Chapter 2
Limits and Continuity

■ Section 2.1 Rates of Change and Limits (pp. 55–65)

Quick Review 2.1

1. $f(2) = 2(2^3) - 5(2)^2 + 4 = 0$

2. $f(2) = \frac{4(2)^2 - 5}{2^3 + 4} = \frac{11}{12}$

3. $f(2) = \sin\left(\pi \cdot \frac{2}{2}\right) = \sin \pi = 0$

4. $f(2) = \frac{1}{2^2 - 1} = \frac{1}{3}$

5. $|x| < 4$
   $-4 < x < 4$

6. $|x| < c^2$
   $-c^2 < x < c^2$

7. $|x - 2| < 3$
   $-3 < x - 2 < 3$
   $-1 < x < 5$

8. $|x - c| < d^2$
   $-d^2 < x - c < d^2$
   $-d^2 + c < x < d^2 + c$

9. $\frac{x^2 - 3x - 18}{x + 3} = \frac{(x + 3)(x - 6)}{x + 3} = x - 6, x \neq -3$

10. $\frac{2x^2 - x}{2x^2 + x - 1} = \frac{x(2x - 1)}{(2x - 1)(x + 1)} = \frac{x}{x + 1}, x \neq \frac{1}{2}$

Section 2.1 Exercises

1. **(a)** $\lim_{x \to 3^-} f(x) = 3$

   (b) $\lim_{x \to 3^+} f(x) = -2$

   (c) $\lim_{x \to 3} f(x)$ does not exist, because the left- and right-hand limits are not equal.

   (d) $f(3) = 1$

2. **(a)** $\lim_{t \to -4^-} g(t) = 5$

   (b) $\lim_{t \to -4^+} g(t) = 2$

   (c) $\lim_{t \to -4} g(t)$ does not exist, because the left- and right-hand limits are not equal.

   (d) $g(-4) = 2$

3. **(a)** $\lim_{h \to 0^-} f(h) = -4$

   (b) $\lim_{h \to 0^+} f(h) = -4$

   (c) $\lim_{h \to 0} f(h) = -4$

   (d) $f(0) = -4$

4. **(a)** $\lim_{s \to -2^-} p(s) = 3$

   (b) $\lim_{s \to -2^+} p(s) = 3$

   (c) $\lim_{s \to -2} p(s) = 3$

   (d) $p(-2) = 3$

5. **(a)** $\lim_{x \to 0^-} F(x) = 4$

   (b) $\lim_{x \to 0^+} F(x) = -3$

   (c) $\lim_{x \to 0} F(x)$ does not exist, because the left- and right-hand limits are not equal.

   (d) $F(0) = 4$

6. **(a)** $\lim_{x \to 2^-} G(x) = 1$

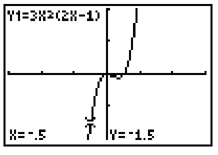
   (b) $\lim_{x \to 2^+} G(x) = 1$

   (c) $\lim_{x \to 2} G(x) = 1$

   (d) $G(2) = 3$

7. $\lim_{x \to -1/2} 3x^2(2x - 1) = 3\left(-\frac{1}{2}\right)^2\left[2\left(-\frac{1}{2}\right) - 1\right] = 3\left(\frac{1}{4}\right)(-2)$
   $= -\frac{3}{2}$

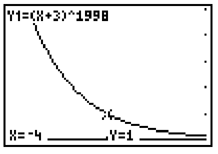
   Graphical support:



   $[-3, 3]$ by $[-2, 2]$

8. $\lim_{x \to -4} (x + 3)^{1998} = (-4 + 3)^{1998} = (-1)^{1998} = 1$

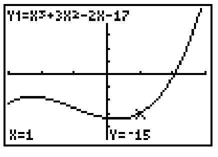
   Graphical support:



   $[-4.001, -3.999]$ by $[0, 5]$

9. $\lim_{x \to 1} (x^3 + 3x^2 - 2x - 17) = (1)^3 + 3(1)^2 - 2(1) - 17$
   $= 1 + 3 - 2 - 17 = -15$

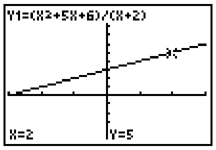
   Graphical support:



   $[-3, 3]$ by $[-25, 25]$

10. $\lim_{y\to 2} \frac{y^2+5y+6}{y+2} = \frac{2^2+5(2)+6}{2+2} = \frac{20}{4} = 5$

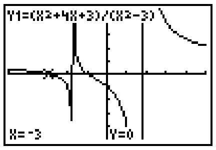
Graphical support:



$[-3, 3]$ by $[-5, 10]$

11. $\lim_{y\to -3} \frac{y^2+4y+3}{y^2-3} = \frac{(-3)^2+4(-3)+3}{(-3)^2-3} = \frac{0}{6} = 0$

Graphical support:

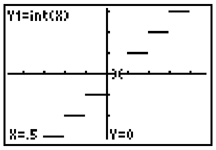


$[-5, 5]$ by $[-5, 5]$

12. $\lim_{x\to 1/2} \operatorname{int} x = \operatorname{int} \frac{1}{2} = 0$

Note that substitution cannot always be used to find limits of the int function. Its use here can be justified by the Sandwich Theorem, using $g(x) = h(x) = 0$ on the interval $(0, 1)$.

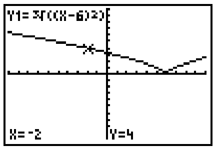
Graphical support:



$[-4.7, 4.7]$ by $[-3.1, 3.1]$

13. $\lim_{x\to -2} (x-6)^{2/3} = (-2-6)^{2/3} = \sqrt[3]{(-8)^2} = \sqrt[3]{64} = 4$

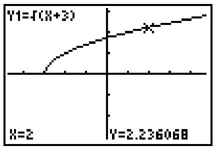
Graphical support:



$[-10, 10]$ by $[-10, 10]$

14. $\lim_{x\to 2} \sqrt{x+3} = \sqrt{2+3} = \sqrt{5}$

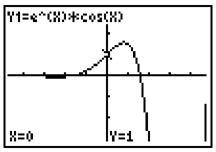
Graphical support:



$[-4.7, 4.7]$ by $[-3.1, 3.1]$

15. $\lim_{x\to 0} (e^x \cos x) = e^0 \cos 0 = 1 \cdot 1 = 1$

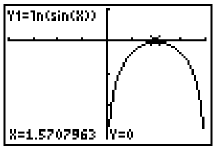
Graphical support:



$[-4.7, 4.7]$ by $[-3.1, 3.1]$

16. $\lim_{x\to \pi/2} \ln(\sin x) = \ln\left(\sin \frac{\pi}{2}\right) = \ln 1 = 0$

Graphical support:



$[-\pi, \pi]$ by $[-3, 1]$

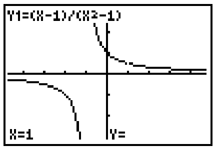
17. You cannot use substitution because the expression $\sqrt{x-2}$ is not defined at $x = -2$. Since the expression is not defined at points near $x = -2$, the limit does not exist.

18. You cannot use substitution because the expression $\frac{1}{x^2}$ is not defined at $x = 0$. Since $\frac{1}{x^2}$ becomes arbitrarily large as x approaches 0 from either side, there is no (finite) limit. (As we shall see in Section 2.2, we may write $\lim_{x\to 0} \frac{1}{x^2} = \infty$.)

19. You cannot use substitution because the expression $\frac{|x|}{x}$ is not defined at $x = 0$. Since $\lim_{x\to 0-} \frac{|x|}{x} = -1$ and $\lim_{x\to 0^+} \frac{|x|}{x} = 1$, the left- and right-hand limits are not equal and so the limit does not exist.

20. You cannot use substitution because the expression $\frac{(4+x)^2-16}{x}$ is not defined at $x = 0$. Since $\frac{(4+x)^2-16}{x} = \frac{8x+x^2}{x} = 8 + x$ for all $x \neq 0$, the limit exists and is equal to $\lim_{x\to 0} (8+x) = 8 + 0 = 8$.

21.



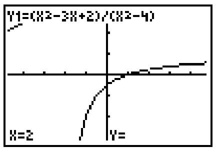
$[-4.7, 4.7]$ by $[-3.1, 3.1]$

$\lim_{x\to 1} \frac{x-1}{x^2-1} = \frac{1}{2}$

Algebraic confirmation:

$$\lim_{x\to 1} \frac{x-1}{x^2-1} = \lim_{x\to 1} \frac{x-1}{(x+1)(x-1)} = \lim_{x\to 1} \frac{1}{x+1} = \frac{1}{1+1} = \frac{1}{2}$$

22.

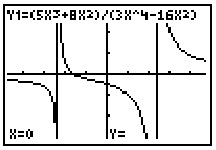


$[-4.7, 4.7]$ by $[-3.1, 3.1]$

$\lim_{t\to 2} \frac{t^2-3t+2}{t^2-4} = \frac{1}{4}$

Algebraic confirmation:

$$\lim_{t\to 2} \frac{t^2-3t+2}{t^2-4} = \lim_{t\to 2} \frac{(t-1)(t-2)}{(t+2)(t-2)} = \lim_{t\to 2} \frac{t-1}{t+2} = \frac{2-1}{2+2} = \frac{1}{4}$$

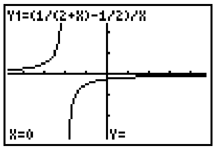
23. 

$[-4.7, 4.7]$ by $[-3.1, 3.1]$

$$\lim_{x\to 0} \frac{5x^3+8x^2}{3x^4-16x^2} = -\frac{1}{2}$$

Algebraic confirmation:

$$\begin{aligned}\lim_{x\to 0} \frac{5x^3+8x^2}{3x^4-16x^2} &= \lim_{x\to 0} \frac{x^2(5x+8)}{x^2(3x^2-16)}\\ &= \lim_{x\to 0} \frac{5x+8}{3x^2-16}\\ &= \frac{5(0)+8}{3(0)^2-16}\\ &= \frac{8}{-16} = -\frac{1}{2}\end{aligned}$$

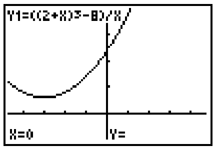
24. 

$[-4.7, 4.7]$ by $[-3.1, 3.1]$

$$\lim_{x\to 0} \frac{\frac{1}{2+x}-\frac{1}{2}}{x} = -\frac{1}{4}$$

Algebraic confirmation:

$$\begin{aligned}\lim_{x\to 0} \frac{\frac{1}{2+x}-\frac{1}{2}}{x} &= \lim_{x\to 0} \frac{2-(2+x)}{x(2+x)(2)}\\ &= \lim_{x\to 0} \frac{-x}{x(2+x)(2)}\\ &= \lim_{x\to 0} \frac{-1}{2(2+x)}\\ &= \frac{-1}{2(2+0)} = -\frac{1}{4}\end{aligned}$$

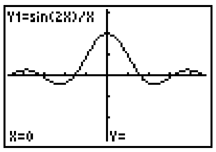
25. 

