)(y+3) = 0$$
$$y = -1 \text{ or } y = -3$$

At $y = -1$, $x = -2y - 3 = 2 - 3 = -1$.
At $y = -3$: $x = -2y - 3 = 6 - 3 = 3$.
The desired points are $(-1, -1)$ and $(3, -3)$.
Finally, we find the desired normals to the curve, which are the lines of slope $-2$ passing through each of these points. At $(-1, -1)$, the normal line is $y = -2(x + 1) - 1$ or $y = -2x - 3$. At $(3, -3)$, the normal line is $y = -2(x - 3) - 3$ or $y = -2x + 3$.

**50.**
$$x = y^2$$

$$\frac{d}{dx}(x) = \frac{d}{dx}(y^2)$$

$$1 = 2y\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{2y}$$

The normal line at $(x, y)$ has slope $-2y$. Thus, the normal line at $(b^2, b)$ is $y = -2b(x - b^2) + b$, or

$y = -2bx + 2b^3 + b$. This line intersects the $x$-axis at

$x = \frac{2b^3 + b}{2b} = b^2 + \frac{1}{2}$, which is the value of $a$ and must be greater than $\frac{1}{2}$ if $b \ne 0$.

The two normals at $(b^2, \pm b)$ will be perpendicular when they have slopes $\pm 1$, which gives $-2y = \pm 1$ or

$y = \pm\frac{1}{2}$ (or $b = \pm\frac{1}{2}$). The corresponding value of $a$ is

$b^2 + \frac{1}{2} = \left(\frac{1}{2}\right)^2 + \frac{1}{2} = \frac{3}{4}$. Thus, the two nonhorizontal

normals are perpendicular when $a = \frac{3}{4}$.

**51. (a)**
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$b^2x^2 + a^2y^2 = a^2b^2$$

$$\frac{d}{dx}(b^2x^2) + \frac{d}{dx}(a^2y^2) = \frac{d}{dx}(a^2b^2)$$

$$2b^2x + 2a^2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{2b^2x}{2a^2y} = -\frac{b^2x}{a^2y}$$

The slope at $(x_1, y_1)$ is $-\frac{b^2x_1}{a^2y_1}$.

The tangent line is $y - y_1 = -\frac{b^2x_1}{a^2y_1}(x - x_1)$. This gives:

$$a^2y_1y - a^2y_1{}^2 = -b^2x_1x + b^2x_1{}^2$$

$$a^2y_1y + b^2x_1x = a^2y_1{}^2 + b^2x_1{}^2.$$

But $a^2y_1{}^2 + b^2x_1{}^2 = a^2b^2$ since $(x_1, y_1)$ is on the

ellipse. Therefore, $a^2y_1y + b^2x_1x = a^2b^2$, and

dividing by $a^2b^2$ gives $\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1$.

**(b)**
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
$$b^2x^2 - a^2y^2 = a^2b^2$$
$$\frac{d}{dx}(b^2x^2) - \frac{d}{dx}(a^2y^2) = \frac{d}{dx}(a^2b^2)$$
$$2b^2x - 2a^2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{-2b^2x}{-2a^2y} = \frac{b^2x}{a^2y}$$

The slope at $(x_1, y_1)$ is $\frac{b^2x_1}{a^2y_1}$.

The tangent line is $y - y_1 = \frac{b^2x_1}{a^2y_1}(x - x_1)$.

This gives:

$$a^2y_1y - a^2y_1{}^2 = b^2x_1x - b^2x_1{}^2$$

$$b^2x_1{}^2 - a^2y_1{}^2 = b^2x_1x - a^2y_1y$$

But $b^2x_1{}^2 - a^2y_1{}^2 = a^2b^2$ since $(x_1, y_1)$ is on the

hyperbola. Therefore, $b^2x_1x - a^2y_1y = a^2b^2$, and

dividing by $a^2b^2$ gives $\frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 1$.

**52. (a)** Solve for $y$:
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
$$-\frac{y^2}{b^2} = -\frac{x^2}{a^2} + 1$$
$$y^2 = \frac{b^2}{a^2}(x^2 - a^2)$$
$$y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$$

**(b)** $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to\infty} \dfrac{\frac{b}{a}\sqrt{x^2 - a^2}}{\frac{b}{a}|x|}$

$\qquad\qquad = \lim\limits_{x\to\infty} \dfrac{\sqrt{x^2 - a^2}}{\sqrt{x^2}}$

$\qquad\qquad = \lim\limits_{x\to\infty} \sqrt{1 - \dfrac{a^2}{x^2}} = 1$

**(c)** $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to\infty} \dfrac{-\frac{b}{a}\sqrt{x^2 - a^2}}{-\frac{b}{a}|x|}$

$\qquad\qquad = \lim\limits_{x\to\infty} \dfrac{\sqrt{x^2 - a^2}}{\sqrt{x^2}}$

$\qquad\qquad = \lim\limits_{x\to\infty} \sqrt{1 - \dfrac{a^2}{x^2}} = 1$

# ■ Section 3.8 Derivatives of Inverse Trigonometric Functions (pp. 157–163)

## Exploration 1   Finding a Derivative on an Inverse Graph Geometrically

1. The graph is shown at the right. It appears to be a one-to-one function
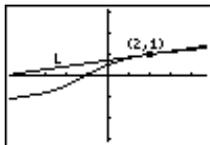


[−4.7, 4.7] by [−3.1, 3.1]

2. $f'(x) = 5x^4 + 2$. The fact that this function is always positive enables us to conclude that $f$ is everywhere increasing, and hence one-to-one.

3. The graph of $f^{-1}$ is shown to the right, along with the graph of $f$. The graph of $f^{-1}$ is obtained from the graph of $f$ by reflecting it in the line $y = x$.
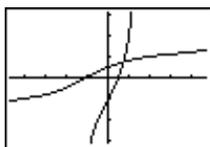


[−4.7, 4.7] by [−3.1, 3.1]

4. The line $L$ is tangent to the graph of $f^{-1}$ at the point (2, 1).



[−4.7, 4.7] by [−3.1, 3.1]

5. The reflection of line $L$ is tangent to the graph of $f$ at the point (1, 2).
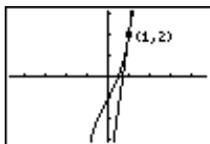


[−4.7, 4.7] by [−3.1, 3.1]

6. The reflection of line $L$ is the tangent line to the graph of $y = x^5 + 2x - 1$ at the point (1, 2). The slope is $\dfrac{dy}{dx}$ at $x = 1$, which is 7.

7. The slope of $L$ is the reciprocal of the slope of its reflection $\left(\text{since } \dfrac{\Delta y}{\Delta x} \text{ gets reflected to become } \dfrac{\Delta x}{\Delta y}\right)$. It is $\dfrac{1}{7}$.

8. $\dfrac{1}{7}$

## Quick Review 3.8

1. Domain: $[-1, 1]$

   Range: $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

   At 1: $\dfrac{\pi}{2}$

2. Domain: $[-1, 1]$
   Range: $[0, \pi]$
   At 1: 0

3. Domain: all reals

   Range: $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

   At 1: $\dfrac{\pi}{4}$

4. Domain: $(-\infty, -1] \cup [1, \infty)$

   Range: $\left[0, \dfrac{\pi}{2}\right) \cup \left(\dfrac{\pi}{2}, \pi\right]$

   At 1: 0

5. Domain: all reals
   Range: all reals
   At 1: 1

6. $f(x) = y = 3x - 8$

   $\qquad y + 8 = 3x$

   $\qquad x = \dfrac{y + 8}{3}$

   Interchange $x$ and $y$:

   $\qquad y = \dfrac{x + 8}{3}$

   $\qquad f^{-1}(x) = \dfrac{x + 8}{3}$

7. $f(x) = y = \sqrt[3]{x + 5}$

   $\qquad y^3 = x + 5$

   $\qquad x = y^3 - 5$

   Interchange $x$ and $y$:

   $\qquad y = x^3 - 5$

   $\qquad f^{-1}(x) = x^3 - 5$

**8.** $f(x) = y = \dfrac{8}{x}$

$x = \dfrac{8}{y}$

Interchange $x$ and $y$:

$y = \dfrac{8}{x}$

$f^{-1}(x) = \dfrac{8}{x}$

**9.** $f(x) = y = \dfrac{3x - 2}{x}$

$xy = 3x - 2$

$(y - 3)x = -2$

$x = \dfrac{-2}{y - 3} = \dfrac{2}{3 - y}$

Interchange $x$ and $y$:

$y = \dfrac{2}{3 - x}$

$f^{-1}(x) = \dfrac{2}{3 - x}$

**10.** $f(x) = y = \arctan \dfrac{x}{3}$

$\tan y = \dfrac{x}{3}, \; -\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$

$x = 3 \tan y, \; -\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$

Interchange $x$ and $y$:

$y = 3 \tan x, \; -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

$f^{-1}(x) = 3 \tan x, \; -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

## Section 3.8 Exercises

**1.** $\dfrac{dy}{dx} = \dfrac{d}{dx} \cos^{-1}(x^2) = -\dfrac{1}{\sqrt{1 - (x^2)^2}} \dfrac{d}{dx}(x^2)$

$= -\dfrac{1}{\sqrt{1 - x^4}} (2x) = -\dfrac{2x}{\sqrt{1 - x^4}}$

**2.** $\dfrac{dy}{dx} = \dfrac{d}{dx} \cos^{-1}\left(\dfrac{1}{x}\right) = -\dfrac{1}{\sqrt{1 - \left(\frac{1}{x}\right)^2}} \dfrac{d}{dx}\left(\dfrac{1}{x}\right)$

$= -\dfrac{1}{\sqrt{1 - \left(\frac{1}{x}\right)^2}} \left(-\dfrac{1}{x^2}\right) = \dfrac{1}{|x| \sqrt{x^2 - 1}}$

**3.** $\dfrac{dy}{dt} = \dfrac{d}{dt} \sin^{-1} \sqrt{2}\,t = \dfrac{1}{\sqrt{1 - (\sqrt{2}\,t)^2}} \dfrac{d}{dt}(\sqrt{2}\,t) = \dfrac{\sqrt{2}}{\sqrt{1 - 2t^2}}$

**4.** $\dfrac{dy}{dt} = \dfrac{d}{dt} \sin^{-1}(1 - t) = \dfrac{1}{\sqrt{1 - (1 - t)^2}} \dfrac{d}{dt}(1 - t)$

$= -\dfrac{1}{\sqrt{2t - t^2}}$

**5.** $\dfrac{dy}{ds} = \dfrac{d}{ds} \sec^{-1}(2s + 1)$

$= \dfrac{1}{|2s + 1| \sqrt{(2s + 1)^2 - 1}} \dfrac{d}{ds}(2s + 1)$

$= \dfrac{1}{|2s + 1| \sqrt{4s^2 + 4s}} (2) = \dfrac{1}{|2s + 1| \sqrt{s^2 + s}}$

**6.** $\dfrac{dy}{ds} = \dfrac{d}{ds} \sec^{-1} 5s = \dfrac{1}{|5s| \sqrt{(5s)^2 - 1}} \dfrac{d}{ds}(5s) = \dfrac{1}{|s| \sqrt{25s^2 - 1}}$

**7.** $\dfrac{dy}{dx} = \dfrac{d}{dx} \csc^{-1}(x^2 + 1)$

$= -\dfrac{1}{|x^2 + 1| \sqrt{(x^2 + 1)^2 - 1}} \dfrac{d}{dx}(x^2 + 1)$

$= -\dfrac{2x}{(x^2 + 1)\sqrt{x^4 + 2x^2}} = -\dfrac{2}{(x^2 + 1)\sqrt{x^2 + 2}}$

Note that the condition $x > 0$ is required in the last step.

**8.** $\dfrac{dy}{dx} = \dfrac{d}{dx} \csc^{-1}\left(\dfrac{x}{2}\right) = -\dfrac{1}{\left|\frac{x}{2}\right| \sqrt{\left(\frac{x}{2}\right)^2 - 1}} \dfrac{d}{dx}\left(\dfrac{x}{2}\right)$

$= -\dfrac{2}{|x| \sqrt{x^2 - 4}}$

**9.** $\dfrac{dy}{dt} = \dfrac{d}{dt} \sec^{-1}\left(\dfrac{1}{t}\right) = \dfrac{1}{\left|\frac{1}{t}\right| \sqrt{\left(\frac{1}{t}\right)^2 - 1}} \dfrac{d}{dt}\left(\dfrac{1}{t}\right)$

$= \dfrac{1}{\left|\frac{1}{t}\right| \sqrt{\left(\frac{1}{t}\right)^2 - 1}} \left(-\dfrac{1}{t^2}\right) = -\dfrac{1}{\sqrt{1 - t^2}}$

Note that the condition $t > 0$ is required in the last step.