$[-4.7, 4.7]$ by $[-5, 20]$

$$\lim_{x\to 0} \frac{(2+x)^3-8}{x} = 12$$

Algebraic confirmation:

$$\begin{aligned}\lim_{x\to 0} \frac{(2+x)^3-8}{x} &= \lim_{x\to 0} \frac{12x+6x^2+x^3}{x}\\ &= \lim_{x\to 0} (12+6x+x^2)\\ &= 12+6(0)+(0)^2 = 12\end{aligned}$$

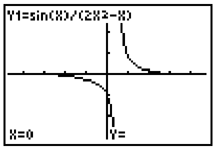
26. 

$[-4.7, 4.7]$ by $[-3.1, 3.1]$

$$\lim_{x\to 0} \frac{\sin 2x}{x} = 2$$

Algebraic confirmation:

$$\lim_{x\to 0} \frac{\sin 2x}{x} = 2\lim_{x\to 0} \frac{\sin 2x}{2x} = 2(1) = 2$$

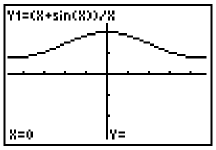
27. 

$[-4.7, 4.7]$ by $[-3.1, 3.1]$

$$\lim_{x\to 0} \frac{\sin x}{2x^2-x} = -1$$

Algebraic confirmation:

$$\begin{aligned}\lim_{x\to 0} \frac{\sin x}{2x^2-x} &= \lim_{x\to 0} \left(\frac{\sin x}{x}\cdot\frac{1}{2x-1}\right)\\ &= \left(\lim_{x\to 0} \frac{\sin x}{x}\right)\left(\lim_{x\to 0} \frac{1}{2x-1}\right) = (1)\left(\frac{1}{2(0)-1}\right) = -1\end{aligned}$$

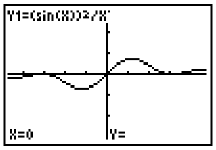
28. 

$[-4.7, 4.7]$ by $[-3.1, 3.1]$

$$\lim_{x\to 0} \frac{x+\sin x}{x} = 2$$

Algebraic confirmation:

$$\begin{aligned}\lim_{x\to 0} \frac{x+\sin x}{x} &= \lim_{x\to 0} \left(1+\frac{\sin x}{x}\right)\\ &= \left(\lim_{x\to 0} 1\right) + \left(\lim_{x\to 0} \frac{\sin x}{x}\right)\\ &= 1+1 = 2\end{aligned}$$

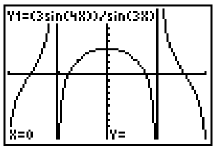
29. 

$[-4.7, 4.7]$ by $[-3.1, 3.1]$

$$\lim_{x\to 0} \frac{\sin^2 x}{x} = 0$$

Algebraic confirmation:

$$\begin{aligned}\lim_{x\to 0} \frac{\sin^2 x}{x} &= \lim_{x\to 0} \left(\sin x \cdot \frac{\sin x}{x}\right)\\ &= \left(\lim_{x\to 0} \sin x\right)\cdot\left(\lim_{x\to 0} \frac{\sin x}{x}\right)\\ &= (\sin 0)(1) = 0\end{aligned}$$

30. 

$[-2, 2]$ by $[-10, 10]$

$$\lim_{x\to 0} \frac{3\sin 4x}{\sin 3x} = 4$$

Algebraic confirmation:

$$\begin{aligned}\lim_{x\to 0} \frac{3\sin 4x}{\sin 3x} &= 4\lim_{x\to 0} \left(\frac{\sin 4x}{4x}\cdot\frac{3x}{\sin 3x}\right)\\ &= 4\left(\lim_{x\to 0} \frac{\sin 4x}{4x}\right) \div \left(\lim_{x\to 0} \frac{\sin 3x}{3x}\right)\\ &= 4(1) \div (1) = 4\end{aligned}$$

31. **(a)** True

(b) True

(c) False, since $\lim_{x \to 0^-} f(x) = 0$.

(d) True, since both are equal to 0.

(e) True, since (d) is true.

(f) True

(g) False, since $\lim_{x \to 0} f(x) = 0$.

(h) False, $\lim_{x \to 1^-} f(x) = 1$, but $\lim_{x \to 1} f(x)$ is undefined.

(i) False, $\lim_{x \to 1^+} f(x) = 0$, but $\lim_{x \to 1} f(x)$ is undefined.

(j) False, since $\lim_{x \to 2^-} f(x) = 0$.

32. **(a)** True

(b) False, since $\lim_{x \to 2} f(x) = 1$.

(c) False, since $\lim_{x \to 2} f(x) = 1$.

(d) True

(e) True

(f) True, since $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$.

(g) True, since both are equal to 0.

(h) True

(i) True, since $\lim_{x \to c} f(x) = 1$ for all c in $(1, 3)$.

33. $y_1 = \frac{x^2 + x - 2}{x - 1} = \frac{(x - 1)(x + 2)}{x - 1} = x + 2, x \neq 1$

(c)

34. $y_1 = \frac{x^2 - x - 2}{x - 1} = \frac{(x + 1)(x - 2)}{x - 1}$

(b)

35. $y_1 = \frac{x^2 - 2x + 1}{x - 1} = \frac{(x - 1)^2}{x - 1} = x - 1, x \neq 1$

(d)

36. $y_1 = \frac{x^2 + x - 2}{x + 1} = \frac{(x - 1)(x + 2)}{x + 1}$

(a)

37. Since $\text{int } x = 0$ for x in $(0, 1)$, $\lim_{x \to 0^+} \text{int } x = 0$.

38. Since $\text{int } x = -1$ for x in $(-1, 0)$, $\lim_{x \to 0^-} \text{int } x = -1$.

39. Since $\text{int } x = 0$ for x in $(0, 1)$, $\lim_{x \to 0.01} \text{int } x = 0$.

40. Since $\text{int } x = 1$ for x in $(1, 2)$, $\lim_{x \to 2^-} \text{int } x = 1$.

41. Since $\frac{x}{|x|} = 1$ for $x > 0$, $\lim_{x \to 0^+} \frac{x}{|x|} = 1$.

42. Since $\frac{x}{|x|} = -1$ for $x < 0$, $\lim_{x \to 0^-} \frac{x}{|x|} = -1$.

43. **(a)** $\lim_{x \to 4} (g(x) + 3) = \left(\lim_{x \to 4} g(x)\right) + \left(\lim_{x \to 4} 3\right) = 3 + 3 = 6$

(b) $\lim_{x \to 4} x f(x) = \left(\lim_{x \to 4} x\right)\left(\lim_{x \to 4} f(x)\right) = 4 \cdot 0 = 0$

(c) $\lim_{x \to 4} g^2(x) = \left(\lim_{x \to 4} g(x)\right)^2 = 3^2 = 9$

(d) $\lim_{x \to 4} \frac{g(x)}{f(x) - 1} = \frac{\lim_{x \to 4} g(x)}{\left(\lim_{x \to 4} f(x)\right) - \left(\lim_{x \to 4} 1\right)} = \frac{3}{0 - 1} = -3$

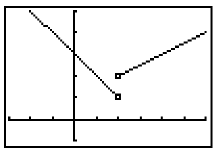
44. **(a)** $\lim_{x \to b} (f(x) + g(x)) = \left(\lim_{x \to b} f(x)\right) + \left(\lim_{x \to b} g(x)\right)$
$= 7 + (-3) = 4$

(b) $\lim_{x \to b} (f(x) \cdot g(x)) = \left(\lim_{x \to b} f(x)\right)\left(\lim_{x \to b} g(x)\right)$
$= (7)(-3) = -21$

(c) $\lim_{x \to b} 4g(x) = 4\lim_{x \to b} g(x) = 4(-3) = -12$

(d) $\lim_{x \to b} \frac{f(x)}{g(x)} = \frac{\lim_{x \to b} f(x)}{\lim_{x \to b} g(x)} = \frac{7}{-3} = -\frac{7}{3}$

45. **(a)**

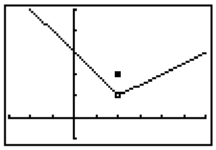


$[-3, 6]$ by $[-1, 5]$

(b) $\lim_{x \to 2^+} f(x) = 2$; $\lim_{x \to 2^-} f(x) = 1$

(c) No, because the two one-sided limits are different.

46. **(a)**

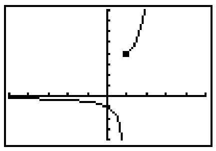


$[-3, 6]$ by $[-1, 5]$

(b) $\lim_{x \to 2^+} f(x) = 1$; $\lim_{x \to 2^-} f(x) = 1$

(c) Yes. The limit is 1.

47. **(a)**

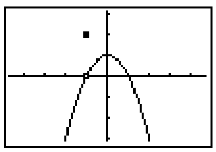


$[-5, 5]$ by $[-4, 8]$

(b) $\lim_{x \to 1^+} f(x) = 4$; $\lim_{x \to 1^-} f(x)$ does not exist.

(c) No, because the left-hand limit does not exist.

48. **(a)**



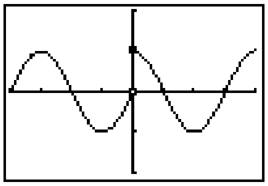
$[-4.7, 4.7]$ by $[-3.1, 3.1]$

48. continued

(b) $\lim_{x\to -1^+} f(x) = 0$; $\lim_{x\to -1^-} f(x) = 0$

(c) Yes. The limit is 0.

49. (a)



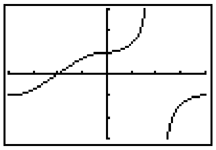
$[-2\pi, 2\pi]$ by $[-2, 2]$

(b) $(-2\pi, 0) \cup (0, 2\pi)$

(c) $c = 2\pi$

(d) $c = -2\pi$

50. (a)

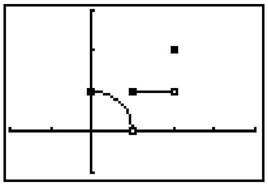


$[-\pi, \pi]$ by $[-3, 3]$

(b) $\left(-\pi, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$

(c) $c = \pi$ **(d)** $c = -\pi$

51. (a)

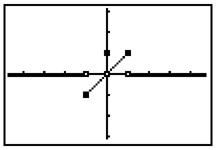


$[-2, 4]$ by $[-1, 3]$

(b) $(0, 1) \cup (1, 2)$

(c) $c = 2$ **(d)** $c = 0$

52. (a)

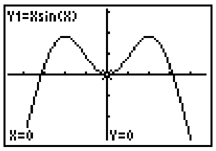


$[-4.7, 4.7]$ by $[-3.1, 3.1]$

(b) $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

(c) None **(d)** None

53.



$[-4.7, 4.7]$ by $[-3.1, 3.1]$

$\lim_{x\to 0} (x \sin x) = 0$

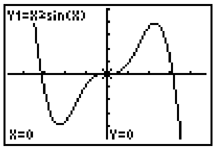
Confirm using the Sandwich Theorem, with $g(x) = -|x|$ and $h(x) = |x|$.

$|x \sin x| = |x| \cdot |\sin x| \le |x| \cdot 1 = |x|$

$-|x| \le x \sin x \le |x|$

Because $\lim_{x\to 0} (-|x|) = \lim_{x\to 0} |x| = 0$, the Sandwich Theorem gives $\lim_{x\to 0} (x \sin x) = 0$.

54.



$[-4.7, 4.7]$ by $[-5, 5]$

$\lim_{x\to 0} (x^2 \sin x) = 0$

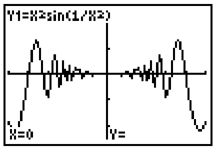
Confirm using the Sandwich Theorem, with $g(x) = -x^2$ and $h(x) = x^2$.

$|x^2 \sin x| = |x^2| \cdot |\sin x| \le |x^2| \cdot 1 = x^2$.

$-x^2 \le x^2 \sin x \le x^2$

Because $\lim_{x\to 0} (-x^2) = \lim_{x\to 0} x^2 = 0$, the Sandwich Theorem gives $\lim_{x\to 0} (x^2 \sin x) = 0$

55.



$[-0.5, 0.5]$ by $[-0.25, 0.25]$

$\lim_{x\to 0} \left(x^2 \sin \frac{1}{x^2}\right) = 0$

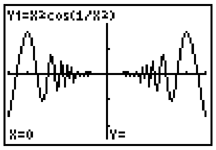
Confirm using the Sandwich Theorem, with $g(x) = -x^2$ and $h(x) = x^2$.

$\left|x^2 \sin \frac{1}{x^2}\right| = |x^2| \cdot \left|\sin \frac{1}{x^2}\right| \le |x^2| \cdot 1 = x^2$.

$-x^2 \le x^2 \sin \frac{1}{x^2} \le x^2$

Because $\lim_{x\to 0} (-x^2) = \lim_{x\to 0} x^2 = 0$, the Sandwich Theorem give $\lim_{x\to 0} \left(x^2 \sin \frac{1}{x^2}\right) = 0$.

56.



$[-0.5, 0.5]$ by $[-0.25, 0.25]$

$\lim_{x\to 0} \left(x^2 \cos \frac{1}{x^2}\right) = 0$

Confirm using the Sandwich Theorem, with $g(x) = -x^2$ and $h(x) = x^2$.

$\left|x^2 \cos \frac{1}{x^2}\right| = |x^2| \cdot \left|\cos \frac{1}{x^2}\right| \le |x^2| \cdot 1 = x^2$.

$-x^2 \le x^2 \cos \frac{1}{x^2} \le x^2$

Because $\lim_{x\to 0} (-x^2) = \lim_{x\to 0} x^2 = 0$, the Sandwich Theorem give $\lim_{x\to 0} \left(x^2 \cos \frac{1}{x^2}\right) = 0$.

57. (a) In three seconds, the ball falls $4.9(3)^2 = 44.1$ m, so its average speed is $\frac{44.1}{3} = 14.7$ m/sec.

(b) The average speed over the interval from time $t = 3$ to time $3 + h$ is

$$\frac{\Delta y}{\Delta t} = \frac{4.9(3+h)^2 - 4.9(3)^2}{(3+h) - 3} = \frac{4.9(6h + h^2)}{h}$$
$$= 29.4 + 4.9h$$

Since $\lim_{h \to 0} (29.4 + 4.9h) = 29.4$, the instantaneous speed is 29.4 m/sec.

58. (a) $y = gt^2$

$$20 = g(4^2)$$
$$g = \frac{20}{16} = \frac{5}{4} \text{ or } 1.25$$

(b) Average speed $= \frac{20}{4} = 5$ m/sec

(c) If the rock had not been stopped, its average speed over the interval from time $t = 4$ to time $t = 4 + h$ is

$$\frac{\Delta y}{\Delta t} = \frac{1.25(4+h)^2 - 1.25(4)^2}{(4+h) - 4} = \frac{1.25(8h + h^2)}{h}$$
$$= 10 + 1.25h$$

Since $\lim_{h \to 0} (10 + 1.25h) = 10$, the instantaneous speed is 10 m/sec.

59. (a)

x	-0.1	-0.01	-0.001	-0.0001
$f(x)$	-0.054402	-0.005064	-0.000827	-0.000031

(b)

x	0.1	0.01	0.001	0.0001
$f(x)$	-0.054402	-0.005064	-0.000827	-0.000031

The limit appears to be 0.

60. (a)

x	-0.1	-0.01	-0.001	-0.0001
$f(x)$	0.5440	0.5064	-0.8269	0.3056

(b)

x	0.1	0.01	0.001	0.0001
$f(x)$	-0.5440	-0.5064	0.8269	-0.3056

There is no clear indication of a limit.

61. (a)

x	-0.1	-0.01	-0.001	-0.0001
$f(x)$	2.0567	2.2763	2.2999	2.3023

(b)

x	0.1	0.01	0.001	0.0001
$f(x)$	2.5893	2.3293	2.3052	2.3029

The limit appears to be approximately 2.3.

62. (a)

x	-0.1	-0.01	-0.001	-0.0001
$f(x)$	0.074398	-0.009943	0.000585	0.000021

(b)

x	0.1	0.01	0.001	0.0001
$f(x)$	-0.074398	0.009943	-0.000585	-0.000021

The limit appears to be 0.

63. (a) Because the right-hand limit at zero depends only on the values of the function for positive x-values near zero.

(b) Area of $\triangle OAP = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(1)(\sin\theta) = \frac{\sin\theta}{2}$

Area of sector $OAP = \frac{(\text{angle})(\text{radius})^2}{2} = \frac{\theta(1)^2}{2} = \frac{\theta}{2}$

Area of $\triangle OAT = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(1)(\tan\theta) = \frac{\tan\theta}{2}$

(c) This is how the areas of the three regions compare.

(d) Multiply by 2 and divide by $\sin\theta$.

(e) Take reciprocals, remembering that all of the values involved are positive.

(f) The limits for $\cos\theta$ and 1 are both equal to 1. Since $\frac{\sin\theta}{\theta}$ is between them, it must also have a limit of 1.

(g) $\frac{\sin(-\theta)}{-\theta} = \frac{-\sin\theta}{-\theta} = \frac{\sin(\theta)}{\theta}$

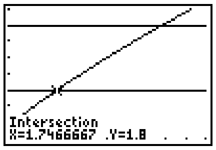
(h) If the function is symmetric about the y-axis, and the right-hand limit at zero is 1, then the left-hand limit at zero must also be 1.

(i) The two one-sided limits both exist and are equal to 1.

64. (a) The limit can be found by substitution.

$$\lim_{x \to 2} f(x) = f(2) = \sqrt{3(2) - 2} = \sqrt{4} = 2$$

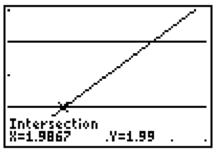
(b) The graphs of $y_1 = f(x)$, $y_2 = 1.8$, and $y_3 = 2.2$ are shown.



[1.5, 2.5] by [1.5, 2.3]

The intersections of y_1 with y_2 and y_3 are at $x \approx 1.7467$ and $x = 2.28$, respectively, so we may choose any value of a in [1.7467, 2) (approximately) and any value of b in (2, 2.28].
One possible answer: $a = 1.75$, $b = 2.28$.

(c) The graphs of $y_1 = f(x)$, $y_2 = 1.99$, and $y_3 = 2.01$ are shown.

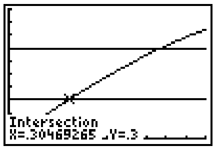


[1.97, 2.03] by [1.98, 2.02]

The intersections of y_1 with y_2 and y_3 are at $x = 1.9867$ and $x \approx 2.0134$, respectively, so we may choose any value of a in [1.9867, 2) and any value of b in (2, 2.0134] (approximately).
One possible answer: $a = 1.99$, $b = 2.01$

65. **(a)** $f\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$

(b) The graphs of $y_1 = f(x)$, $y_2 = 0.3$, and $y_3 = 0.7$ are shown.

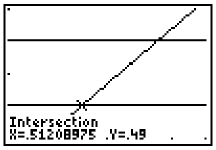


[0, 1] by [0, 1]

The intersections of y_1 with y_2 and y_3 are at $x \approx 0.3047$ and $x \approx 0.7754$, respectively, so we may choose any value of a in $\left[0.3047, \frac{\pi}{6}\right)$ and any value of b in $\left(\frac{\pi}{6}, 0.7754\right]$, where the interval endpoints are approximate.

One possible answer: $a = 0.305$, $b = 0.775$

(c) The graphs of $y_1 = f(x)$, $y_2 = 0.49$, and $y_3 = 0.51$ are shown.



[0.49, 0.55] by [0.48, 0.52]

The intersections of y_1 with y_2 and y_3 are at $x \approx 0.5121$ and $x \approx 0.5352$, respectively, so we may choose any value of a in $\left[0.5121, \frac{\pi}{6}\right)$, and any value of b in $\left(\frac{\pi}{6}, 0.5352\right]$, where the interval endpoints are approximate.

One possible answer: $a = 0.513$, $b = 0.535$

66. Line segment OP has endpoints $(0, 0)$ and (a, a^2), so its midpoint is $\left(\frac{0+a}{2}, \frac{0+a^2}{2}\right) = \left(\frac{a}{2}, \frac{a^2}{2}\right)$ and its slope is $\frac{a^2-0}{a-0} = a$. The perpendicular bisector is the line through $\left(\frac{a}{2}, \frac{a^2}{2}\right)$ with slope $-\frac{1}{a}$, so its equation is $y = -\frac{1}{a}\left(x - \frac{a}{2}\right) + \frac{a^2}{2}$, which is equivalent to $y = -\frac{1}{a}x + \frac{1+a^2}{2}$. Thus the y-intercept is $b = \frac{1+a^2}{2}$. As the point P approaches the origin along the parabola, the value of a approaches zero. Therefore,

$$\lim_{P\to 0} b = \lim_{a\to 0}\frac{1+a^2}{2} = \frac{1+0^2}{2} = \frac{1}{2}.$$

Section 2.2 Limits Involving Infinity (pp. 65–73)

Exploration 1 Exploring Theorem 5

1. Neither $\lim_{x\to\infty} f(x)$ or $\lim_{x\to\infty} g(x)$ exist. In this case, we can describe the behavior of f and g as $x \to \infty$ by writing $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$. We cannot apply the quotient rule because both limits must exist. However, from Example 5,

$$\lim_{x\to\infty}\frac{5x+\sin x}{x} = \lim_{x\to\infty}\left(5 + \frac{\sin x}{x}\right) = 5 + 0 = 5,$$

so the limit of the quotient exists.