**10.** $\dfrac{dy}{dt} = \dfrac{d}{dt} \sin^{-1}\left(\dfrac{3}{t^2}\right) = \dfrac{1}{\sqrt{1 - \left(\frac{3}{t^2}\right)^2}} \dfrac{d}{dt}\left(\dfrac{3}{t^2}\right)$

$= \dfrac{1}{\sqrt{1 - \frac{9}{t^4}}} \left(-\dfrac{6}{t^3}\right) = -\dfrac{6}{t\sqrt{t^4 - 9}}$

**11.** $\dfrac{dy}{dt} = \dfrac{d}{dt} \cot^{-1} \sqrt{t} = -\dfrac{1}{1 + (\sqrt{t})^2} \dfrac{d}{dt} \sqrt{t}$

$= -\dfrac{1}{2\sqrt{t}(t + 1)}$

**12.** $\dfrac{dy}{dt} = \dfrac{d}{dt} \cot^{-1} \sqrt{t - 1} = -\dfrac{1}{1 + (\sqrt{t - 1})^2} \dfrac{d}{dt} \sqrt{t - 1}$

$= -\left(\dfrac{1}{1 + t - 1}\right)\left(\dfrac{1}{2\sqrt{t - 1}}\right) = -\dfrac{1}{2t\sqrt{t - 1}}$

**13.** $\dfrac{dy}{ds} = \dfrac{d}{ds}(s\sqrt{1-s^2}) + \dfrac{d}{ds}(\cos^{-1}s)$

$= (s)\left(\dfrac{1}{2\sqrt{1-s^2}}\right)(-2s) + (\sqrt{1-s^2})(1) - \dfrac{1}{\sqrt{1-s^2}}$

$= -\dfrac{s^2}{\sqrt{1-s^2}} + \sqrt{1-s^2} - \dfrac{1}{\sqrt{1-s^2}}$

$= \dfrac{-s^2 + (1-s^2) - 1}{\sqrt{1-s^2}}$

$= -\dfrac{2s^2}{\sqrt{1-s^2}}$

**14.** $\dfrac{dy}{ds} = \dfrac{d}{ds}\sqrt{s^2-1} - \dfrac{d}{ds}\sec^{-1}s$

$= \dfrac{1}{2\sqrt{s^2-1}}(2s) - \dfrac{1}{|s|\sqrt{s^2-1}}$

$= \dfrac{s|s|-1}{|s|\sqrt{s^2-1}}$

**15.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(\tan^{-1}\sqrt{x^2-1}) + \dfrac{d}{dx}(\csc^{-1}x)$

$= \dfrac{1}{1+(\sqrt{x^2-1})^2}\dfrac{d}{dx}(\sqrt{x^2-1}) - \dfrac{1}{|x|\sqrt{x^2-1}}$

$= \dfrac{1}{x^2}\dfrac{1}{2\sqrt{x^2-1}}(2x) - \dfrac{1}{|x|\sqrt{x^2-1}}$

$= \dfrac{1}{x\sqrt{x^2-1}} - \dfrac{1}{|x|\sqrt{x^2-1}}$

$= 0$

Note that the condition $x > 1$ is required in the last step.

**16.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\cot^{-1}\dfrac{1}{x}\right) - \dfrac{d}{dx}(\tan^{-1}x)$

$= -\dfrac{1}{1+\left(\dfrac{1}{x^2}\right)}\dfrac{d}{dx}\left(\dfrac{1}{x}\right) - \dfrac{1}{1+x^2}$

$= \left(-\dfrac{1}{1+\dfrac{1}{x^2}}\right)\left(-\dfrac{1}{x^2}\right) - \dfrac{1}{1+x^2}$

$= \dfrac{1}{x^2+1} - \dfrac{1}{1+x^2}$

$= 0,\ x \neq 0$

The condition $x \neq 0$ is required because the original function was undefined when $x \neq 0$.

**17.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(x\sin^{-1}x) + \dfrac{d}{dx}(\sqrt{1-x^2})$

$= (x)\left(\dfrac{1}{\sqrt{1-x^2}}\right) + (\sin^{-1}x)(1) + \dfrac{1}{2\sqrt{1-x^2}}(-2x)$

$= \sin^{-1}x$

**18.** $\dfrac{dy}{dx} = \dfrac{d}{dx}[\sin^{-1}(2x)]^{-1}$

$= -[\sin^{-1}(2x)]^{-2}\dfrac{d}{dx}\sin^{-1}(2x)$

$= -[\sin^{-1}(2x)]^{-2}\dfrac{1}{\sqrt{1-4x^2}}(2)$

$= -\dfrac{2}{[\sin^{-1}(2x)]^2\sqrt{1-4x^2}}$

**19. (a)** Since $\dfrac{dy}{dx} = \sec^2 x$, the slope at $\left(\dfrac{\pi}{4},1\right)$ is $\sec^2\left(\dfrac{\pi}{4}\right) = 2$.

The tangent line is given by $y = 2\left(x - \dfrac{\pi}{4}\right) + 1$, or

$y = 2x - \dfrac{\pi}{2} + 1$.

**(b)** Since $\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$, the slope at $\left(1,\dfrac{\pi}{4}\right)$ is $\dfrac{1}{1+1^2} = \dfrac{1}{2}$.

The tangent line is given by $y = \dfrac{1}{2}(x-1) + \dfrac{\pi}{4}$, or

$y = \dfrac{1}{2}x - \dfrac{1}{2} + \dfrac{\pi}{4}$.

**20. (a)** Note that $f'(x) = 5x^4 + 6x^2 + 1$. Thus $f(1) = 3$ and $f'(1) = 12$.

**(b)** Since the graph of $y = f(x)$ includes the point $(1, 3)$ and the slope of the graph is 12 at this point, the graph of $y = f^{-1}(x)$ will include $(3, 1)$ and the slope will be $\dfrac{1}{12}$. Thus, $f^{-1}(3) = 1$ and $(f^{-1})'(3) = \dfrac{1}{12}$. (We have assumed that $f^{-1}(x)$ is defined and differentiable at $x = 3$. This is true by Theorem 3, because $f'(x) = 5x^4 + 6x^2 + 1$, which is never zero.)

**21. (a)** Note that $f'(x) = -\sin x + 3$, which is always between 2 and 4. Thus $f$ is differentiable at every point on the interval $(-\infty, \infty)$ and $f'(x)$ is never zero on this interval, so $f$ has a differentiable inverse by Theorem 3.

**(b)** $f(0) = \cos 0 + 3(0) = 1$;
$f'(0) = -\sin 0 + 3 = 3$

**(c)** Since the graph of $y = f(x)$ includes the point $(0, 1)$ and the slope of the graph is 3 at this point, the graph of $y = f^{-1}(x)$ will include $(1, 0)$ and the slope will be $\dfrac{1}{3}$. Thus, $f^{-1}(1) = 0$ and $(f^{-1})'(1) = \dfrac{1}{3}$.

**22.**
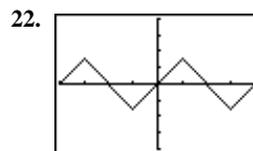


$[-2\pi, 2\pi]$ by $[-4, 4]$

**(a)** All reals

**(b)** $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

**22. continued**

(c)  At the points $x = k\dfrac{\pi}{2}$, where $k$ is an odd integer.

(d)



(e)  $f'(x) = \dfrac{d}{dx}\sin^{-1}(\sin x)$
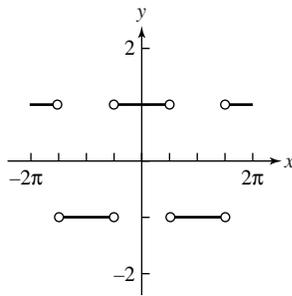
$\qquad = \dfrac{1}{\sqrt{1 - \sin^2 x}}\,\dfrac{d}{dx}\sin x$

$\qquad = \dfrac{\cos x}{\sqrt{1 - \sin^2 x}}$

which is $\pm 1$ depending on whether

$\cos x$ is positive or negative.

**23. (a)** $v(t) = \dfrac{dx}{dt} = \dfrac{1}{1 + t^2}$ which is always positive.

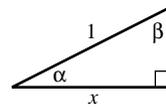**(b)** $a(t) = \dfrac{dv}{dt} = -\dfrac{2t}{(1 + t^2)^2}$ which is always negative.

**(c)** $\dfrac{\pi}{2}$

**24.** $\dfrac{d}{dx}\cos^{-1}(x) = \dfrac{d}{dx}\left(\dfrac{\pi}{2} - \sin^{-1}x\right)$

$\qquad = 0 - \dfrac{d}{dx}\sin^{-1}(x)$

$\qquad = -\dfrac{1}{\sqrt{1 - x^2}}$

**25.** $\dfrac{d}{dx}\cot^{-1}x = \dfrac{d}{dx}\left(\dfrac{\pi}{2} - \tan^{-1}(x)\right)$

$\qquad = 0 - \dfrac{d}{dx}\tan^{-1}(x)$

$\qquad = -\dfrac{1}{1 + x^2}$

**26.** $\dfrac{d}{dx}\csc^{-1}(x) = \dfrac{d}{dx}\left(\dfrac{\pi}{2} - \sec^{-1}(x)\right)$

$\qquad = 0 - \dfrac{d}{dx}\sec^{-1}(x)$

$\qquad = -\dfrac{1}{|x|\sqrt{x^2 - 1}}$

**27. (a)** $y = \dfrac{\pi}{2}$

**(b)** $y = -\dfrac{\pi}{2}$

**(c)**  None, since $\dfrac{d}{dx}\tan^{-1}x = \dfrac{1}{1 + x^2} \neq 0$.

**28. (a)** $y = 0$

**(b)** $y = \pi$

**(c)**  None, since $\dfrac{d}{dx}\cot^{-1}x = -\dfrac{1}{1 + x^2} \neq 0$.

**29. (a)** $y = \dfrac{\pi}{2}$

**(b)** $y = \dfrac{\pi}{2}$

**(c)**  None, since $\dfrac{d}{dx}\sec^{-1}x = \dfrac{1}{|x|\sqrt{x^2 - 1}} \neq 0$.

**30. (a)** $y = 0$

**(b)** $y = 0$

**(c)**  None, since $\dfrac{d}{dx}\csc^{-1}x = -\dfrac{1}{|x|\sqrt{x^2 - 1}} \neq 0$.

**31. (a)**  None, since $\sin^{-1}x$ is undefined for $x > 1$.

**(b)**  None, since $\sin^{-1}x$ is undefined for $x < -1$.

**(c)**  None, since $\dfrac{d}{dx}\sin^{-1}x = \dfrac{1}{\sqrt{1 - x^2}} \neq 0$.

**32. (a)**  None, since $\cos^{-1}x$ is undefined for $x > 1$.

**(b)**  None, since $\cos^{-1}x$ is undefined for $x < -1$.

**(c)**  None, since $\dfrac{d}{dx}\cos^{-1}x = -\dfrac{1}{\sqrt{1 - x^2}} \neq 0$.

**33. (a)**



$\alpha = \cos^{-1}x, \beta = \sin^{-1}x$

So $\cos^{-1}x + \sin^{-1}x = \alpha + \beta = \dfrac{\pi}{2}$.

**(b)**
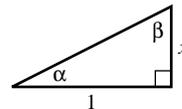


$\alpha = \tan^{-1}x, \beta = \cot^{-1}x$

So $\tan^{-1}x + \cot^{-1}x = \alpha + \beta = \dfrac{\pi}{2}$.

**(c)**
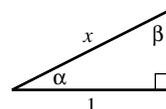


$\alpha = \sec^{-1}x, \beta = \csc^{-1}x$

So $\sec^{-1}x + \csc^{-1}x = \alpha + \beta = \dfrac{\pi}{2}$.

**34.**
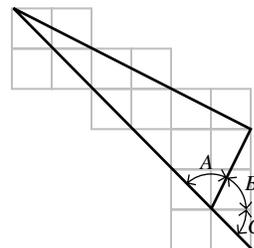


The "straight angle" with the arrows in it is the sum of the three angles $A$, $B$, and $C$.

$A$ is equal to $\tan^{-1}3$ since the opposite side is 3 times as long as the adjacent side.

$B$ is equal to $\tan^{-1}2$ since the side opposite it is 2 units and the adjacent side is one unit.

$C$ is equal to $\tan^{-1}1$ since both the opposite and adjacent sides are one unit long.

But the sum of these three angles is the "straight angle," which has measure $\pi$ radians.

**35.**
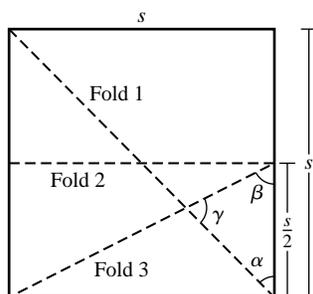


If $s$ is the length of a side of the square, then

$$\tan \alpha = \frac{s}{s} = 1, \text{ so } \alpha = \tan^{-1} 1 \text{ and}$$

$$\tan \beta = \frac{s}{\frac{s}{2}} = 2, \text{ so } \beta = \tan^{-1} 2.$$

From Exercise 34, we have

$$\gamma = \pi - \alpha - \beta = \pi - \tan^{-1} 1 - \tan^{-1} 2 = \tan^{-1} 3.$$

## ■ Section 3.9 Derivatives of Exponential and Logarithmic Functions (pp. 163–171)

### Exploration 1   Leaving Milk on the Counter

1. The temperature of the refrigerator is 42°F, the temperature of the milk at time $t = 0$.

2. The temperature of the room is 72°F, the limit to which $y$ tends as $t$ increases.

3. The milk is warming up the fastest at $t = 0$. The second derivative $y'' = -30(\ln(0.98))^2(0.98)^t$ is negative, so $y'$ (the rate at which the milk is warming) is maximized at the lowest value of $t$.