2. Both f and g oscillate between 0 and 1 as $x \to \infty$, taking on each value infinitely often. We cannot apply the sum rule because neither limit exists. However,

$$\lim_{x\to\infty}(\sin^2 x + \cos^2 x) = \lim_{x\to\infty}(1) = 1,$$

so the limit of the sum exists.

3. The limt of f and g as $x \to \infty$ do not exist, so we cannot apply the difference rule to $f - \text{g}$. We can say that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$. We can write the difference as

$$f(x) - g(x) = \ln(2x) - \ln(x+1) = \ln\frac{2x}{x+1}.$$

We can use graphs or tables to convince ourselves that this limit is equal to ln 2.

4. The fact that the limits of f and g as $x \to \infty$ do not exist does not necessarily mean that the limits of $f + g$, $f - g$ or $\frac{f}{g}$ do not exist, just that Theorem 5 cannot be applied.

Quick Review 2.2

1. $y = 2x - 3$

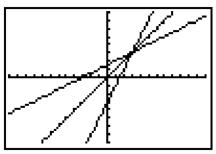
$y + 3 = 2x$

$\frac{y+3}{2} = x$

Interchange x and y.

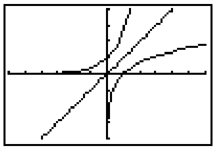
$\frac{x+3}{2} = y$

$f^{-1}(x) = \frac{x+3}{2}$



[−12, 12] by [−8, 8]

2. $y = e^x$
$\ln y = x$
Interchange x and y.
$\ln x = y$
$f^{-1}(x) = \ln x$



$[-6, 6]$ by $[-4, 4]$

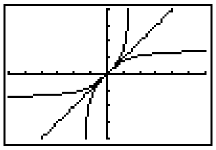
3. $y = \tan^{-1} x$

$\tan y = x,\ -\frac{\pi}{2} < y < \frac{\pi}{2}$

Interchange x and y.

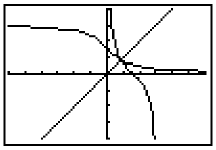
$\tan x = y,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$

$f^{-1}(x) = \tan x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$



$[-6, 6]$ by $[-4, 4]$

4. $y = \cot^{-1} x$
$\cot y = x,\ 0 < x < \pi$
Interchange x and y.
$\cot x = y,\ 0 < y < \pi$
$f^{-1}(x) = \cot x,\ 0 < x < \pi$



$[-6, 6]$ by $[-4, 4]$

5.

$$\begin{array}{r} \frac{2}{3} \\ 3x^3 + 4x - 5 \overline{)2x^3 - 3x^2 + x - 1} \\ \underline{2x^3 + 0x^2 + \frac{8}{3}x - \frac{10}{3}} \\ -3x^2 - \frac{5}{3}x + \frac{7}{3} \end{array}$$

$q(x) = \frac{2}{3}$

$r(x) = -3x^2 - \frac{5}{3}x + \frac{7}{3}$

6.

$$\begin{array}{r} 2x^2 + 2x + 1 \\ x^3 - x^2 + 1 \overline{)2x^5 + 0x^4 - x^3 + 0x^2 + x - 1} \\ \underline{2x^5 - 2x^4 + 0x^3 + 2x^2} \\ 2x^4 - x^3 - 2x^2 + x - 1 \\ \underline{2x^4 - 2x^3 + 0x^2 + 2x} \\ x^3 - 2x^2 - x - 1 \\ \underline{x^3 - x^2 + 0x + 1} \\ -x^2 - x - 2 \end{array}$$

$q(x) = 2x^2 + 2x + 1$
$r(x) = -x^2 - x - 2$

7. **(a)** $f(-x) = \cos(-x) = \cos x$

(b) $f\left(\frac{1}{x}\right) = \cos\left(\frac{1}{x}\right)$

8. **(a)** $f(-x) = e^{-(-x)} = e^x$

(b) $f\left(\frac{1}{x}\right) = e^{-1/x}$

9. **(a)** $f(-x) = \frac{\ln(-x)}{-x} = -\frac{\ln(-x)}{x}$

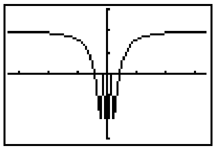
(b) $f\left(\frac{1}{x}\right) = \frac{\ln 1/x}{1/x} = x \ln x^{-1} = -x \ln x$

10. **(a)** $f(-x) = \left(-x + \frac{1}{-x}\right)\sin(-x) = -\left(x + \frac{1}{x}\right)(-\sin x)$
$= \left(x + \frac{1}{x}\right)\sin x$

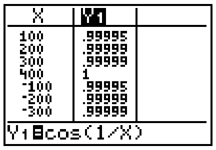
(b) $f\left(\frac{1}{x}\right) = \left(\frac{1}{x} + \frac{1}{1/x}\right)\sin\left(\frac{1}{x}\right) = \left(\frac{1}{x} + x\right)\sin\left(\frac{1}{x}\right)$

Section 2.2 Exercises

1.



$[-5, 5]$ by $[-1.5, 1.5]$



X	Y1
100	.99995
200	.99999
300	.99999
400	1
-100	.99995
-200	.99999
-300	.99999

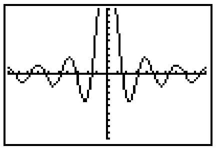
Y1=cos(1/X)

(a) $\lim_{x\to\infty} f(x) = 1$

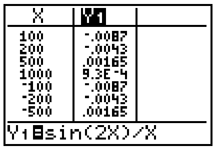
(b) $\lim_{x\to-\infty} f(x) = 1$

(c) $y = 1$

2.



$[-10, 10]$ by $[-1, 1]$



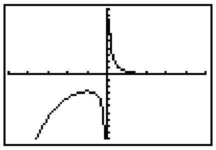
X	Y1
100	-.0087
200	-.0043
500	.00165
1000	9.3E-4
-100	-.0087
-200	-.0043
-500	.00165

Y1=sin(2X)/X

(a) $\lim_{x\to\infty} f(x) = 0$

(b) $\lim_{x\to-\infty} f(x) = 0$

(c) $y = 0$

3. 

[−5, 5] by [−10, 10]

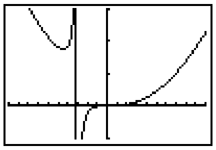
X	Y1
100	4E-46
200	7E-90
300	0
400	0
-100	-3E41
-200	-4E84
-300	ERROR

Y1=e^(-X)/X

(a) $\lim_{x\to\infty} f(x) = 0$

(b) $\lim_{x\to-\infty} f(x) = -\infty$

(c) $y = 0$

4. 

[−10, 10] by [−100, 300]

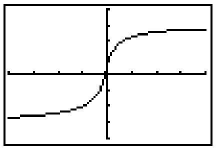
X	Y1
100	29125
200	118226
300	267326
400	476426
-100	30927
-200	121826
-300	272726

Y1=(3X³-X+1)/(X...

(a) $\lim_{x\to\infty} f(x) = \infty$

(b) $\lim_{x\to-\infty} f(x) = \infty$

(c) No horizontal asymptotes.

5. 

[−20, 20] by [−4, 4]

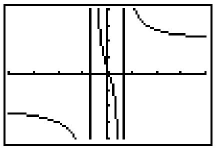
X	Y1
100	2.951
200	2.9752
300	2.9834
400	2.9876
-100	-2.931
-200	-2.965
-300	-2.977

Y1=(3X+1)/(abs(...

(a) $\lim_{x\to\infty} f(x) = 3$

(b) $\lim_{x\to-\infty} f(x) = -3$

(c) $y = 3, y = -3$

6. 

[−20, 20] by [−4, 4]

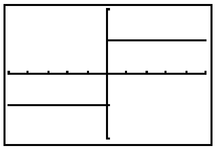
X	Y1
100	2.0515
200	2.0254
500	2.0101
1000	2.005
-100	-2.072
-200	-2.036
-500	-2.014

Y1=(2X-1)/(abs(...

(a) $\lim_{x\to\infty} f(x) = 2$

(b) $\lim_{x\to-\infty} f(x) = -2$

(c) $y = 2, y = -2$

7. 

[−5, 5] by [−2, 2]

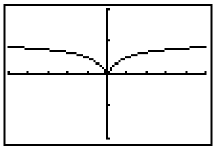
X	Y1
100	1
200	1
500	1
1000	1
-100	-1
-200	-1
-500	-1

Y1=X/abs(X)

(a) $\lim_{x\to\infty} f(x) = 1$

(b) $\lim_{x\to-\infty} f(x) = -1$

(c) $y = 1, y = -1$

8. 

[−5, 5] by [−2, 2]

X	Y1
100	.9901
200	.99502
500	.998
1000	.999
-100	.9901
-200	.99502
-500	.998

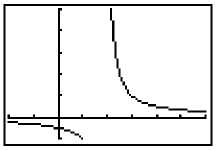
Y1=abs(X)/(abs(...

(a) $\lim_{x\to\infty} f(x) = 1$

(b) $\lim_{x\to-\infty} f(x) = 1$

(c) $y = 1$

9.



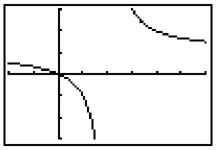
[−2, 6] by [−1, 5]

X	Y1
2.8	1.25
2.4	2.5
2.2	5
2.1	10
2.01	100
2.001	1000
2.0001	10000

Y1=1/(X-2)

$$\lim_{x\to 2^+} \frac{1}{x-2} = \infty$$

10.



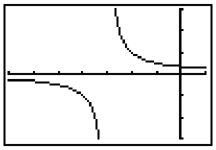
[−2, 6] by [−3, 3]

X	Y1
1.2	-1.5
1.6	-4
1.8	-9
1.9	-19
1.99	-199
1.999	-1999
1.9999	-19999

Y1=X/(X-2)

$$\lim_{x\to 2^-} \frac{x}{x-2} = -\infty$$

11.



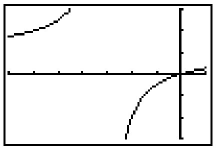
[−7, 1] by [−3, 3]

X	Y1
-3.8	-1.25
-3.4	-2.5
-3.3	-3.333
-3.2	-5
-3.1	-10
-3.01	-100
-3.001	-1000

Y1=1/(X+3)

$$\lim_{x\to -3^-} \frac{1}{x+3} = -\infty$$

12.



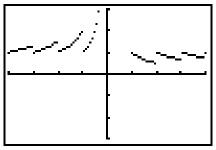
[−7, 1] by [−3, 3]

X	Y1
-2.2	-2.75
-2.6	-6.5
-2.7	-9
-2.8	-14
-2.9	-29
-2.99	-299
-2.999	-2999

Y1=X/(X+3)

$$\lim_{x\to -3^+} \frac{x}{x+3} = -\infty$$

13.



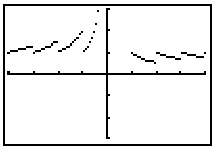
[−4, 4] by [−3, 3]

X	Y1
.8	0
.4	0
.2	0
.1	0
.01	0
.001	0
1E-4	0

Y1=int(X)/X

$$\lim_{x\to 0^+} \frac{\text{int } x}{x} = 0$$

14.



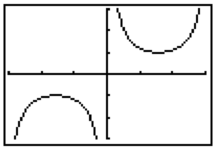
[−4, 4] by [−3, 3]

X	Y1
-.8	1.25
-.4	2.5
-.2	5
-.1	10
-.01	100
-.001	1000
-1E-4	10000

Y1=int(X)/X

$$\lim_{x\to 0^-} \frac{\text{int } x}{x} = \infty$$

15.



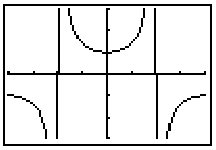
[−3, 3] by [−3, 3]

X	Y1
.8	1.394
.4	2.5679
.2	5.0335
.1	10.017
.01	100
.001	1000
1E-4	10000

Y1=1/sin(X)

$$\lim_{x\to 0^+} \csc x = \infty$$

16.



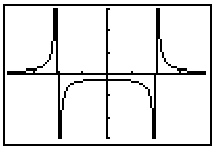
[−π, π] by [−3, 3]

X	Y1
1.6	-34.25
1.59	-52.08
1.58	-108.7
1.576	-192.2
1.572	-830.8
1.571	-4910
1.5708	-2.7E5

Y1=1/cos(X)

$$\lim_{x\to (\pi/2)^+} \sec x = -\infty$$

17.

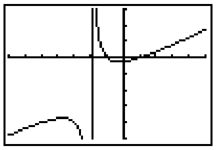


$[-4, 4]$ by $[-3, 3]$

(a) $x = -2, x = 2$

(b) Left-hand limit at -2 is ∞.
Right-hand limit at -2 is $-\infty$.
Left-hand limit at 2 is $-\infty$.
Right-hand limit at 2 is ∞.

18.

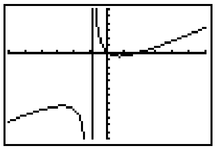


$[-7, 5]$ by $[-5, 3]$

(a) $x = -2$

(b) Left-hand limit at -2 is $-\infty$.
Right-hand limit at -2 is ∞.

19.

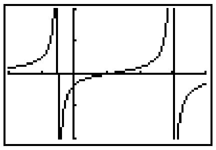


$[-6, 6]$ by $[-12, 6]$

(a) $x = -1$

(b) Left-hand limit at -1 is $-\infty$.
Right-hand limit at -1 is ∞.

20.



$[-2, 4]$ by $[-2, 2]$

(a) $x = -\frac{1}{2}, x = 3$

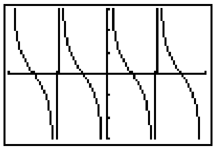
(b) Left-hand limit at $-\frac{1}{2}$ is ∞.

Right-hand limit at $-\frac{1}{2}$ is $-\infty$.

Left-hand limit at 3 is ∞.

Right-hand limit at 3 is $-\infty$.

21.

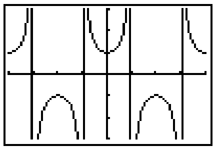


$[-2\pi, 2\pi]$ by $[-3, 3]$

(a) $x = k\pi$, k any integer

(b) at each vertical asymptote:
Left-hand limit is $-\infty$.
Right-hand limit is ∞.

22.



$[-2\pi, 2\pi]$ by $[-3, 3]$

(a) $x = \frac{\pi}{2} + n\pi$, n any integer

(b) If n is even:
Left-hand limit is ∞.
Right-hand limit is $-\infty$.

If n is odd:
Left-hand limit is $-\infty$.
Right-hand limit is ∞.

23. $$y = \left(2 - \frac{x}{x+1}\right)\left(\frac{x^2}{5+x^2}\right) = \left(\frac{2(x+1) - x}{x+1}\right)\left(\frac{x^2}{5+x^2}\right)$$
$$= \left(\frac{x+2}{x+1}\right)\left(\frac{x^2}{5+x^2}\right) = \frac{x^3 + 2x^2}{x^3 + x^2 + 5x + 5}$$

An end behavior model for y is $\frac{x^3}{x^3} = 1$.

$$\lim_{x\to\infty} y = \lim_{x\to\infty} 1 = 1$$
$$\lim_{x\to-\infty} y = \lim_{x\to-\infty} 1 = 1$$

24. $$y = \left(\frac{2}{x} + 1\right)\left(\frac{5x^2 - 1}{x^2}\right) = \left(\frac{2+x}{x}\right)\left(\frac{5x^2 - 1}{x^2}\right)$$
$$= \frac{5x^3 + 10x^2 - x - 2}{x^3}$$

An end behavior model for y is $\frac{5x^3}{x^3} = 5$.

$$\lim_{x\to\infty} y = \lim_{x\to\infty} 5 = 5$$
$$\lim_{x\to-\infty} y = \lim_{x\to-\infty} 5 = 5$$

25. Use the method of Example 10 in the text.

$$\lim_{x\to\infty} \frac{\cos\left(\frac{1}{x}\right)}{1 + \frac{1}{x}} = \lim_{x\to0^+} \frac{\cos x}{1+x} = \frac{\cos(0)}{1+0} = \frac{1}{1} = 1$$
$$\lim_{x\to-\infty} \frac{\cos\left(\frac{1}{x}\right)}{1 + \frac{1}{x}} = \lim_{x\to0^-} \frac{\cos x}{1+x} = \frac{\cos(0)}{1+0} = \frac{1}{1} = 1$$

26. Note that $y = \frac{2x + \sin x}{x} = 2 + \frac{\sin x}{x}$.

So, $\lim_{x\to\infty} y = \lim_{x\to\infty} 2 + \lim_{x\to\infty} \frac{\sin x}{x} = 2 + 0 = 2$.

Similarly, $\lim_{x\to-\infty} y = 2$.

27. Use $y = \frac{\sin x}{2x^2 + x} = \frac{\sin x}{x} \cdot \frac{1}{2x+1}$

$\lim_{x\to\pm\infty} \frac{\sin x}{x} = 0$

$\lim_{x\to\pm\infty} \frac{1}{2x+1} = 0$

So, $\lim_{x\to\infty} y = 0$ and $\lim_{x\to-\infty} y = 0$.

28. $y = 2\frac{\sin x}{x} + \frac{1}{x}\frac{\sin x}{x}$

So, $\lim_{x\to\infty} y = 0$ and $\lim_{x\to-\infty} y = 0$.

29. An end behavior model is $\frac{2x^3}{x} = 2x^2$. (a)

30. An end behavior model is $\frac{x^5}{2x^2} = 0.5x^3$. (c)

31. An end behavior model is $\frac{2x^4}{-x} = -2x^3$. (d)

32. An end behavior model is $\frac{x^4}{-x^2} = -x^2$. (b)

33. **(a)** $3x^2$

(b) None

34. **(a)** $-4x^3$

(b) None

35. **(a)** $\frac{x}{2x^2} = \frac{1}{2x}$

(b) $y = 0$

36. **(a)** $\frac{3x^2}{x^2} = 3$

(b) $y = 3$

37. **(a)** $\frac{4x^3}{x} = 4x^2$

(b) None

38. **(a)** $\frac{-x^4}{x^2} = -x^2$

(b) None

39. **(a)** The function $y = e^x$ is a right end behavior model because $\lim_{x\to\infty} \frac{e^x - 2x}{e^x} = \lim_{x\to\infty}\left(1 - \frac{2x}{e^x}\right) = 1 - 0 = 1$.

(b) The function $y = -2x$ is a left end behavior model because $\lim_{x\to-\infty} \frac{e^x - 2x}{-2x} = \lim_{x\to-\infty}\left(-\frac{e^x}{2x} + 1\right) = 0 + 1 = 1$.

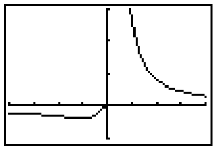
40. **(a)** The function $y = x^2$ is a right end behavior model because $\lim_{x\to\infty} \frac{x^2 + e^{-x}}{x^2} = \lim_{x\to\infty}\left(1 + \frac{e^{-x}}{x^2}\right) = 1 + 0 = 1$.

(b) The function $y = e^{-x}$ is a left end behavior model because $\lim_{x\to-\infty} \frac{x^2 + e^{-x}}{e^{-x}} = \lim_{x\to-\infty}\left(\frac{x^2}{e^{-x}} + 1\right)$
$= \lim_{x\to-\infty} (x^2e^x + 1) = 0 + 1 = 1$.

41. **(a, b)** The function $y = x$ is both a right end behavior model and a left end behavior model because $\lim_{x\to\pm\infty}\left(\frac{x + \ln|x|}{x}\right) = \lim_{x\to\pm\infty}\left(1 + \frac{\ln|x|}{x}\right) = 1 + 0 = 1$.

42. **(a, b)** The function $y = x^2$ is both a right end behavior model and a left end behavior model because $\lim_{x\to\pm\infty}\left(\frac{x^2 + \sin x}{x^2}\right) = \lim_{x\to\pm\infty}\left(1 + \frac{\sin x}{x^2}\right) = 1$.

43.



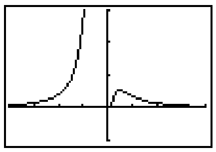
$[-4, 4]$ by $[-1, 3]$

The graph of $y = f\left(\frac{1}{x}\right) = \frac{1}{x}e^{1/x}$ is shown.

$\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} f\left(\frac{1}{x}\right) = \infty$

$\lim_{x\to\infty^-} f(x) = \lim_{x\to 0^-} f\left(\frac{1}{x}\right) = 0$

44.