4. We set $y = 55$ and solve;
$$72 - 30(0.98)^t = 55$$
$$(0.98)^t = \frac{17}{30}$$
$$t \ln (0.98) = \ln \left(\frac{17}{30}\right)$$
$$t = \frac{\ln \left(\frac{17}{30}\right)}{\ln (0.98)} \approx 28.114$$

The milk reaches a temperature of 55°F after about 28 minutes.

5. $\frac{dy}{dt} = -30 \ln (0.98) \cdot (0.98)^t$. At $t = \frac{\ln \left(\frac{17}{30}\right)}{\ln (0.98)}$,
$$\frac{dy}{dt} \approx 0.343 \text{ degrees/minute.}$$

### Quick Review 3.9

1. $\log_5 8 = \dfrac{\ln 8}{\ln 5}$

2. $7^x = e^{\ln 7^x} = e^{x \ln 7}$

3. $\ln (e^{\tan x}) = \tan x$

4. $\ln (x^2 - 4) - \ln (x + 2) = \ln \dfrac{x^2 - 4}{x + 2} = \ln \dfrac{(x + 2)(x - 2)}{x + 2}$
$= \ln (x - 2)$

5. $\log_2 (8^{x-5}) = \log_2 (2^3)^{x-5} = \log_2 2^{3x-15} = 3x - 15$

6. $\dfrac{\log_4 x^{15}}{\log_4 x^{12}} = \dfrac{15 \log_4 x}{12 \log_4 x} = \dfrac{15}{12} = \dfrac{5}{4}, x > 0$

7. $3 \ln x - \ln 3x + \ln (12x^2) = \ln x^3 - \ln 3x + \ln (12x^2)$
$= \ln \dfrac{(x^3)(12x^2)}{3x} = \ln (4x^4)$

8.  $3^x = 19$
$\ln 3^x = \ln 19$
$x \ln 3 = \ln 19$
$x = \dfrac{\ln 19}{\ln 3} \approx 2.68$

9.  $5^t \ln 5 = 18$
$5^t = \dfrac{18}{\ln 5}$
$\ln 5^t = \ln \dfrac{18}{\ln 5}$
$t \ln 5 = \ln 18 - \ln (\ln 5)$
$t = \dfrac{\ln 18 - \ln (\ln 5)}{\ln 5} \approx 1.50$

10.  $3^{x+1} = 2^x$
$\ln 3^{x+1} = \ln 2^x$
$(x + 1) \ln 3 = x \ln 2$
$x(\ln 3 - \ln 2) = -\ln 3$
$x = \dfrac{\ln 3}{\ln 2 - \ln 3} \approx -2.71$

### Section 3.9 Exercises

1. $\dfrac{dy}{dx} = \dfrac{d}{dx}(2e^x) = 2e^x$

2. $\dfrac{dy}{dx} = \dfrac{d}{dx}(e^{2x}) = e^{2x} \dfrac{d}{dx}(2x) = 2e^{2x}$

3. $\dfrac{dy}{dx} = \dfrac{d}{dx}e^{-x} = e^{-x} \dfrac{d}{dx}(-x) = -e^{-x}$

4. $\dfrac{dy}{dx} = \dfrac{d}{dx}e^{-5x} = e^{-5x} \dfrac{d}{dx}(-5x) = -5e^{-5x}$

5. $\dfrac{dy}{dx} = \dfrac{d}{dx}e^{2x/3} = e^{2x/3} \dfrac{d}{dx}\left(\dfrac{2x}{3}\right) = \dfrac{2}{3}e^{2x/3}$

6. $\dfrac{dy}{dx} = \dfrac{d}{dx}e^{-x/4} = e^{-x/4} \dfrac{d}{dx}\left(-\dfrac{x}{4}\right) = -\dfrac{1}{4}e^{-x/4}$

7. $\dfrac{dy}{dx} = \dfrac{d}{dx}(xe^2) - \dfrac{d}{dx}(e^x) = e^2 - e^x$

8. $\dfrac{dy}{dx} = \dfrac{d}{dx}(x^2e^x) - \dfrac{d}{dx}(xe^x)$
$= (x^2)(e^x) + (e^x)(2x) - [(x)(e^x) + (e^x)(1)]$
$= x^2e^x + xe^x - e^x$

9. $\dfrac{dy}{dx} = \dfrac{d}{dx}e^{\sqrt{x}} = e^{\sqrt{x}} \dfrac{d}{dx}(\sqrt{x}) = \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$

10. $\dfrac{dy}{dx} = \dfrac{d}{dx}e^{(x^2)} = e^{(x^2)} \dfrac{d}{dx}(x^2) = 2xe^{(x^2)}$

**11.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(x^\pi) = \pi x^{\pi-1}$

**12.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(x^{1+\sqrt{2}}) = (1+\sqrt{2})x^{1+\sqrt{2}-1} = (1+\sqrt{2})x^{\sqrt{2}}$

**13.** $\dfrac{dy}{dx} = \dfrac{d}{dx}x^{-\sqrt{2}} = -\sqrt{2}x^{-\sqrt{2}-1}$

**14.** $\dfrac{dy}{dx} = \dfrac{d}{dx}x^{1-e} = (1-e)x^{1-e-1} = (1-e)x^{-e}$

**15.** $\dfrac{dy}{dx} = \dfrac{d}{dx}8^x = 8^x \ln 8$

**16.** $\dfrac{dy}{dx} = \dfrac{d}{dx}9^{-x} = 9^{-x}(\ln 9)\dfrac{d}{dx}(-x) = -9^{-x}\ln 9$

**17.** $\dfrac{dy}{dx} = \dfrac{d}{dx}3^{\csc x} = 3^{\csc x}(\ln 3)\dfrac{d}{dx}(\csc x)$

$\qquad = 3^{\csc x}(\ln 3)(-\csc x \cot x)$

$\qquad = -3^{\csc x}(\ln 3)(\csc x \cot x)$

**18.** $\dfrac{dy}{dx} = \dfrac{d}{dx}3^{\cot x} = 3^{\cot x}(\ln 3)\dfrac{d}{dx}(\cot x)$

$\qquad = 3^{\cot x}(\ln 3)(-\csc^2 x)$

$\qquad = -3^{\cot x}(\ln 3)(\csc^2 x)$

**19.** Use logarithmic differentiation.

$\qquad y = x^{\ln x}$

$\qquad \ln y = \ln x^{\ln x}$

$\qquad \ln y = \ln x \ln x$

$\qquad \dfrac{d}{dx}(\ln y) = \dfrac{d}{dx}(\ln x)^2$

$\qquad \dfrac{1}{y}\dfrac{dy}{dx} = (2\ln x)\left(\dfrac{1}{x}\right)$

$\qquad \dfrac{dy}{dx} = \dfrac{2y\ln x}{x}$

$\qquad \dfrac{dy}{dx} = \dfrac{2x^{\ln x}\ln x}{x}$

**20.** Use logarithmic differentiation.

$\qquad y = x^{1/\ln x}$

$\qquad \ln y = \ln x^{1/\ln x}$

$\qquad \ln y = \dfrac{1}{\ln x}\ln x$

$\qquad \ln y = 1$

$\qquad y = e$

$\qquad \dfrac{dy}{dx} = \dfrac{d}{dx}(e) = 0,\ x > 0$

**21.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\ln(x^2) = \dfrac{1}{x^2}\dfrac{d}{dx}(x^2) = \dfrac{1}{x^2}(2x) = \dfrac{2}{x}$

**22.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(\ln x)^2 = 2\ln x \dfrac{d}{dx}(\ln x) = \dfrac{2\ln x}{x}$

**23.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\ln(x^{-1}) = \dfrac{d}{dx}(-\ln x) = -\dfrac{1}{x},\ x > 0$

**24.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\ln\dfrac{10}{x} = \dfrac{d}{dx}(\ln 10 - \ln x) = 0 - \dfrac{1}{x} = -\dfrac{1}{x},\ x > 0$

**25.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\ln(x+2) = \dfrac{1}{x+2}\dfrac{d}{dx}(x+2) = \dfrac{1}{x+2},\ x > -2$

**26.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\ln(2x+2) = \dfrac{1}{2x+2}\dfrac{d}{dx}(2x+2) = \dfrac{2}{2x+2}$

$\qquad = \dfrac{1}{x+1},\ x > -1$

**27.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\ln(2-\cos x) = \dfrac{1}{2-\cos x}\dfrac{d}{dx}(2-\cos x)$

$\qquad = \dfrac{\sin x}{2-\cos x}$

**28.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\ln(x^2+1) = \dfrac{1}{x^2+1}\dfrac{d}{dx}(x^2+1) = \dfrac{2x}{x^2+1}$

**29.** $\dfrac{d}{dx}\ln(\ln x) = \dfrac{1}{\ln x}\dfrac{d}{dx}\ln x = \dfrac{1}{\ln x}\cdot\dfrac{1}{x} = \dfrac{1}{x\ln x}$

**30.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(x\ln x - x) = (x)\left(\dfrac{1}{x}\right) + (\ln x)(1) - 1$

$\qquad = 1 + \ln x - 1 = \ln x$

**31.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(\log_4 x^2) = \dfrac{d}{dx}\dfrac{\ln x^2}{\ln 4} = \dfrac{d}{dx}\left[\left(\dfrac{2}{\ln 4}\right)(\ln x)\right]$

$\qquad = \dfrac{2}{\ln 4}\cdot\dfrac{1}{x} = \dfrac{2}{x\ln 4} = \dfrac{1}{x\ln 2}$

**32.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(\log_5\sqrt{x}) = \dfrac{d}{dx}\dfrac{\ln x^{1/2}}{\ln 5} = \dfrac{d}{dx}\dfrac{\frac{1}{2}\ln x}{\ln 5}$

$\qquad = \dfrac{1}{2\ln 5}\dfrac{d}{dx}(\ln x) = \dfrac{1}{2\ln 5}\cdot\dfrac{1}{x} = \dfrac{1}{2x\ln 5},\ x > 0$

**33.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\log_2(3x+1) = \dfrac{1}{(3x+1)\ln 2}\dfrac{d}{dx}(3x+1)$

$\qquad = \dfrac{3}{(3x+1)\ln 2},\ x > -\dfrac{1}{3}$

**34.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\log_{10}(x+1)^{1/2} = \dfrac{1}{2}\dfrac{d}{dx}\log_{10}(x+1)$

$\qquad = \dfrac{1}{2}\dfrac{1}{(x+1)\ln 10}\dfrac{d}{dx}(x+1) = \dfrac{1}{2(x+1)\ln 10},\ x > -1$

**35.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\log_2\left(\dfrac{1}{x}\right) = \dfrac{d}{dx}(-\log_2 x) = -\dfrac{1}{x\ln 2},\ x > 0$

**36.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\dfrac{1}{\log_2 x} = -\dfrac{1}{(\log_2 x)^2}\dfrac{d}{dx}(\log_2 x)$

$\qquad = -\dfrac{1}{(\log_2 x)^2}\dfrac{1}{x\ln 2} = -\dfrac{1}{x(\ln 2)(\log_2 x)^2}$ or $-\dfrac{\ln 2}{x(\ln x)^2}$

**37.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(\ln 2 \cdot \log_2 x) = (\ln 2)\dfrac{d}{dx}(\log_2 x)$

$\qquad = (\ln 2)\left(\dfrac{1}{x\ln 2}\right) = \dfrac{1}{x},\ x > 0$

**38.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\log_3(1 + x\ln 3) = \dfrac{1}{(1 + x\ln 3)\ln 3}\dfrac{d}{dx}(1 + x\ln 3)$

$\qquad = \dfrac{\ln 3}{(1 + x\ln 3)\ln 3} = \dfrac{1}{1 + x\ln 3},\ x > -\dfrac{1}{\ln 3}$

**39.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(\log_{10} e^x) = \dfrac{d}{dx}(x\log_{10} e) = \log_{10} e = \dfrac{\ln e}{\ln 10}$

$\qquad = \dfrac{1}{\ln 10}$

**40.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\ln 10^x = \dfrac{d}{dx}(x\ln 10) = \ln 10$

**41.** The line passes through $(a, e^a)$ for some value of $a$ and has

slope $m = e^a$. Since the line also passes through the origin,

the slope is also given by $m = \dfrac{e^a - 0}{a - 0}$ and we have

$e^a = \dfrac{e^a}{a}$, so $a = 1$. Hence, the slope is $e$ and the equation is

$y = ex$.