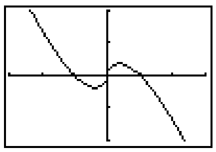
$[-4, 4]$ by $[-1, 3]$

The graph of $y = f\left(\frac{1}{x}\right) = \frac{1}{x^2}e^{-1/x}$ is shown.

$\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} f\left(\frac{1}{x}\right) = 0$

$\lim_{x\to-\infty} f(x) = \lim_{x\to 0^-} f\left(\frac{1}{x}\right) = \infty$

45.



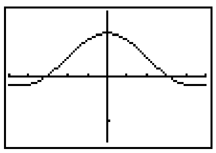
$[-3, 3]$ by $[-2, 2]$

The graph of $y = f\left(\frac{1}{x}\right) = x \ln\left|\frac{1}{x}\right|$ is shown.

$\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} f\left(\frac{1}{x}\right) = 0$

$\lim_{x\to-\infty} f(x) = \lim_{x\to 0^-} f\left(\frac{1}{x}\right) = 0$

46.



$[-5, 5]$ by $[-1.5, 1.5]$

The graph of $y = f\left(\frac{1}{x}\right) = \frac{\sin x}{x}$ is shown.

$\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} f\left(\frac{1}{x}\right) = 1$

$\lim_{x\to-\infty} f(x) = \lim_{x\to 0^-} f\left(\frac{1}{x}\right) = 1$

47. **(a)** $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(\frac{1}{x}\right) = 0$

(b) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty}(-1) = -1$

(c) $\lim_{x\to0^-} f(x) = \lim_{x\to0^-} \frac{1}{x} = -\infty$

(d) $\lim_{x\to0^+} f(x) = \lim_{x\to0^+}(-1) = -1$

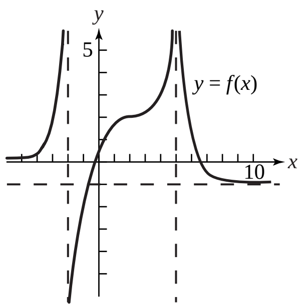
48. **(a)** $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x-2}{x-1} = \lim_{x\to-\infty} \frac{x}{x} = 1$

(b) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{x^2} = 0$

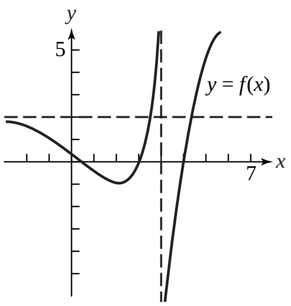
(c) $\lim_{x\to0^-} f(x) = \lim_{x\to0^-} \frac{x-2}{x-1} = \frac{0-2}{0-1} = 2$

(d) $\lim_{x\to0^+} f(x) = \lim_{x\to0^+} \frac{1}{x^2} = \infty$

49. One possible answer:



50. One possible answer:

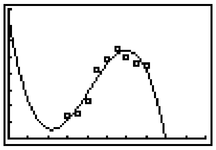


51. Note that $\frac{f_1(x)/f_2(x)}{g_1(x)/g_2(x)} = \frac{f_1(x)g_2(x)}{g_1(x)f_2(x)} = \frac{f_1(x)/g_1(x)}{f_2(x)/g_2(x)}$.

As x becomes large, $\frac{f_1}{g_1}$ and $\frac{f_2}{g_2}$ both approach 1. Therefore, using the above equation, $\frac{f_1/f_2}{g_1/g_2}$ must also approach 1.

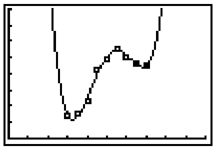
52. Yes. The limit of $(f+g)$ will be the same as the limit of g. This is because adding numbers that are very close to a given real number L will not have a significant effect on the value of $(f+g)$ since the values of g are becoming arbitrarily large.

53. **(a)** Using 1980 as $x = 0$:
$y = -2.2316x^3 + 54.7134x^2 - 351.0933x + 733.2224$



[0, 20] by [0, 800]

(b) Again using 1980 as $x = 0$:
$y = 1.458561x^4 - 60.5740x^3 + 905.8877x^2 - 5706.0943x + 12967.6288$



[0, 20] by [0, 800]

(c) Cubic: approximately -2256 dollars
Quartic: approximately 9979 dollars

(d) Cubic: End behavior model is $-2.2316x^3$. This model predicts that the grants will become negative by 1996.

Quartic: End behavior model is $1.458561x^4$. This model predicts that the size of the grants will grow very rapidly after 1995.

Neither of these seems reasonable. There is no reason to expect the grants to disappear (become negative) based on the data. Similarly, the data give no indication that a period of rapid growth is about to occur.

54. **(a)** Note that $fg = f(x)g(x) = 1$.
$f \to -\infty$ as $x \to 0^-$, $f \to \infty$ as $x \to 0^+$, $g \to 0$, $fg \to 1$

(b) Note that $fg = f(x)g(x) = -8$.
$f \to \infty$ as $x \to 0^-$, $f \to -\infty$ as $x \to 0^+$, $g \to 0$, $fg \to -8$

(c) Note that $fg = f(x)g(x) = 3(x-2)^2$.
$f \to -\infty$ as $x \to 2^-$, $f \to \infty$ as $x \to 2^+$, $g \to 0$, $fg \to 0$

(d) Note that $fg = f(x)g(x) = \frac{5}{(x-3)^2}$.

$f \to \infty$, $g \to 0$, $fg \to \infty$

(e) Nothing – you need more information to decide.

55. **(a)** This follow from $x - 1 < \text{int } x \le x$, which is true for all x. Dividing by x gives the result.

(b, c) Since $\lim_{x\to\pm\infty} \frac{x-1}{x} = \lim_{x\to\pm\infty} 1 = 1$, the Sandwich Theorem gives $\lim_{x\to\infty} \frac{\text{int } x}{x} = \lim_{x\to-\infty} \frac{\text{int } x}{x} = 1$.

56. For $x > 0$, $0 < e^{-x} < 1$, so $0 < \frac{e^{-x}}{x} < \frac{1}{x}$.

Since both 0 and $\frac{1}{x}$ approach zero as $x \to \infty$, the Sandwich Theorem states that $\frac{e^{-x}}{x}$ must also approach zero.

57. This is because as x approaches infinity, $\sin x$ continues to oscillate between 1 and -1 and doesn't approach any given real number.

58. $\lim_{x\to\infty} \frac{\ln x^2}{\ln x} = 2$, because $\frac{\ln x^2}{\ln x} = \frac{2 \ln x}{x} = 2$.

59. $\lim_{x\to\infty} \frac{\ln x}{\log x} = \ln(10)$, since $\frac{\ln x}{\log x} = \frac{\ln x}{(\ln x)/(\ln 10)}$
$= \ln 10$.

60. $\lim_{x\to\infty} \frac{\ln(x+1)}{\ln x} = 1$

Since $\ln(x+1) = \ln\left[x\left(1+\frac{1}{x}\right)\right] = \ln x + \ln\left(1+\frac{1}{x}\right)$,

$$\frac{\ln(x+1)}{\ln x} = \frac{\ln x + \ln\left(1+\frac{1}{x}\right)}{\ln x} = 1 + \frac{\ln\left(1+\frac{1}{x}\right)}{\ln x}$$

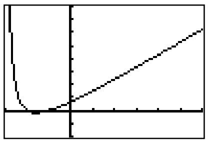
But as $x \to \infty$, $1 + 1/x$ approaches 1, so $\ln(1 + 1/x)$ approaches $\ln(1) = 0$. Also, as $x \to \infty$, $\ln x$ approaches infinity. This means the second term above approaches 0 and the limit is 1.

■ Section 2.3 Continuity
(pp. 73–81)

Exploration 1 Removing a Discontinuity

1. $x^2 - 9 = (x-3)(x+3)$. The domain of f is $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$ or all $x \neq \pm 3$.

2. It appears that the limit of f as $x \to 3$ exists and is a little more than 3.



$[-3, 6]$ by $[-2, 8]$

3. $f(3)$ should be defined as $\frac{10}{3}$.

4. $x^3 - 7x - 6 = (x-3)(x+1)(x+2)$, $x^2 - 9$
$= (x-3)(x+3)$, so $f(x) = \frac{(x+1)(x+2)}{x+3}$ for $x \neq 3$.
Thus, $\lim_{x\to 3} \frac{(x+1)(x+2)}{x+3} = \frac{20}{6} = \frac{10}{3}$.

5. $\lim_{x\to 3} g(x) = \frac{10}{3} = g(3)$, so g is continuous at $x = 3$.

Quick Review 2.3

1. $\lim_{x\to -1} \frac{3x^2 - 2x + 1}{x^3 + 4} = \frac{3(-1)^2 - 2(-1) + 1}{(-1)^3 + 4} = \frac{6}{3} = 2$

2. **(a)** $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \text{int}(x) = -2$

(b) $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} f(x) = -1$

(c) $\lim_{x\to -1} f(x)$ does not exist, because the left- and right-hand limits are not equal.

(d) $f(-1) = \text{int}(-1) = -1$

3. **(a)** $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^2 - 4x + 5) = 2^2 - 4(2) + 5 = 1$

(b) $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (4 - x) = 4 - 2 = 2$

(c) $\lim_{x\to 2} f(x)$ does not exist, because the left- and right-hand limits are not equal.

(d) $f(2) = 4 - 2 = 2$

4. $(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{x} + 1\right) = \frac{2\left(\frac{1}{x}+1\right) - 1}{\left(\frac{1}{x}+1\right) + 5}$
$= \frac{2(1+x) - x}{(1+x) + 5x} = \frac{x+2}{6x+1}$, $x \neq 0$

$(g \circ f)(x) = g(f(x)) = g\left(\frac{2x-1}{x+5}\right) = \frac{1}{\frac{2x-1}{x+5}} + 1$
$= \frac{x+5}{2x-1} + \frac{2x-1}{2x-1} = \frac{3x+4}{2x-1}$, $x \neq -5$

5. Note that $\sin x^2 = (g \circ f)(x) = g(f(x)) = g(x^2)$.
Therefore: $g(x) = \sin x$, $x \geq 0$
$(f \circ g)(x) = f(g(x)) = f(\sin x) = (\sin x)^2$ or $\sin^2 x$, $x \geq 0$

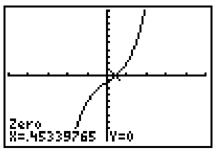
6. Note that $\frac{1}{x} = (g \circ f)(x) = g(f(x)) = \sqrt{f(x) - 1}$.
Therefore, $\sqrt{f(x) - 1} = \frac{1}{x}$ for $x > 0$. Squaring both sides gives $f(x) - 1 = \frac{1}{x^2}$. Therefore, $f(x) = \frac{1}{x^2} + 1$, $x > 0$.
$(f \circ g)(x) = f(g(x)) = \frac{1}{(\sqrt{x-1})^2} + 1 = \frac{1}{x-1} + 1$
$= \frac{1 + x - 1}{x - 1} = \frac{x}{x-1}$, $x > 1$

7. $2x^2 + 9x - 5 = 0$

$(2x - 1)(x + 5) = 0$

Solutions: $x = \frac{1}{2}$, $x = -5$

8.

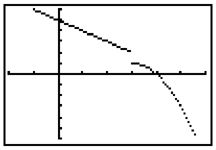


$[-5, 5]$ by $[-10, 10]$

Solution: $x \approx 0.453$

9. For $x \leq 3$, $f(x) = 4$ when $5 - x = 4$, which gives $x = 1$. (Note that this value is, in fact, ≤ 3.)
For $x > 3$, $f(x) = 4$ when $-x^2 + 6x - 8 = 4$, which gives $x^2 - 6x + 12 = 0$. The discriminant of this equation is $b^2 - 4ac = (-6)^2 - 4(1)(12) = -12$. Since the discriminant is negative, the quadratic equation has no solution.
The only solution to the original equation is $x = 1$.

10.



$[-2.7, 6.7]$ by $[-6, 6]$

A graph of $f(x)$ is shown. The range of $f(x)$ is $(-\infty, 1) \cup [2, \infty)$. The values of c for which $f(x) = c$ has no solution are the values that are excluded from the range. Therefore, c can be any value in $[1, 2)$.

Section 2.3 Exercises

1. The function $y = \frac{1}{(x+2)^2}$ is continuous because it is a quotient of polynomials, which are continuous. Its only point of discontinuity occurs where it is undefined. There is an infinite discontinuity at $x = -2$.

2. The function $y = \frac{x+1}{x^2 - 4x + 3}$ is continuous because it is a quotient of polynomials, which are continuous. Its only points of discontinuity occur where it is undefined, that is, where the denominator $x^2 - 4x + 3 = (x-1)(x-3)$ is zero. There are infinite discontinuities at $x = 1$ and at $x = 3$.

3. The function $y = \frac{1}{x^2 + 1}$ is continuous because it is a quotient of polynomials, which are continuous. Furthermore, the domain is all real numbers because the denominator, $x^2 + 1$, is never zero. Since the function is continuous and has domain $(-\infty, \infty)$, there are no points of discontinuity.

4. The function $y = |x - 1|$ is a composition $(f \circ g)(x)$ of the continuous functions $f(x) = |x|$ and $g(x) = x - 1$, so it is continuous. Since the function is continuous and has domain $(-\infty, \infty)$, there are no points of discontinuity.

5. The function $y = \sqrt{2x+3}$ is a composition $(f \circ g)(x)$ of the continuous functions $f(x) = \sqrt{x}$ and $g(x) = 2x + 3$, so it is continuous. Its points of discontinuity are the points not in the domain, i.e., all $x < -\frac{3}{2}$.

6. The function $y = \sqrt[3]{2x-1}$ is a composition $(f \circ g)(x)$ of the continuous functions $f(x) = \sqrt[3]{x}$ and $g(x) = 2x - 1$, so it is continuous. Since the function is continuous and has domain $(-\infty, \infty)$, there are no points of discontinuity.

7. The function $y = \frac{|x|}{x}$ is equivalent to
$$y = \begin{cases} -1, & x < 0 \\ 1, & x > 0. \end{cases}$$
It has a jump discontinuity at $x = 0$.

8. The function $y = \cot x$ is equivalent to $y = \frac{\cos x}{\sin x}$, a quotient of continuous functions, so it is continuous. Its only points of discontinuity occur where it is undefined. It has infinite discontinuities at $x = k\pi$ for all integers k.

9. The function $y = e^{1/x}$ is a composition $(f \circ g)(x)$ of the continuous functions $f(x) = e^x$ and $g(x) = \frac{1}{x}$, so it is continuous. Its only point of discontinuity occurs at $x = 0$, where it is undefined. Since $\lim_{x\to 0^+} e^{1/x} = \infty$, this may be considered an infinite discontinuity.

10. The function $y = \ln(x+1)$ is a composition $(f \circ g)(x)$ of the continuous functions $f(x) = \ln x$ and $g(x) = x + 1$, so it is continuous. Its points of discontinuity are the points not in the domain, i.e., $x < -1$.

11. **(a)** Yes, $f(-1) = 0$.

(b) Yes, $\lim_{x\to -1^+} = 0$.

(c) Yes

(d) Yes, since -1 is a left endpoint of the domain of f and $\lim_{x\to -1^+} f(x) = f(-1)$, f is continuous at $x = -1$.

12. **(a)** Yes, $f(1) = 1$.

(b) Yes, $\lim_{x\to 1} f(x) = 2$.

(c) No

(d) No

13. **(a)** No

(b) No, since $x = 2$ is not in the domain.

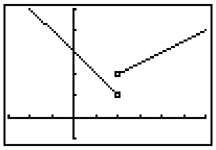
14. Everywhere in $[-1, 3)$ except for $x = 0, 1, 2$.

15. Since $\lim_{x\to 2} f(x) = 0$, we should assign $f(2) = 0$.

16. Since $\lim_{x\to 1} f(x) = 2$, we should reassign $f(1) = 2$.

17. No, because the right-hand and left-hand limits are not the same at zero.

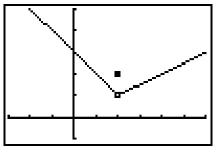
18. Yes. Assign the value 0 to $f(3)$. Since 3 is a right endpoint of the extended function and $\lim_{x\to 3^-} f(x) = 0$, the extended function is continuous at $x = 3$.

19. 

$[-3, 6]$ by $[-1, 5]$

(a) $x = 2$

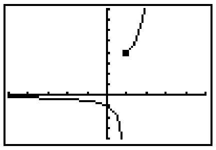
(b) Not removable, the one-sided limits are different.

20. 

$[-3, 6]$ by $[-1, 5]$

(a) $x = 2$

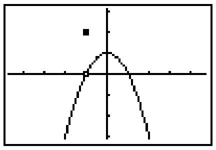
(b) Removable, assign the value 1 to $f(2)$.

21. 

$[-5, 5]$ by $[-4, 8]$

(a) $x = 1$

(b) Not removable, it's an infinite discontinuity.

22. 

$[-4.7, 4.7]$ by $[-3.1, 3.1]$

(a) $x = -1$

(b) Removable, assign the value 0 to $f(-1)$.

23. **(a)** All points not in the domain along with $x = 0, 1$

(b) $x = 0$ is a removable discontinuity, assign $f(0) = 0$. $x = 1$ is not removable, the one-sided limits are different.

24. **(a)** All points not in the domain along with $x = 1, 2$

(b) $x = 1$ is not removable, the one-sided limits are different.
$x = 2$ is a removable discontinuity, assign $f(2) = 1$.

25. For $x \neq -3$, $f(x) = \dfrac{x^2 - 9}{x + 3} = \dfrac{(x + 3)(x - 3)}{x + 3} = x - 3$.

The extended function is $y = x - 3$.

26. For $x \neq 1$,
$$f(x) = \frac{x^3 - 1}{x^2 - 1} = \frac{(x - 1)(x^2 + x + 1)}{(x + 1)(x - 1)} = \frac{x^2 + x + 1}{x + 1}.$$

The extended function is $y = \dfrac{x^2 + x + 1}{x + 1}$.

27. Since $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1$, the extended function is
$$y = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0. \end{cases}$$

28. Since $\displaystyle\lim_{x\to 0} \frac{\sin 4x}{x} = 4 \lim_{x\to 0} \frac{\sin 4x}{4x} = 4(1) = 4$, the extended function is
$$y = \begin{cases} \dfrac{\sin 4x}{x}, & x \neq 0 \\ 4, & x = 0. \end{cases}$$

29. For $x \neq 4$ (and $x > 0$),
$$f(x) = \frac{x - 4}{\sqrt{x} - 2} = \frac{(\sqrt{x} + 2)(\sqrt{x} - 2)}{\sqrt{x} - 2} = \sqrt{x} + 2.$$

The extended function is $y = \sqrt{x} + 2$.

30. For $x \neq 2$ (and $x \neq -2$),
$$\begin{aligned} f(x) &= \frac{x^3 - 4x^2 - 11x + 30}{x^2 - 4} \\ &= \frac{(x - 2)(x - 5)(x + 3)}{(x - 2)(x + 2)} \\ &= \frac{(x - 5)(x + 3)}{x + 2} \\ &= \frac{x^2 - 2x - 15}{x + 2}. \end{aligned}$$

The extended function is $y = \dfrac{x^2 - 2x - 15}{x + 2}$.

31. One possible answer:
Assume $y = x$, constant functions, and the square root function are continuous.

By the sum theorem, $y = x + 2$ is continuous.

By the composite theorem, $y = \sqrt{x + 2}$ is continuous.

By the quotient theorem, $y = \dfrac{1}{\sqrt{x + 2}}$ is continuous.

Domain: $(-2, \infty)$

32. One possible answer:
Assume $y = x$, constant functions, and the cube root function are continuous.

By the difference theorem, $y = 4 - x$ is continuous.
By the composite theorem, $y = \sqrt[3]{4 - x}$ is continuous.
By the product theorem, $y = x^2 = x \cdot x$ is continuous.
By the sum theorem, $y = x^2 + \sqrt[3]{4 - x}$ is continuous.
Domain: $(-\infty, \infty)$

33. Possible answer:
Assume $y = x$ and $y = |x|$ are continuous.

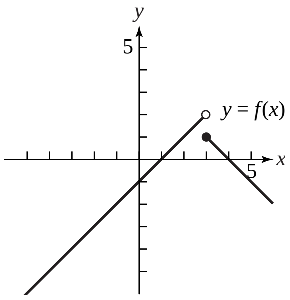
By the product theorem, $y = x^2 = x \cdot x$ is continuous.
By the constant multiple theorem, $y = 4x$ is continuous.
By the difference theorem, $y = x^2 - 4x$ is continuous.
By the composite theorem, $y = |x^2 - 4x|$ is continuous.
Domain: $(-\infty, \infty)$

34. One possible answer:
Assume $y = x$ and $y = 1$ are continuous.

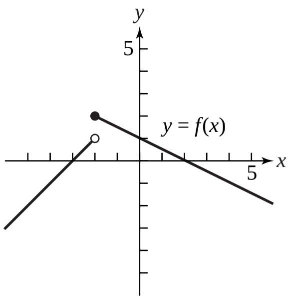
Use the product, difference, and quotient theorems. One also needs to verify that the limit of this function as x approaches 1 is 2.