**42.** For $y = xe^x$, we have $y' = (x)(e^x) + (e^x)(1) = (x + 1)e^x$, so

the normal line through the point $(a, ae^a)$

has slope $m = -\dfrac{1}{(a + 1)e^a}$ and its equation is

$y = -\dfrac{1}{(a + 1)e^a}(x - a) + ae^a$. The desired normal line

includes the point $(0, 0)$, so we have:

$$0 = -\frac{1}{(a + 1)e^a}(0 - a) + ae^a$$

$$0 = \frac{a}{(a + 1)e^a} + ae^a$$

$$0 = a\left(\frac{1}{(a + 1)e^a} + e^a\right)$$

$$a = 0 \text{ or } \frac{1}{(a + 1)e^a} + e^a = 0$$

The equation $\dfrac{1}{(a + 1)e^a} + e^a = 0$ has no solution, so we

need to use $a = 0$. The equation of the normal line is

$y = -\dfrac{1}{(0 + 1)e^0}(x - 0) + 0e^0$, or $y = -x$.

**43.**
$$y = (\sin x)^x$$

$$\ln y = \ln (\sin x)^x$$

$$\ln y = x \ln (\sin x)$$

$$\frac{d}{dx} \ln y = \frac{d}{dx}[x \ln (\sin x)]$$

$$\frac{1}{y}\frac{dy}{dx} = (x)\left(\frac{1}{\sin x}\right)(\cos x) + \ln (\sin x)(1)$$

$$\frac{dy}{dx} = y[x \cot x + \ln (\sin x)]$$

$$\frac{dy}{dx} = (\sin x)^x[x \cot x + \ln (\sin x)]$$

**44.**
$$y = x^{\tan x}$$

$$\ln y = \ln (x^{\tan x})$$

$$\ln y = (\tan x)(\ln x)$$

$$\frac{d}{dx} \ln y = \frac{d}{dx}[(\tan x)(\ln x)]$$

$$\frac{1}{y}\frac{dy}{dx} = (\tan x)\left(\frac{1}{x}\right) + (\ln x)(\sec^2 x)$$

$$\frac{dy}{dx} = y\left[\frac{\tan x}{x} + (\ln x)(\sec^2 x)\right]$$

$$\frac{dy}{dx} = x^{\tan x}\left[\frac{\tan x}{x} + (\ln x)(\sec^2 x)\right]$$

**45.**
$$y = \sqrt[5]{\frac{(x - 3)^4(x^2 + 1)}{(2x + 5)^3}} = \left(\frac{(x - 3)^4(x^2 + 1)}{(2x + 5)^3}\right)^{1/5}$$

$$\ln y = \ln\left(\frac{(x - 3)^4(x^2 + 1)}{(2x + 5)^3}\right)^{1/5}$$

$$\ln y = \frac{1}{5} \ln \frac{(x - 3)^4(x^2 + 1)}{(2x + 5)^3}$$

$$\ln y = \frac{1}{5}[4 \ln (x - 3) + \ln (x^2 + 1) - 3 \ln (2x + 5)]$$

$$\frac{d}{dx}(\ln y) = \frac{4}{5}\frac{d}{dx} \ln (x - 3) +$$

$$\frac{1}{5}\frac{d}{dx} \ln (x^2 + 1) - \frac{3}{5}\frac{d}{dx} \ln (2x + 5)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{4}{5}\frac{1}{x - 3} + \frac{1}{5}\frac{1}{x^2 + 1}(2x) - \frac{3}{5}\frac{1}{2x + 5} \quad (2)$$

$$\frac{dy}{dx} = y\left(\frac{4}{5(x - 3)} + \frac{2x}{5(x^2 + 1)} - \frac{6}{5(2x + 5)}\right)$$

$$\frac{dy}{dx} = \left(\frac{(x - 3)^4(x^2 + 1)}{(2x + 5)^3}\right)^{1/5} \cdot$$

$$\left(\frac{4}{5(x - 3)} + \frac{2x}{5(x^2 + 1)} - \frac{6}{5(2x + 5)}\right)$$

**46.**
$$y = \frac{x\sqrt{x^2 + 1}}{(x + 1)^{2/3}} = \frac{x(x^2 + 1)^{1/2}}{(x + 1)^{2/3}}$$

$$\ln y = \ln \frac{x(x^2 + 1)^{1/2}}{(x + 1)^{2/3}}$$

$$\ln y = \ln x + \frac{1}{2} \ln (x^2 + 1) - \frac{2}{3} \ln (x + 1)$$

$$\frac{d}{dx} \ln y = \frac{d}{dx} \ln x + \frac{1}{2}\frac{d}{dx} \ln (x^2 + 1) - \frac{2}{3}\frac{d}{dx} \ln (x + 1)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{1}{2}\frac{1}{x^2 + 1}(2x) - \frac{2}{3}\frac{1}{x + 1} \quad (1)$$

$$\frac{dy}{dx} = y\left(\frac{1}{x} + \frac{x}{x^2 + 1} - \frac{2}{3(x + 1)}\right)$$

$$\frac{dy}{dx} = \frac{x\sqrt{x^2 + 1}}{(x + 1)^{2/3}}\left(\frac{1}{x} + \frac{x}{x^2 + 1} - \frac{2}{3(x + 1)}\right)$$

**47.**
$$\frac{dA}{dt} = 20\frac{d}{dt}\left(\frac{1}{2}\right)^{t/140}$$

$$= 20\frac{d}{dt} 2^{-t/140}$$

$$= 20 (2^{-t/140})(\ln 2)\frac{d}{dt}\left(-\frac{t}{140}\right)$$

$$= 20(2^{-t/140})(\ln 2)\left(-\frac{1}{140}\right)$$

$$= -\frac{(2^{-t/140})(\ln 2)}{7}$$

At $t = 2$ days, we have

$$\frac{dA}{dt} = -\frac{(2^{-1/70})(\ln 2)}{7} \approx -0.098 \text{ grams/day}.$$

This means that the rate of *decay* is the positive rate of

approximately 0.098 grams/day.

**48. (a)** $\dfrac{d}{dx} \ln (kx) = \dfrac{1}{kx}\dfrac{d}{dx} kx = \dfrac{k}{kx} = \dfrac{1}{x}$

**(b)** $\dfrac{d}{dx} \ln (kx) = \dfrac{d}{dx} (\ln k + \ln x)$

$= 0 + \dfrac{d}{dx} \ln x = \dfrac{1}{x}$

**49. (a)** Since $f'(x) = 2^x \ln 2$, $f'(0) = 2^0 \ln 2 = \ln 2$.

**(b)** $f'(0) = \lim_{h \to 0} \dfrac{f(h) - f(0)}{h} = \lim_{h \to 0} \dfrac{2^h - 2^0}{h} = \lim_{h \to 0} \dfrac{2^h - 1}{h}$

**(c)** Since quantities in parts (a) and (b) are equal,

$\lim_{h \to 0} \dfrac{2^h - 1}{h} = \ln 2$.

**(d)** By following the same procedure as above using

$g(x) = 7^x$, we may see that $\lim_{h \to 0} \dfrac{7^h - 1}{h} = \ln 7$.

**50.** Recall that a point $(a, b)$ is on the graph of $y = e^x$ if and only if the point $(b, a)$ is on the graph of $y = \ln x$. Since there are points $(x, e^x)$ on the graph of $y = e^x$ with arbitrarily large $x$-coordinates, there will be points $(x, \ln x)$ on the graph of $y = \ln x$ with arbitrarily large $y$-coordinates.

**51. (a)** The graph $y_4$ is a horizontal line at $y = a$.

**(b)** The graph of $y_3$ is always a horizontal line.

| $a$ | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| $y_3$ | 0.693147 | 1.098613 | 1.386295 | 1.609439 |
| $\ln a$ | 0.693147 | 1.098612 | 1.386294 | 1.609438 |

We conclude that the graph of $y_3$ is a horizontal line at $y = \ln a$.

**(c)** $\dfrac{d}{dx} a^x = a^x$ if and only if $y_3 = \dfrac{y_2}{y_1} = 1$.

So if $y_3 = \ln a$, then $\dfrac{d}{dx} a^x$ will equal $a^x$ if and only if $\ln a = 1$, or $a = e$.

**(d)** $y_2 = \dfrac{d}{dx} a^x = a^x \ln a$. This will equal $y_1 = a^x$ if and only if $\ln a = 1$, or $a = e$.

**52.** $\dfrac{d}{dx}\left(-\dfrac{1}{2}x^2 + k\right) = -x$ and $\dfrac{d}{dx}(\ln x + c) = \dfrac{1}{x}$.

Therefore, at any given value of $x$, these two curves will have perpendicular tangent lines.

**53. (a)** Since the line passes through the origin and has slope $\dfrac{1}{e}$, its equation is $y = \dfrac{x}{e}$.

**(b)** The graph of $y = \ln x$ lies below the graph of the line $y = \dfrac{x}{e}$ for all positive $x \neq e$. Therefore, $\ln x < \dfrac{x}{e}$ for all positive $x \neq e$.

**(c)** Multiplying by $e$, $e \ln x < x$ or $\ln x^e < x$.

**(d)** Exponentiating both sides of $\ln x^e < x$, we have $e^{\ln x^e} < e^x$, or $x^e < e^x$ for all positive $x \neq e$.

**(e)** Let $x = \pi$ to see that $\pi^e < e^\pi$. Therefore, $e^\pi$ is bigger.

## ■ Chapter 3 Review Exercises

(pp. 172–175)

**1.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(x^5 - \dfrac{1}{8}x^2 + \dfrac{1}{4}x\right) = 5x^4 - \dfrac{1}{4}x + \dfrac{1}{4}$

**2.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(3 - 7x^3 + 3x^7) = -21x^2 + 21x^6$

**3.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(2 \sin x \cos x)$

$= 2(\sin x)\dfrac{d}{dx}(\cos x) + 2(\cos x)\dfrac{d}{dx}(\sin x)$

$= -2 \sin^2 x + 2 \cos^2 x$

Alternate solution:

$\dfrac{dy}{dx} = \dfrac{d}{dx}(2 \sin x \cos x) = \dfrac{d}{dx} \sin 2x = (\cos 2x)(2)$

$= 2 \cos 2x$

**4.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\dfrac{2x + 1}{2x - 1} = \dfrac{(2x - 1)(2) - (2x + 1)(2)}{(2x - 1)^2} = -\dfrac{4}{(2x - 1)^2}$

**5.** $\dfrac{ds}{dt} = \dfrac{d}{dt}\cos(1 - 2t) = -\sin(1 - 2t)(-2) = 2\sin(1 - 2t)$

**6.** $\dfrac{ds}{dt} = \dfrac{d}{dt}\cot\left(\dfrac{2}{t}\right) = -\csc^2\left(\dfrac{2}{t}\right)\dfrac{d}{dt}\left(\dfrac{2}{t}\right) = -\csc^2\left(\dfrac{2}{t}\right)\left(-\dfrac{2}{t^2}\right)$

$= \dfrac{2}{t^2}\csc^2\left(\dfrac{2}{t}\right)$

**7.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\sqrt{x} + 1 + \dfrac{1}{\sqrt{x}}\right) = \dfrac{d}{dx}(x^{1/2} + 1 + x^{-1/2})$

$= \dfrac{1}{2}x^{-1/2} - \dfrac{1}{2}x^{-3/2} = \dfrac{1}{2\sqrt{x}} - \dfrac{1}{2x^{3/2}}$

**8.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(x\sqrt{2x + 1}) = (x)\left(\dfrac{1}{2\sqrt{2x + 1}}\right)(2) + (\sqrt{2x + 1})(1)$

$= \dfrac{x + (2x + 1)}{\sqrt{2x + 1}} = \dfrac{3x + 1}{\sqrt{2x + 1}}$

**9.** $\dfrac{dr}{d\theta} = \dfrac{d}{d\theta}\sec(1 + 3\theta) = \sec(1 + 3\theta)\tan(1 + 3\theta)(3)$

$= 3\sec(1 + 3\theta)\tan(1 + 3\theta)$

**10.** $\dfrac{dr}{d\theta} = \dfrac{d}{d\theta}\tan^2(3 - \theta^2)$

$= 2\tan(3 - \theta^2)\dfrac{d}{d\theta}\tan(3 - \theta^2)$

$= 2\tan(3 - \theta^2)\sec^2(3 - \theta^2)(-2\theta)$

$= -4\theta\tan(3 - \theta^2)\sec^2(3 - \theta^2)$

**11.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(x^2 \csc 5x)$

$= (x^2)(-\csc 5x \cot 5x)(5) + (\csc 5x)(2x)$

$= -5x^2 \csc 5x \cot 5x + 2x \csc 5x$

**12.** $\dfrac{dy}{dx} = \dfrac{d}{dx} \ln \sqrt{x} = \dfrac{1}{\sqrt{x}}\dfrac{d}{dx}\sqrt{x} = \dfrac{1}{\sqrt{x}} \cdot \dfrac{1}{2\sqrt{x}} = \dfrac{1}{2x}, \; x > 0$

**13.** $\dfrac{dy}{dx} = \dfrac{d}{dx} \ln(1 + e^x) = \dfrac{1}{1 + e^x}\dfrac{d}{dx}(1 + e^x) = \dfrac{e^x}{1 + e^x}$

**14.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(xe^{-x}) = (x)(e^{-x})(-1) + (e^{-x})(1) = -xe^{-x} + e^{-x}$

**15.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(e^{1+\ln x}) = \dfrac{d}{dx}(e^{1}e^{\ln x}) = \dfrac{d}{dx}(ex) = e$

**16.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\ln(\sin x) = \dfrac{1}{\sin x}\dfrac{d}{dx}(\sin x) = \dfrac{\cos x}{\sin x} = \cot x$, for

values of $x$ in the intervals $(k\pi, (k+1)\pi)$, where $k$ is even.

**17.** $\dfrac{dr}{dx} = \dfrac{d}{dx}\ln(\cos^{-1}x) = \dfrac{1}{\cos^{-1}x}\dfrac{d}{dx}\cos^{-1}x$

$= \dfrac{1}{\cos^{-1}x}\left(-\dfrac{1}{\sqrt{1-x^2}}\right) = -\dfrac{1}{\cos^{-1}x\sqrt{1-x^2}}$

**18.** $\dfrac{dr}{d\theta} = \dfrac{d}{d\theta}\log_2(\theta^2) = \dfrac{1}{\theta^2\ln 2}\dfrac{d}{d\theta}(\theta^2) = \dfrac{2\theta}{\theta^2\ln 2} = \dfrac{2}{\theta\ln 2}$

**19.** $\dfrac{ds}{dt} = \dfrac{d}{dt}\log_5(t-7) = \dfrac{1}{(t-7)\ln 5}\dfrac{d}{dt}(t-7) = \dfrac{1}{(t-7)\ln 5}$,

$t > 7$

**20.** $\dfrac{ds}{dt} = \dfrac{d}{dt}(8^{-t}) = 8^{-t}(\ln 8)\dfrac{d}{dt}(-t) = -8^{-t}\ln 8$

**21.** Use logarithmic differentiation.

$$y = x^{\ln x}$$

$$\ln y = \ln(x^{\ln x})$$

$$\ln y = (\ln x)(\ln x)$$

$$\dfrac{d}{dx}\ln y = \dfrac{d}{dx}(\ln x)^2$$

$$\dfrac{1}{y}\dfrac{dy}{dx} = 2\ln x\dfrac{d}{dx}\ln x$$

$$\dfrac{dy}{dx} = \dfrac{2y\ln x}{x}$$

$$\dfrac{dy}{dx} = \dfrac{2x^{\ln x}\ln x}{x}$$

**22.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\dfrac{(2x)2^x}{\sqrt{x^2+1}}$

$= \dfrac{\sqrt{x^2+1}\dfrac{d}{dx}[(2x)2^x] - (2x)(2^x)\dfrac{d}{dx}\sqrt{x^2+1}}{x^2+1}$

$= \dfrac{\sqrt{x^2+1}[(2x)(2^x)(\ln 2) + (2^x)(2)] - (2x)(2^x)\dfrac{1}{2\sqrt{x^2+1}}(2x)}{x^2+1}$

$= \dfrac{(x^2+1)(2^x)(2x\ln 2 + 2) - 2x^2(2^x)}{(x^2+1)^{3/2}}$

$= \dfrac{(2\cdot 2^x)[(x^2+1)(x\ln 2 + 1) - x^2]}{(x^2+1)^{3/2}}$

$= \dfrac{(2\cdot 2^x)(x^3\ln 2 + x^2 + x\ln 2 + 1 - x^2)}{(x^2+1)^{3/2}}$

$= \dfrac{(2\cdot 2^x)(x^3\ln 2 + x\ln 2 + 1)}{(x^2+1)^{3/2}}$

Alternate solution, using logarithmic differentiation:

$$y = \dfrac{(2x)2^x}{\sqrt{x^2+1}}$$

$$\ln y = \ln(2x) + \ln(2^x) - \ln\sqrt{x^2+1}$$

$$\ln y = \ln 2 + \ln x + x\ln 2 - \dfrac{1}{2}\ln(x^2+1)$$

$$\dfrac{d}{dx}\ln y = \dfrac{d}{dx}[\ln 2 + \ln x + x\ln 2 - \dfrac{1}{2}\ln(x^2+1)]$$

$$\dfrac{1}{y}\dfrac{dy}{dx} = 0 + \dfrac{1}{x} + \ln 2 - \dfrac{1}{2}\dfrac{1}{x^2+1}(2x)$$

$$\dfrac{dy}{dx} = y\left(\dfrac{1}{x} + \ln 2 - \dfrac{x}{x^2+1}\right)$$

$$\dfrac{dy}{dx} = \dfrac{(2x)2^x}{\sqrt{x^2+1}}\left(\dfrac{1}{x} + \ln 2 - \dfrac{x}{x^2+1}\right)$$

**23.** $\dfrac{dy}{dx} = \dfrac{d}{dx}e^{\tan^{-1}x} = e^{\tan^{-1}x}\dfrac{d}{dx}\tan^{-1}x = \dfrac{e^{\tan^{-1}x}}{1+x^2}$

**24.** $\dfrac{dy}{du} = \dfrac{d}{du}\sin^{-1}\sqrt{1-u^2}$

$= \dfrac{1}{\sqrt{1-(\sqrt{1-u^2})^2}}\dfrac{d}{du}\sqrt{1-u^2}$

$= \dfrac{1}{\sqrt{u^2}}\dfrac{1}{2\sqrt{1-u^2}}(-2u) = -\dfrac{u}{|u|\sqrt{1-u^2}}$

**25.** $\dfrac{dy}{dt} = \dfrac{d}{dt}(t\sec^{-1}t - \dfrac{1}{2}\ln t)$

$= (t)\left(\dfrac{1}{|t|\sqrt{t^2-1}}\right) + (\sec^{-1}t)(1) - \dfrac{1}{2t}$

$= \dfrac{t}{|t|\sqrt{t^2-1}} + \sec^{-1}t - \dfrac{1}{2t}$

**26.** $\dfrac{dy}{dt} = \dfrac{d}{dt}[(1+t^2)\cot^{-1}2t]$

$= (1+t^2)\left(-\dfrac{1}{1+(2t)^2}\right)(2) + (\cot^{-1}2t)(2t)$

$= -\dfrac{2+2t^2}{1+4t^2} + 2t\cot^{-1}2t$

**27.** $\dfrac{dy}{dz} = \dfrac{d}{dz}(z\cos^{-1}z - \sqrt{1-z^2})$

$= (z)\left(-\dfrac{1}{\sqrt{1-z^2}}\right) + (\cos^{-1}z)(1) - \dfrac{1}{2\sqrt{1-z^2}}(-2z)$

$= -\dfrac{z}{\sqrt{1-z^2}} + \cos^{-1}z + \dfrac{z}{\sqrt{1-z^2}}$

$= \cos^{-1}z$

**28.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(2\sqrt{x-1}\csc^{-1}\sqrt{x})$

$= (2\sqrt{x-1})\left(-\dfrac{1}{|\sqrt{x}|\sqrt{(\sqrt{x})^2-1}}\right)\left(\dfrac{1}{2\sqrt{x}}\right)$

$+ (2\csc^{-1}\sqrt{x})\left(\dfrac{1}{2\sqrt{x-1}}\right)$

$= -\dfrac{\sqrt{x-1}}{(\sqrt{x})^2\sqrt{x-1}} + \dfrac{\csc^{-1}\sqrt{x}}{\sqrt{x-1}}$

$= -\dfrac{1}{x} + \dfrac{\csc^{-1}\sqrt{x}}{\sqrt{x-1}}$

**29.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\csc^{-1}(\sec x)$

$= \left(-\dfrac{1}{|\sec x|\,\sqrt{\sec^2 x - 1}}\right)\dfrac{d}{dx}(\sec x)$

$= -\dfrac{1}{|\sec x|\,\sqrt{\tan^2 x}}\,\sec x\,\tan x$

$= -\dfrac{\sec x\,\tan x}{|\sec x\,\tan x|}$

$= -\dfrac{\dfrac{1}{\cos x}\dfrac{\sin x}{\cos x}}{\left|\dfrac{1}{\cos x}\dfrac{\sin x}{\cos x}\right|} = -\dfrac{\sin x}{|\sin x|}$

$= -\text{sign}(\sin x),\; x \neq \dfrac{\pi}{2},\, \pi,\, \dfrac{3\pi}{2}$

Alternate method:

On the domain $0 \leq x \leq 2\pi$, $x \neq \dfrac{\pi}{2}$, $x \neq \dfrac{3\pi}{2}$, we may

rewrite the function as follows:

$y = \csc^{-1}(\sec x)$

$= \dfrac{\pi}{2} - \sec^{-1}(\sec x)$

$= \dfrac{\pi}{2} - \cos^{-1}(\cos x)$

$= \begin{cases} \dfrac{\pi}{2} - x, & 0 \leq x \leq \pi, \quad x \neq \dfrac{\pi}{2} \\ \dfrac{\pi}{2} - (\pi - x), & \pi < x \leq 2\pi, \quad x \neq \dfrac{3\pi}{2} \end{cases}$

$= \begin{cases} \dfrac{\pi}{2} - x, & 0 \leq x \leq \pi, \quad x \neq \dfrac{\pi}{2} \\ -\dfrac{\pi}{2} + x, & \pi < x \leq 2\pi, \quad x \neq \dfrac{3\pi}{2} \end{cases}$

Therefore, $\dfrac{dy}{dx} = \begin{cases} -1, & 0 \leq x < \pi, \quad x \neq \dfrac{\pi}{2} \\ 1, & \pi < x \leq 2\pi, \; x \neq \dfrac{\pi}{2} \end{cases}$

Note that the derivative exists at 0 and $2\pi$ only because these are the endpoints of the given domain; the two-sided derivative of $y = \csc^{-1}(\sec x)$ does not exist at these points.

**30.** $\dfrac{dr}{d\theta} = \dfrac{d}{d\theta}\left(\dfrac{1 + \sin\theta}{1 - \cos\theta}\right)^2$

$= 2\left(\dfrac{1 + \sin\theta}{1 - \cos\theta}\right)\left(\dfrac{(1 - \cos\theta)(\cos\theta) - (1 + \sin\theta)(\sin\theta)}{(1 - \cos\theta)^2}\right)$

$= 2\left(\dfrac{1 + \sin\theta}{1 - \cos\theta}\right)\left(\dfrac{\cos\theta - \cos^2\theta - \sin\theta - \sin^2\theta}{(1 - \cos\theta)^2}\right)$

$= 2\left(\dfrac{1 + \sin\theta}{1 - \cos\theta}\right)\left(\dfrac{\cos\theta - \sin\theta - 1}{(1 - \cos\theta)^2}\right)$

**31.** Since $y = \ln x^2$ is defined for all $x \neq 0$ and

$\dfrac{dy}{dx} = \dfrac{1}{x^2}\dfrac{d}{dx}(x^2) = \dfrac{2x}{x^2} = \dfrac{2}{x}$, the function is differentiable for

all $x \neq 0$.

**32.** Since $y = \sin x - x \cos x$ is defined for all real $x$ and

$\dfrac{dy}{dx} = \cos x - (x)(-\sin x) - (\cos x)(1) = x \sin x$, the

function is differentiable for all real $x$.

**33.** Since $y = \sqrt{\dfrac{1 - x}{1 + x^2}}$ is defined for all $x < 1$ and

$\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{\dfrac{1 - x}{1 + x^2}}}\dfrac{(1 + x^2)(-1) - (1 - x)(2x)}{(1 + x^2)^2}$

$= \dfrac{x^2 - 2x - 1}{2\sqrt{1 - x}\,(1 + x^2)^{3/2}}$, which is defined only for $x < 1$,

the function is differentiable for all $x < 1$.

**34.** Since $y = (2x - 7)^{-1}(x + 5) = \dfrac{x + 5}{2x - 7}$ is defined for all

$x \neq \dfrac{7}{2}$ and $\dfrac{dy}{dx} = \dfrac{(2x - 7)(1) - (x + 5)(2)}{(2x - 7)^2} = -\dfrac{17}{(2x - 7)^2}$, the

function is differentiable for all $x \neq \dfrac{7}{2}$.

**35.** Use implicit differentiation.

$$xy + 2x + 3y = 1$$

$$\dfrac{d}{dx}(xy) + \dfrac{d}{dx}(2x) + \dfrac{d}{dx}(3y) = \dfrac{d}{dx}(1)$$

$$x\dfrac{dy}{dx} + (y)(1) + 2 + 3\dfrac{dy}{dx} = 0$$

$$(x + 3)\dfrac{dy}{dx} = -(y + 2)$$

$$\dfrac{dy}{dx} = -\dfrac{y + 2}{x + 3}$$

**36.** Use implicit differentiation.

$$5x^{4/5} + 10y^{6/5} = 15$$

$$\dfrac{d}{dx}(5x^{4/5}) + \dfrac{d}{dx}(10y^{6/5}) = \dfrac{d}{dx}(15)$$

$$4x^{-1/5} + 12y^{1/5}\dfrac{dy}{dx} = 0$$

$$\dfrac{dy}{dx} = -\dfrac{4x^{-1/5}}{12y^{1/5}} = -\dfrac{1}{3(xy)^{1/5}}$$

**37.** Use implicit differentiation.

$$\sqrt{xy} = 1$$

$$\dfrac{d}{dx}\sqrt{xy} = \dfrac{d}{dx}(1)$$

$$\dfrac{1}{2\sqrt{xy}}[x\dfrac{dy}{dx} + (y)(1)] = 0$$

$$x\dfrac{dy}{dx} + y = 0$$

$$\dfrac{dy}{dx} = -\dfrac{y}{x}$$

Alternate method:

Since $\sqrt{xy} = 1$, we have $xy = 1$ and $y = \dfrac{1}{x}$.

Therefore, $\dfrac{dy}{dx} = -\dfrac{1}{x^2}$.

**38.** Use implicit differentiation.

$$y^2 = \frac{x}{x+1}$$

$$\frac{d}{dx}y^2 = \frac{d}{dx}\frac{x}{x+1}$$

$$2y\frac{dy}{dx} = \frac{(x+1)(1) - (x)(1)}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{1}{2y(x+1)^2}$$

**39.**         $x^3 + y^3 = 1$

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = \frac{d}{dx}(1)$$

$$3x^2 + 3y^2y' = 0$$

$$y' = -\frac{x^2}{y^2}$$

$$y'' = \frac{d}{dx}\left(-\frac{x^2}{y^2}\right)$$

$$= -\frac{(y^2)(2x) - (x^2)(2y)(y')}{y^4}$$

$$= -\frac{(y^2)(2x) - (x^2)(2y)\left(-\frac{x^2}{y^2}\right)}{y^4}$$

$$= -\frac{2xy^3 + 2x^4}{y^5}$$

$$= -\frac{2x(x^3 + y^3)}{y^5}$$

$$= -\frac{2x}{y^5}$$

since $x^3 + y^3 = 1$

**40.**        $y^2 = 1 - \frac{2}{x}$

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(1) - \frac{d}{dx}\left(\frac{2}{x}\right)$$

$$2yy' = \frac{2}{x^2}$$

$$y' = \frac{2}{x^2(2y)} = \frac{1}{x^2y}$$

$$y'' = \frac{d}{dx}\left(\frac{1}{x^2y}\right)$$

$$= -\frac{1}{(x^2y)^2}\frac{d}{dx}(x^2y)$$

$$= -\frac{1}{(x^2y)^2}[(x^2)(y') + (y)(2x)]$$

$$= -\frac{1}{(x^2y)^2}\left[(x^2)\left(\frac{1}{x^2y}\right) + 2xy\right]$$

$$= -\frac{1}{x^4y^2}\left(\frac{1}{y} + 2xy\right)$$

$$= -\frac{1 + 2xy^2}{x^4y^3}$$

**41.**         $y^3 + y = 2\cos x$

$$\frac{d}{dx}(y^3) + \frac{d}{dx}(y) = \frac{d}{dx}(2\cos x)$$

$$3y^2y' + y' = -2\sin x$$

$$(3y^2 + 1)y' = -2\sin x$$

$$y' = -\frac{2\sin x}{3y^2 + 1}$$

$$y'' = \frac{d}{dx}\left(-\frac{2\sin x}{3y^2 + 1}\right)$$

$$= -\frac{(3y^2 + 1)(2\cos x) - (2\sin x)(6yy')}{(3y^2 + 1)^2}$$

$$= -\frac{(3y^2 + 1)(2\cos x) - (12y\sin x)\left(-\frac{2\sin x}{3y^2+1}\right)}{(3y^2 + 1)^2}$$

$$= -2\frac{(3y^2 + 1)^2\cos x + 12y\sin^2 x}{(3y^2 + 1)^3}$$

**42.**         $x^{1/3} + y^{1/3} = 4$

$$\frac{d}{dx}(x^{1/3}) + \frac{d}{dx}(y^{1/3}) = \frac{d}{dx}(4)$$

$$\frac{1}{3}x^{-2/3} + \frac{1}{3}y^{-2/3}y' = 0$$

$$y' = -\frac{x^{-2/3}}{y^{-2/3}} = -\left(\frac{y}{x}\right)^{2/3}$$

$$y'' = \frac{d}{dx}\left[-\left(\frac{y}{x}\right)^{2/3}\right]$$

$$= -\frac{2}{3}\left(\frac{y}{x}\right)^{-1/3}\left(\frac{xy' - (y)(1)}{x^2}\right)$$

$$= -\frac{2}{3}\left(\frac{y}{x}\right)^{-1/3}\left(\frac{(x)\left[-\left(\frac{y}{x}\right)^{2/3}\right] - y}{x^2}\right)$$

$$= -\frac{2}{3}x^{1/3}y^{-1/3}(-x^{-5/3}y^{2/3} - x^{-2}y)$$

$$= \frac{2}{3}x^{-4/3}y^{1/3} + \frac{2}{3}x^{-5/3}y^{2/3}$$

**43.** $y' = 2x^3 - 3x - 1$,
$y'' = 6x^2 - 3$,
$y''' = 12x$,
$y^{(4)} = 12$, and the rest are all zero.

**44.** $y' = \frac{x^4}{24}$,

$y'' = \frac{x^3}{6}$,

$y''' = \frac{x^2}{2}$,

$y^{(4)} = x$,

$y^{(5)} = 1$, and the rest are all zero.

**45.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\sqrt{x^2 - 2x} = \dfrac{1}{2\sqrt{x^2 - 2x}}(2x - 2) = \dfrac{x - 1}{\sqrt{x^2 - 2x}}$

At $x = 3$, we have $y = \sqrt{3^2 - 2(3)} = \sqrt{3}$

and $\dfrac{dy}{dx} = \dfrac{3 - 1}{\sqrt{3^2 - 2(3)}} = \dfrac{2}{\sqrt{3}}$.

(a) Tangent: $y = \dfrac{2}{\sqrt{3}}(x - 3) + \sqrt{3}$ or $y = \dfrac{2}{\sqrt{3}}x - \sqrt{3}$

(b) Normal: $y = -\dfrac{\sqrt{3}}{2}(x - 3) + \sqrt{3}$

or $y = -\dfrac{\sqrt{3}}{2}x + \dfrac{5\sqrt{3}}{2}$

**46.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(4 + \cot x - 2 \csc x)$

$= -\csc^2 x + 2 \csc x \cot x$

At $x = \dfrac{\pi}{2}$, we have

$y = 4 + \cot \dfrac{\pi}{2} - 2 \csc \dfrac{\pi}{2} = 4 + 0 - 2 = 2$ and

$\dfrac{dy}{dx} = -\csc^2 \dfrac{\pi}{2} + 2 \csc \dfrac{\pi}{2} \cot \dfrac{\pi}{2} = -1 + 2(1)(0) = -1$.

(a) Tangent: $y = -1\left(x - \dfrac{\pi}{2}\right) + 2$ or $y = -x + \dfrac{\pi}{2} + 2$

(b) Normal: $y = 1\left(x - \dfrac{\pi}{2}\right) + 2$ or $y = x - \dfrac{\pi}{2} + 2$

**47.** Use implicit differentiation.

$x^2 + 2y^2 = 9$

$\dfrac{d}{dx}(x^2) + \dfrac{d}{dx}(2y^2) = \dfrac{d}{dx}(9)$

$2x + 4y\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = -\dfrac{2x}{4y} = -\dfrac{x}{2y}$

Slope at $(1, 2)$: $-\dfrac{1}{2(2)} = -\dfrac{1}{4}$

(a) Tangent: $y = -\dfrac{1}{4}(x - 1) + 2$ or $y = -\dfrac{1}{4}x + \dfrac{9}{4}$

(b) Normal: $y = 4(x - 1) + 2$ or $y = 4x - 2$

**48.** Use implicit differentiation.

$x + \sqrt{xy} = 6$

$\dfrac{d}{dx}(x) + \dfrac{d}{dx}(\sqrt{xy}) = \dfrac{d}{dx}(6)$

$1 + \dfrac{1}{2\sqrt{xy}}\left[(x)\left(\dfrac{dy}{dx}\right) + (y)(1)\right] = 0$

$\dfrac{x}{2\sqrt{xy}}\dfrac{dy}{dx} = -1 - \dfrac{y}{2\sqrt{xy}}$

$\dfrac{dy}{dx} = \dfrac{2\sqrt{xy}}{x}\left(-1 - \dfrac{y}{2\sqrt{xy}}\right)$

$= -2\sqrt{\dfrac{y}{x}} - \dfrac{y}{x}$

Slope at $(4, 1)$: $-2\sqrt{\dfrac{1}{4}} - \dfrac{1}{4} = -\dfrac{2}{2} - \dfrac{1}{4} = -\dfrac{5}{4}$

(a) Tangent: $y = -\dfrac{5}{4}(x - 4) + 1$ or $y = -\dfrac{5}{4}x + 6$

(b) Normal: $y = \dfrac{4}{5}(x - 4) + 1$ or $y = \dfrac{4}{5}x - \dfrac{11}{5}$

**49.** $\dfrac{dy}{dx} = \dfrac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \dfrac{-2 \sin t}{2 \cos t} = -\tan t$

At $t = \dfrac{3\pi}{4}$, we have $x = 2 \sin \dfrac{3\pi}{4} = \sqrt{2}$,

$y = 2 \cos \dfrac{3\pi}{4} = -\sqrt{2}$, and $\dfrac{dy}{dx} = -\tan \dfrac{3\pi}{4} = 1$.

The equation of the tangent line is

$y = 1(x - \sqrt{2}) + (-\sqrt{2})$, or $y = x - 2\sqrt{2}$.

**50.** $\dfrac{dy}{dx} = \dfrac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \dfrac{4 \cos t}{-3 \sin t} = -\dfrac{4}{3}\cot t$

At $t = \dfrac{3\pi}{4}$, we have $x = 3 \cos \dfrac{3\pi}{4} = -\dfrac{3\sqrt{2}}{2}$,

$y = 4 \sin \dfrac{3\pi}{4} = 2\sqrt{2}$, and $\dfrac{dy}{dx} = -\dfrac{4}{3}\cot \dfrac{3\pi}{4} = \dfrac{4}{3}$.

The equation of the tangent line is

$y = \dfrac{4}{3}\left(x + \dfrac{3\sqrt{2}}{2}\right) + 2\sqrt{2}$, or $y = \dfrac{4}{3}x + 4\sqrt{2}$.

**51.** $\dfrac{dy}{dx} = \dfrac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \dfrac{5 \sec^2 t}{3 \sec t \tan t} = \dfrac{5 \sec t}{3 \tan t} = \dfrac{5}{3 \sin t}$

At $t = \dfrac{\pi}{6}$, we have $x = 3 \sec \dfrac{\pi}{6} = 2\sqrt{3}$,

$y = 5 \tan \dfrac{\pi}{6} = \dfrac{5\sqrt{3}}{3}$, and $\dfrac{dy}{dx} = \dfrac{5}{3 \sin\left(\dfrac{\pi}{6}\right)} = \dfrac{10}{3}$.

The equation of the tangent line is

$y = \dfrac{10}{3}(x - 2\sqrt{3}) + \dfrac{5\sqrt{3}}{3}$, or $y = \dfrac{10}{3}x - 5\sqrt{3}$.

**52.** $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{1 + \cos t}{-\sin t}$

At $t = -\dfrac{\pi}{4}$, we have $x = \cos\left(-\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}$,

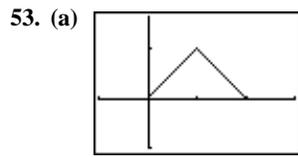$y = -\dfrac{\pi}{4} + \sin\left(-\dfrac{\pi}{4}\right) = -\dfrac{\pi}{4} - \dfrac{\sqrt{2}}{2}$, and

$\dfrac{dy}{dx} = \dfrac{1 + \cos\left(-\frac{\pi}{4}\right)}{-\sin\left(-\frac{\pi}{4}\right)} = \dfrac{1 + \frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \sqrt{2} + 1$.

The equation of the tangent line is

$y = (\sqrt{2} + 1)\left(x - \dfrac{\sqrt{2}}{2}\right) - \dfrac{\pi}{4} - \dfrac{\sqrt{2}}{2}$, or

$y = (1 + \sqrt{2})x - \sqrt{2} - 1 - \dfrac{\pi}{4}$.

This is approximately $y = 2.414x - 3.200$.

**53. (a)**



$[-1, 3]$ by $\left[-1, \dfrac{5}{3}\right]$

**(b)** Yes, because both of the one-sided limits as $x \to 1$ are equal to $f(1) = 1$.

**(c)** No, because the left-hand derivative at $x = 1$ is $+1$ and the right-hand derivative at $x = 1$ is $-1$.

**54. (a)** The function is continuous for all values of $m$, because the right-hand limit as $x \to 0$ is equal to $f(0) = 0$ for any value of $m$.

**(b)** The left-hand derivative at $x = 0$ is $2 \cos (2 \cdot 0) = 2$, and the right-hand derivative at $x = 0$ is $m$, so in order for the function to be differentiable at $x = 0$, $m$ must be 2.

**55. (a)** For all $x \neq 0$      **(b)** At $x = 0$

**(c)** Nowhere

**56. (a)** For all $x$      **(b)** Nowhere

**(c)** Nowhere

**57.** Note that $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} (2x - 3) = -3$ and

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} (x - 3) = -3$. Since these values agree

with $f(0)$, the function is continuous at $x = 0$. On the other

hand,

$f'(x) = \begin{cases} 2, & -1 \le x < 0 \\ 1, & 0 < x \le 4 \end{cases}$, so the derivative is undefined at

$x = 0$.

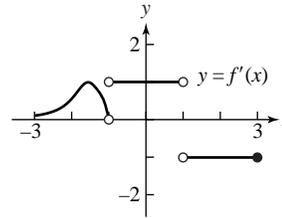**(a)** $[-1, 0) \cup (0, 4]$      **(b)** At $x = 0$

**(c)** Nowhere in its domain
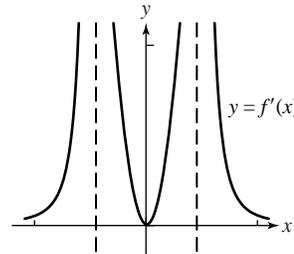
**58.** Note that the function is undefined at $x = 0$.

**(a)** $[-2, 0) \cup (0, 2]$      **(b)** Nowhere

**(c)** Nowhere in its domain

**59.**



**60.**



**61. (a)** iii                     **(b)** i
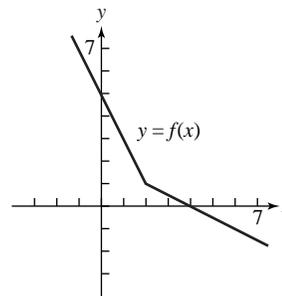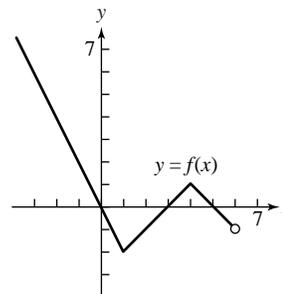
**(c)** ii

**62.** The graph passes through $(0, 5)$ and has slope $-2$ for $x < 2$ and slope $-0.5$ for $x > 2$.



**63.** The graph passes through $(-1, 2)$ and has slope $-2$ for $x < 1$, slope 1 for $1 < x < 4$, and slope $-1$ for $4 < x < 6$.



**64. i.** If $f(x) = \dfrac{9}{28}x^{7/3} + 9$, then $f'(x) = \dfrac{3}{4}x^{4/3}$ and $f''(x) = x^{1/3}$, which matches the given equation.
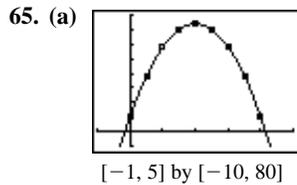
**ii.** If $f'(x) = \dfrac{9}{28}x^{7/3} - 2$, then $f''(x) = \dfrac{3}{4}x^{4/3}$, which contradicts the given equation $f''(x) = x^{1/3}$.

**iii.** If $f'(x) = \dfrac{3}{4}x^{4/3} + 6$, then $f''(x) = x^{1/3}$, which matches the given equation.
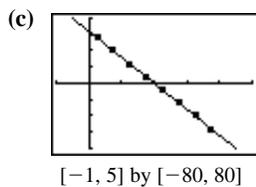
**64. continued**

**iv.** If $f(x) = \frac{3}{4}x^{4/3} - 4$, then $f'(x) = x^{1/3}$ and

$f''(x) = \frac{1}{3}x^{-2/3}$, which contradicts the given equation

$f''(x) = x^{1/3}$.

Answer is **D**: **i** and **iii** only could be true. Note,

however that **i** and **iii** could not simultaneously be true.

**65. (a)**



[−1, 5] by [−10, 80]

**(b)** 

| $t$ interval | avg. vel. |
|---|---|
| [0, 0.5] | $\frac{38 - 10}{0.5 - 0} = 56$ |
| [0.5, 1] | $\frac{58 - 38}{1 - 0.5} = 40$ |
| [1, 1.5] | $\frac{70 - 58}{1.5 - 1} = 24$ |
| [1.5, 2] | $\frac{74 - 70}{2 - 1.5} = 8$ |
| [2, 2.5] | $\frac{70 - 74}{2.5 - 2} = -8$ |
| [2.5, 3] | $\frac{58 - 70}{3 - 2.5} = -24$ |
| [3, 3.5] | $\frac{38 - 58}{3.5 - 3} = -40$ |
| [3.5, 4] | $\frac{10 - 38}{4 - 3.5} = -56$ |

**(c)**



[−1, 5] by [−80, 80]

**(d)** Average velocity is a good approximation to velocity.

**66. (a)** $\frac{d}{dx}[\sqrt{x}\,f(x)] = \sqrt{x}\,f'(x) + \frac{1}{2\sqrt{x}}f(x)$

At $x = 1$, the derivative is

$\sqrt{1}\,f'(1) + \frac{1}{2\sqrt{1}}f(1) = 1\left(\frac{1}{5}\right) + \left(\frac{1}{2}\right)(-3) = -\frac{13}{10}$.

**(b)** $\frac{d}{dx}\sqrt{f(x)} = \frac{1}{2\sqrt{f(x)}}f'(x) = \frac{f'(x)}{2\sqrt{f(x)}}$

At $x = 0$, the derivative is $\frac{f'(0)}{2\sqrt{f(0)}} = -\frac{2}{2\sqrt{9}} = -\frac{1}{3}$.

**(c)** $\frac{d}{dx}f(\sqrt{x}) = f'(\sqrt{x})\frac{d}{dx}\sqrt{x} = \frac{f'(\sqrt{x})}{2\sqrt{x}}$

At $x = 1$, the derivative is $\frac{f'(\sqrt{1})}{2\sqrt{1}} = \frac{f'(1)}{2} = \frac{\frac{1}{5}}{2} = \frac{1}{10}$.

**(d)** $\frac{d}{dx}f(1 - 5\tan x) = f'(1 - 5\tan x)(-5\sec^2 x)$

At $x = 0$, the derivative is

$f'(1 - 5\tan 0)(-5\sec^2 0) = f'(1)(-5) = \left(\frac{1}{5}\right)(-5)$

$= -1.$

**(e)** $\frac{d}{dx}\frac{f(x)}{2 + \cos x} = \frac{(2 + \cos x)(f'(x)) - (f(x))(-\sin x)}{(2 + \cos x)^2}$

At $x = 0$, the derivative is

$\frac{(2 + \cos 0)(f'(0)) - (f(0))(-\sin 0)}{(2 + \cos 0)^2} = \frac{3f'(0)}{3^2} = -\frac{2}{3}.$

**(f)** $\frac{d}{dx}[10\sin\left(\frac{\pi x}{2}\right)f^2(x)]$

$= 10\left(\sin\frac{\pi x}{2}\right)(2f(x)f'(x)) + 10f^2(x)\left(\cos\frac{\pi x}{2}\right)\left(\frac{\pi}{2}\right)$

$= 20\,f(x)f'(x)\sin\frac{\pi x}{2} + 5\pi f^2(x)\cos\frac{\pi x}{2}$

At $x = 1$, the derivative is

$20\,f(1)f'(1)\sin\frac{\pi}{2} + 5\pi f^2(1)\cos\frac{\pi}{2}$

$= 20(-3)\left(\frac{1}{5}\right)(1) + 5\pi(-3)^2(0)$

$= -12.$

**67. (a)** $\frac{d}{dx}[3f(x) - g(x)] = 3f'(x) - g'(x)$

At $x = -1$, the derivative is

$3f'(-1) - g'(-1) = 3(2) - 1 = 5.$

**(b)** $\frac{d}{dx}[f^2(x)g^3(x)]$

$= f^2(x) \cdot 3g^2(x)g'(x) + g^3(x) \cdot 2f(x)f'(x)$

$= f(x)g^2(x)[3f(x)g'(x) + 2g(x)f'(x)]$

At $x = 0$, the derivative is

$f(0)g^2(0)[3f(0)g'(0) + 2g(0)f'(0)]$

$= (-1)(-3)^2[3(-1)(4) + 2(-3)(-2)]$

$= -9[-12 + 12] = 0.$

**(c)** $\frac{d}{dx}g(f(x)) = g'(f(x))f'(x)$

At $x = -1$, the derivative is

$g'(f(-1))f'(-1) = g'(0)f'(-1) = (4)(2) = 8.$

**(d)** $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

At $x = -1$, the derivative is

$f'(g(-1))g'(-1) = f'(-1)g'(-1) = (2)(1) = 2.$

**(e)** $\frac{d}{dx}\frac{f(x)}{g(x) + 2} = \frac{(g(x) + 2)f'(x) - f(x)\,g'(x)}{(g(x) + 2)^2}$

At $x = 0$, the derivative is

$\frac{(g(0) + 2)f'(0) - f(0)g'(0)}{(g(0) + 2)^2} = \frac{(-3 + 2)(-2) - (-1)(4)}{(-3 + 2)^2}$

$= 6.$

**(f)** $\dfrac{d}{dx} g(x + f(x)) = g'(x + f(x))\dfrac{d}{dx}(x + f(x))$

$= g'(x + f(x))(1 + f'(x))$

At $x = 0$, the derivative is $g'(0 + f(0))[1 + f'(0)]$

$= g'(0 - 1)[1 + (-2)] = (1)(-1) = -1$

**68.** $\dfrac{dw}{ds} = \dfrac{dw}{dr}\dfrac{dr}{ds} = \dfrac{d}{dr}[\sin(\sqrt{r} - 2)]\dfrac{d}{ds}\left[8\sin\left(s + \dfrac{\pi}{6}\right)\right]$

$= \left[\cos(\sqrt{r} - 2)\dfrac{1}{2\sqrt{r}}\right]\left[8\cos\left(s + \dfrac{\pi}{6}\right)\right]$

At $s = 0$, we have $r = 8\sin\left(0 + \dfrac{\pi}{6}\right) = 4$ and so

$\dfrac{dw}{ds} = \left[\cos(\sqrt{4} - 2)\dfrac{1}{2\sqrt{4}}\right]\left[8\cos\left(0 + \dfrac{\pi}{6}\right)\right]$

$= \left(\dfrac{\cos 0}{4}\right)\left(8\cos\dfrac{\pi}{6}\right) = \left(\dfrac{1}{4}\right)\left(\dfrac{8\sqrt{3}}{2}\right) = \sqrt{3}$

**69.** Solving $\theta^2 t + \theta = 1$ for $t$, we have

$t = \dfrac{1 - \theta}{\theta^2} = \theta^{-2} - \theta^{-1}$, and we may write:

$$\dfrac{dr}{d\theta} = \dfrac{dr}{dt}\dfrac{dt}{d\theta}$$

$$\dfrac{d}{d\theta}(\theta^2 + 7)^{1/3} = \dfrac{dr}{dt}\dfrac{d}{d\theta}(\theta^{-2} - \theta^{-1})$$

$$\dfrac{1}{3}(\theta^2 + 7)^{-2/3}(2\theta) = \left(\dfrac{dr}{dt}\right)(-2\theta^{-3} + \theta^{-2})$$

$$\dfrac{dr}{dt} = \dfrac{2\theta(\theta^2 + 7)^{-2/3}}{3(-2\theta^{-3} + \theta^{-2})} = \dfrac{2\theta^4(\theta^2 + 7)^{-2/3}}{3(\theta - 2)}$$

At $t = 0$, we may solve $\theta^2 t + \theta = 1$ to obtain $\theta = 1$,

and so $\dfrac{dr}{dt} = \dfrac{2(1)^4(1^2 + 7)^{-2/3}}{3(1 - 2)} = \dfrac{2(8)^{-2/3}}{-3} = -\dfrac{1}{6}$.

**70. (a)** One possible answer:

$x(t) = 10\cos\left(t + \dfrac{\pi}{4}\right)$, $y(t) = 1$

**(b)** $s(0) = 10\cos\dfrac{\pi}{4} = 5\sqrt{2}$

**(c)** Farthest left:

When $\cos\left(t + \dfrac{\pi}{4}\right) = -1$, we have $s(t) = -10$.

Farthest right:

When $\cos\left(t + \dfrac{\pi}{4}\right) = 1$, we have $s(t) = 10$.

**(d)** Since $\cos\dfrac{\pi}{2} = 0$, the particle first reaches the origin at

$t = \dfrac{\pi}{4}$. The velocity is given by

$v(t) = -10\sin\left(t + \dfrac{\pi}{4}\right)$, so the velocity at

$t = \dfrac{\pi}{4}$ is $-10\sin\dfrac{\pi}{2} = -10$, and the

speed at $t = \dfrac{\pi}{4}$ is $|-10| = 10$. The acceleration is given

by $a(t) = -10\cos\left(t + \dfrac{\pi}{4}\right)$, so the acceleration at

$t = \dfrac{\pi}{4}$ is $-10\cos\dfrac{\pi}{2} = 0$.

**71. (a)** $\dfrac{ds}{dt} = \dfrac{d}{dt}(64t - 16t^2) = 64 - 32t$

$\dfrac{d^2s}{dt^2} = \dfrac{d}{dt}(64 - 32t) = -32$

**(b)** The maximum height is reached when $\dfrac{ds}{dt} = 0$, which

occurs at $t = 2$ sec.

**(c)** When $t = 0$, $\dfrac{ds}{dt} = 64$, so the velocity is 64 ft/sec.

**(d)** Since $\dfrac{ds}{dt} = \dfrac{d}{dt}(64t - 2.6t^2) = 64 - 5.2t$, the

maximum height is reached at $t = \dfrac{64}{5.2} \approx 12.3$ sec.

The maximum height is $s\left(\dfrac{64}{5.2}\right) \approx 393.8$ ft.

**72. (a)** Solving $160 = 490t^2$, it takes $\dfrac{4}{7}$ sec.

The average velocity is $\dfrac{160}{\frac{4}{7}} = 280$ cm/sec.

**(b)** Since $v(t) = \dfrac{ds}{dt} = 980t$, the velocity is

$(980)\left(\dfrac{4}{7}\right) = 560$ cm/sec. Since $a(t) = \dfrac{dv}{dt} = 980$, the

acceleration is 980 cm/sec$^2$.

**73.** $\dfrac{dV}{dx} = \dfrac{d}{dx}\left[\pi\left(10 - \dfrac{x}{3}\right)x^2\right] = \dfrac{d}{dx}\left[\pi\left(10x^2 - \dfrac{1}{3}x^3\right)\right]$

$= \pi(20x - x^2)$

**74. (a)** $r(x) = \left(3 - \dfrac{x}{40}\right)^2 x = 9x - \dfrac{3}{20}x^2 + \dfrac{1}{1600}x^3$

**(b)** The marginal revenue is

$r'(x) = 9 - \dfrac{3}{10}x + \dfrac{3}{1600}x^2$

$= \dfrac{3}{1600}(x^2 - 160x + 4800)$

$= \dfrac{3}{1600}(x - 40)(x - 120)$,

which is zero when $x = 40$ or $x = 120$. Since the bus
holds only 60 people, we require $0 \le x \le 60$. The
marginal revenue is 0 when there are 40 people, and the
corresponding fare is $p(40) = \left(3 - \dfrac{40}{40}\right)^2 = \$4.00$.

**74. continued**

(c) One possible answer:
If the current ridership is less than 40, then the proposed plan may be good. If the current ridership is greater than or equal to 40, then the plan is not a good idea. Look at the graph of $y = r(x)$.
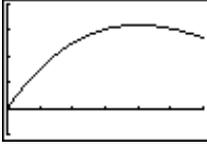


[0, 60] by [−50, 200]

**75. (a)** Since $x = \tan \theta$, we have

$\dfrac{dx}{dt} = (\sec^2 \theta)\dfrac{d\theta}{dt} = -0.6 \sec^2 \theta$. At point $A$, we have

$\theta = 0$ and $\dfrac{dx}{dt} = -0.6 \sec^2 0 = -0.6$ km/sec.

**(b)** $0.6 \dfrac{\text{rad}}{\text{sec}} \cdot \dfrac{1 \text{ revolution}}{2\pi \text{ rad}} \cdot \dfrac{60 \text{ sec}}{1 \text{ min}} = \dfrac{18}{\pi}$ revolutions per

minute or approximately 5.73 revolutions per minute.

**76.** Let $f(x) = \sin (x - \sin x)$. Then

$f'(x) = \cos (x - \sin x)\dfrac{d}{dx}(x - \sin x)$

$= \cos (x - \sin x)(1 - \cos x)$. This derivative is zero when

$\cos (x - \sin x) = 0$ (which we need not solve) or when

$\cos x = 1$, which occurs at $x = 2k\pi$ for integers $k$. For each

of these values, $f(x) = f(2k\pi) = \sin (2k\pi - \sin 2k\pi)$

$= \sin (2k\pi - 0) = 0$. Thus, $f(x) = f'(x) = 0$ for $x = 2k\pi$,

which means that the graph has a horizontal tangent at each

of these values of $x$.

**77.** $y'(r) = \dfrac{d}{dr}\left(\dfrac{1}{2rl}\sqrt{\dfrac{T}{\pi d}}\right) = \left(\dfrac{1}{2l}\sqrt{\dfrac{T}{\pi d}}\right)\dfrac{d}{dr}\left(\dfrac{1}{r}\right) = -\dfrac{1}{2r^2 l}\sqrt{\dfrac{T}{\pi d}}$

$y'(l) = \dfrac{d}{dl}\left(\dfrac{1}{2rl}\sqrt{\dfrac{T}{\pi d}}\right) = \left(\dfrac{1}{2r}\sqrt{\dfrac{T}{\pi d}}\right)\dfrac{d}{dl}\left(\dfrac{1}{l}\right) = -\dfrac{1}{2rl^2}\sqrt{\dfrac{T}{\pi d}}$

$y'(d) = \dfrac{d}{dd}\left(\dfrac{1}{2rl}\sqrt{\dfrac{T}{\pi d}}\right) = \left(\dfrac{1}{2rl}\sqrt{\dfrac{T}{\pi}}\right)\dfrac{d}{dd}(d^{-1/2})$

$= \dfrac{1}{2rl}\sqrt{\dfrac{T}{\pi}}\left(-\dfrac{1}{2}d^{-3/2}\right) = -\dfrac{1}{4rl}\sqrt{\dfrac{T}{\pi d^3}}$

$y'(T) = \dfrac{d}{dT}\left(\dfrac{1}{2rl}\sqrt{\dfrac{T}{\pi d}}\right) = \left(\dfrac{1}{2rl}\sqrt{\dfrac{1}{\pi d}}\right)\dfrac{d}{dT}(\sqrt{T})$

$= \dfrac{1}{2rl}\sqrt{\dfrac{1}{\pi d}}\left(\dfrac{1}{2\sqrt{T}}\right) = \dfrac{1}{4rl\sqrt{\pi d T}}$

Since $y'(r) < 0$, $y'(l) < 0$, and $y'(d) < 0$, increasing $r$, $l$, or

$d$ would decrease the frequency. Since $y'(T) > 0$,

increasing $T$ would increase the frequency.

**78. (a)** $P(0) = \dfrac{200}{1 + e^5} \approx 1$ student

**(b)** $\lim\limits_{t\to\infty} P(t) = \lim\limits_{t\to\infty} \dfrac{200}{1 + e^{5-t}} = \dfrac{200}{1} = 200$ students

**(c)** $P'(t) = \dfrac{d}{dt} 200(1 + e^{5-t})^{-1}$

$= -200(1 + e^{5-t})^{-2}(e^{5-t})(-1)$

$= \dfrac{200e^{5-t}}{(1 + e^{5-t})^2}$

$P''(t) = \dfrac{(1 + e^{5-t})^2(200e^{5-t})(-1) - (200e^{5-t})(2)(1 + e^{5-t})(e^{5-t})(-1)}{(1 + e^{5-t})^4}$

$= \dfrac{(1 + e^{5-t})(-200e^{5-t}) + 400(e^{5-t})^2}{(1 + e^{5-t})^3}$

$= \dfrac{(200e^{5-t})(e^{5-t} - 1)}{(1 + e^{5-t})^3}$

Since $P'' = 0$ when $t = 5$, the critical point of $y = P'(t)$
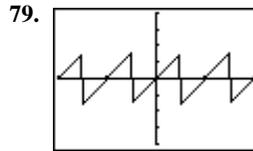
occurs at $t = 5$. To confirm that this corresponds to the

maximum value of $P'(t)$, note that $P''(t) > 0$ for $t < 5$

and $P''(t) < 0$ for $t > 5$. The maximum rate occurs at

$t = 5$, and this rate is

$P'(5) = \dfrac{200e^0}{(1 + e^0)^2} = \dfrac{200}{2^2} = 50$ students per day.

Note: This problem can also be solved graphically.

**79.**



[−π, π] by [−4, 4]

**(a)** $x \neq k\dfrac{\pi}{4}$, where $k$ is an odd integer

**(b)** $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

**(c)** Where it's not defined, at $x = k\dfrac{\pi}{4}$, $k$ an odd integer

**(d)** It has period $\dfrac{\pi}{2}$ and continues to repeat the pattern seen

in this window.

**80.** Use implicit differentiation.

$$x^2 - y^2 = 1$$

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(y^2) = \frac{d}{dx}(1)$$

$$2x - 2yy' = 0$$

$$y' = \frac{2x}{2y} = \frac{x}{y}$$

$$y'' = \frac{d}{dx}\frac{x}{y}$$

$$= \frac{(y)(1) - (x)(y')}{y^2}$$

$$= \frac{y - x\left(\frac{x}{y}\right)}{y^2}$$

$$= \frac{y^2 - x^2}{y^3}$$

$$= -\frac{1}{y^3}$$

(since the given equation is $x^2 - y^2 = 1$)

At $(2, \sqrt{3})$, $\dfrac{d^2y}{dx^2} = -\dfrac{1}{y^3} = -\dfrac{1}{(\sqrt{3})^3} = -\dfrac{1}{3\sqrt{3}}$.

# Chapter 4
## Applications of Derivatives

## ■ Section 4.1 Extreme Values of Functions
(pp. 177– 185)

### Exploration 1    Finding Extreme Values

**1.** From the graph we can see that there are three critical
points: $x = -1, 0, 1$.
Critical point values: $f(-1) = 0.5, f(0) = 0, f(1) = 0.5$
Endpoint values: $f(-2) = 0.4, f(2) = 0.4$
Thus $f$ has absolute maximum value of 0.5 at $x = -1$ and
$x = 1$, absolute minimum value of 0 at $x = 0$, and local
minimum value of 0.4 at $x = -2$ and $x = 2$.
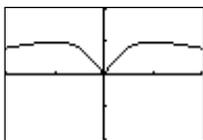
[−2, 2] by [−1, 1]

**2.** The graph of $f'$ has zeros at $x = -1$ and $x = 1$ where the
graph of $f$ has local extreme values. The graph of $f'$ is not
defined at $x = 0$, another extreme value of the graph of $f$.
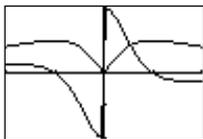
[−2, 2] by [−1, 1]

**3.** Using the chain rule and $\dfrac{d}{dx}(|x|) = \dfrac{|x|}{x}$, we find

$$\frac{df}{dx} = \frac{|x|}{x} \cdot \frac{1 - x^2}{(x^2 + 1)^2}.$$

## Quick Review 4.1

**1.** $f'(x) = \dfrac{1}{2\sqrt{4 - x}} \cdot \dfrac{d}{dx}(4 - x) = \dfrac{-1}{2\sqrt{4 - x}}$

**2.** $f'(x) = \dfrac{3}{4}x^{-1/4}$

**3.** $f'(x) = \dfrac{d}{dx}2(9 - x^2)^{-1/2} = -(9 - x^2)^{-3/2} \cdot \dfrac{d}{dx}(9 - x^2)$

$\quad = -(9 - x^2)^{-3/2}(-2x) = \dfrac{2x}{(9 - x^2)^{3/2}}$

**4.** $f'(x) = \dfrac{d}{dx}(x^2 - 1)^{-1/3} = -\dfrac{1}{3}(x^2 - 1)^{-4/3} \cdot \dfrac{d}{dx}(x^2 - 1)$

$\quad = -\dfrac{1}{3}(x^2 - 1)^{-4/3}(2x) = \dfrac{-2x}{3(x^2 - 1)^{4/3}}$

**5.** $g'(x) = \dfrac{1}{x^2 + 1} \cdot \dfrac{d}{dx}(x^2 + 1) = \dfrac{2x}{x^2 + 1}$

**6.** $g'(x) = -\sin(\ln x) \cdot \dfrac{d}{dx}\ln x = -\dfrac{\sin(\ln x)}{x}$

**7.** $h'(x) = e^{2x} \cdot \dfrac{d}{dx}2x = 2e^{2x}$

**8.** $h'(x) = \dfrac{d}{dx}e^{\ln x} = \dfrac{d}{dx}x = 1$

**9.** As $x \to 3^-$, $\sqrt{9 - x^2} \to 0^+$. Therefore, $\displaystyle\lim_{x \to 3^-} f(x) = \infty$.

**10.** As $x \to -3^+$, $\sqrt{9 - x^2} \to 0^+$. Therefore, $\displaystyle\lim_{x \to -3^+} f(x) = \infty$.

**11. (a)** $\dfrac{d}{dx}(x^3 - 2x) = 3x^2 - 2$

$\quad f'(1) = 3(1)^2 - 2 = 1$

**(b)** $\dfrac{d}{dx}(x + 2) = 1$

$\quad f'(3) = 1$

**(c)** Left-hand derivative:

$$\lim_{h \to 0^-} \frac{f(2 + h) - f(2)}{h} = \lim_{h \to 0^-} \frac{[(2 + h)^3 - 2(2 + h)] - 4}{h}$$

$$= \lim_{h \to 0^-} \frac{h^3 + 6h^2 + 10h}{h}$$

$$= \lim_{h \to 0^-} (h^2 + 6h + 10)$$

$$= 10$$

Right-hand derivative:

$$\lim_{h \to 0^+} \frac{f(2 + h) - f(2)}{h} = \lim_{h \to 0^+} \frac{[(2 + h) + 2] - 4}{h}$$

$$= \lim_{h \to 0^+} \frac{h}{h}$$

$$= \lim_{h \to 0^+} 1$$

$$= 1$$

Since the left- and right-hand derivatives are not equal,

$f'(2)$ is undefined.