Alternately, observe that the function is equivalent to $y = x + 1$ (for all x), which is continuous by the sum theorem.
Domain: $(-\infty, \infty)$

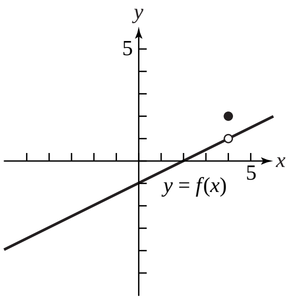
35. One possible answer:



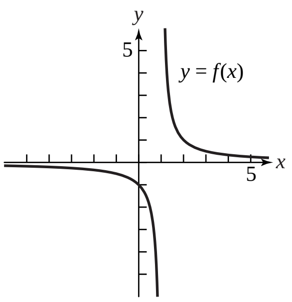
36. One possible answer:



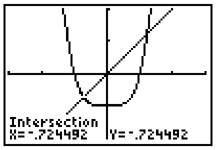
37. One possible answer:



38. One possible answer:



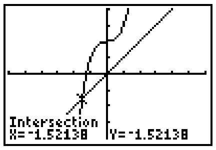
39.



$[-3, 3]$ by $[-2, 2]$

Solving $x = x^4 - 1$, we obtain the solutions $x \approx -0.724$ and $x \approx 1.221$.

40.



$[-6, 6]$ by $[-4, 4]$

Solving $x = x^3 + 2$, we obtain the solution $x \approx -1.521$.

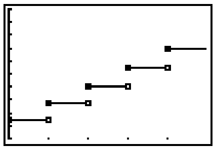
41. We require that $\lim\limits_{x\to 3^+} 2ax = \lim\limits_{x\to 3^-} (x^2 - 1)$:

$$\begin{aligned} 2a(3) &= 3^2 - 1 \\ 6a &= 8 \\ a &= \frac{4}{3} \end{aligned}$$

42. Consider $f(x) = x - e^{-x}$. f is continuous, $f(0) = -1$, and $f(1) = 1 - \frac{1}{e} > 0.5$. By the Intermediate Value Theorem, for some c in $(0, 1)$, $f(c) = 0$ and $e^{-c} = c$.

43. **(a)** Sarah's salary is $\$36{,}500 = \$36{,}500(1.035)^0$ for the first year $(0 \le t < 1)$, $\$36{,}500(1.035)$ for the second year $(1 \le t < 2)$, $\$36{,}500(1.035)^2$ for the third year $(2 \le t < 3)$, and so on. This corresponds to $y = 36{,}500(1.035)^{\text{int } t}$.

(b)



$[0, 4.98]$ by $[35{,}000, 45{,}000]$

The function is continuous at all points in the domain $[0, 5)$ except at $t = 1, 2, 3, 4$.

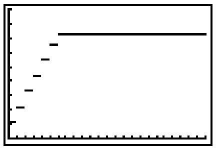
44. **(a)** We require:

$$f(x) = \begin{cases} 0 & x = 0 \\ 1.10, & 0 < x \le 1 \\ 2.20, & 1 < x \le 2 \\ 3.30, & 2 < x \le 3 \\ 4.40, & 3 < x \le 4 \\ 5.50, & 4 < x \le 5 \\ 6.60, & 5 < x \le 6 \\ 7.25, & 6 < x \le 24. \end{cases}$$

This may be written more compactly as

$$f(x) = \begin{cases} -1.10 \text{ int}(-x), & 0 \le x \le 6 \\ 7.25, & 6 < x \le 24 \end{cases}$$

(b)

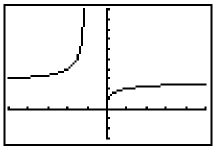


[0, 24] by [0, 9]

This is continuous for all values of x in the domain $[0, 24]$ except for $x = 0, 1, 2, 3, 4, 5, 6$.

45. (a) The function is defined when $1 + \frac{1}{x} > 0$, that is, on $(-\infty, -1) \cup (0, \infty)$. (It can be argued that the domain should also include certain values in the interval $(-1, 0)$, namely, those rational numbers that have odd denominators when expressed in lowest terms.)

(b)

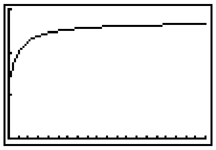


[−5, 5] by [−3, 10]

(c) If we attempt to evaluate $f(x)$ at these values, we obtain $f(-1) = \left(1 + \frac{1}{-1}\right)^{-1} = 0^{-1} = \frac{1}{0}$ (undefined) and $f(0) = \left(1 + \frac{1}{0}\right)^{0}$ (undefined). Since f is undefined at these values due to division by zero, both values are points of discontinuity.

(d) The discontinuity at $x = 0$ is removable because the right-hand limit is 0. The discontinuity at $x = -1$ is not removable because it is an infinite discontinuity.

(e)



[0, 20] by [0, 3]

X	Y1
10	2.5937
100	2.7048
1000	2.7169
10000	2.7181
100000	2.7183
1E6	2.7183
1E7	2.7183

Y1=(1+1/X)^X

The limit is about 2.718, or e.

46. This is because $\lim_{h\to 0} f(a + h) = \lim_{x\to a} f(x)$.

47. Suppose not. Then f would be negative somewhere in the interval and positive somewhere else in the interval. So, by the Intermediate Value Theorem, it would have to be zero somewhere in the interval, which contradicts the hypothesis.

48. Since the absolute value function is continuous, this follows from the theorem about continuity of composite functions.

49. For any real number a, the limit of this function as x approaches a cannot exist. This is because as x approaches a, the values of the function will continually oscillate between 0 and 1.

■ Section 2.4 Rates of Change and Tangent Lines (pp. 82–90)

Quick Review 2.4

1. $\Delta x = 3 - (-5) = 8$
$\Delta y = 5 - 2 = 3$

2. $\Delta x = a - 1$
$\Delta y = b - 3$

3. $m = \frac{-1 - 3}{5 - (-2)} = \frac{-4}{7} = -\frac{4}{7}$

4. $m = \frac{3 - (-1)}{3 - (-3)} = \frac{4}{6} = \frac{2}{3}$

5. $y = \frac{3}{2}[x - (-2)] + 3$
$y = \frac{3}{2}x + 6$

6. $m = \frac{-1 - 6}{4 - 1} = \frac{-7}{3} = -\frac{7}{3}$
$y = -\frac{7}{3}(x - 1) + 6$
$y = -\frac{7}{3}x + \frac{25}{3}$

7. $y = -\frac{3}{4}(x - 1) + 4$
$y = -\frac{3}{4}x + \frac{19}{4}$

8. $m = -\frac{1}{-3/4} = \frac{4}{3}$
$y = \frac{4}{3}(x - 1) + 4$
$y = \frac{4}{3}x + \frac{8}{3}$

9. Since $2x + 3y = 5$ is equivalent to $y = -\frac{2}{3}x + \frac{5}{3}$, we use $m = -\frac{2}{3}$.
$y = -\frac{2}{3}[x - (-1)] + 3$
$y = -\frac{2}{3}x + \frac{7}{3}$

10. $\frac{b - 3}{4 - 2} = \frac{5}{3}$
$b - 3 = \frac{10}{3}$
$b = \frac{19}{3}$

Section 2.4 Exercises

1. (a) $\frac{\Delta f}{\Delta x} = \frac{f(3) - f(2)}{3 - 2} = \frac{28 - 9}{1} = 19$

(b) $\frac{\Delta f}{\Delta x} = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{2 - 0}{2} = 1$

2. (a) $\frac{\Delta f}{\Delta x} = \frac{f(2) - f(0)}{2 - 0} = \frac{3 - 1}{2} = 1$

(b) $\frac{\Delta f}{\Delta x} = \frac{f(12) - f(10)}{12 - 10} = \frac{7 - \sqrt{41}}{2} \approx 0.298$

3. (a) $\frac{\Delta f}{\Delta x} = \frac{f(0) - f(-2)}{0 - (-2)} = \frac{1 - e^{-2}}{2} \approx 0.432$

(b) $\frac{\Delta f}{\Delta x} = \frac{f(3) - f(1)}{3 - 1} = \frac{e^3 - e}{2} \approx 8.684$

4. (a) $\frac{\Delta f}{\Delta x} = \frac{f(4) - f(1)}{4 - 1} = \frac{\ln 4 - 0}{3} = \frac{\ln 4}{3} \approx 0.462$

(b) $\frac{\Delta f}{\Delta x} = \frac{f(103) - f(100)}{103 - 100} = \frac{\ln 103 - \ln 100}{3} = \frac{1}{3} \ln \frac{103}{100}$
$= \frac{1}{3} \ln 1.03 \approx 0.0099$

5. (a) $\frac{\Delta f}{\Delta x} = \frac{f(3\pi/4) - f(\pi/4)}{(3\pi/4) - (\pi/4)} = \frac{-1 - 1}{\pi/2} = -\frac{4}{\pi} \approx -1.273$

(b) $\frac{\Delta f}{\Delta x} = \frac{f(\pi/2) - f(\pi/6)}{(\pi/2) - (\pi/6)} = \frac{0 - \sqrt{3}}{\pi/3} = -\frac{3\sqrt{3}}{\pi} \approx -1.654$

6. (a) $\frac{\Delta f}{\Delta x} = \frac{f(\pi) - f(0)}{\pi - 0} = \frac{1 - 3}{\pi} = -\frac{2}{\pi} \approx -0.637$

(b) $\frac{\Delta f}{\Delta x} = \frac{f(\pi) - f(-\pi)}{\pi - (-\pi)} = \frac{1 - 1}{2\pi} = 0$

7. We use $Q_1 = (10, 225)$, $Q_2 = (14, 375)$, $Q_3 = (16.5, 475)$, $Q_4 = (18, 550)$, and $P = (20, 650)$.

(a) Slope of PQ_1: $\frac{650 - 225}{20 - 10} \approx 43$
Slope of PQ_2: $\frac{650 - 375}{20 - 14} \approx 46$
slope of PQ_3: $\frac{650 - 475}{20 - 16.5} = 50$
Slope of PQ_4: $\frac{650 - 550}{20 - 18} = 50$

Secant	Slope
PQ_1	43
PQ_2	46
PQ_3	50
PQ_4	50

The appropriate units are meters per second.

(b) Approximately 50 m/sec

8. We use $Q_1 = (5, 20)$, $Q_2 = (7, 38)$, $Q_3 = (8.5, 56)$, $Q_4 = (9.5, 72)$, and $P = (10, 80)$.

(a) Slope of PQ_1: $\frac{80 - 20}{10 - 5} = 12$
Slope of PQ_2: $\frac{80 - 38}{10 - 7} = 14$
Slope of PQ_3: $\frac{80 - 56}{10 - 8.5} = 16$
Slope of PQ_4: $\frac{80 - 72}{10 - 9.5} = 16$

Secant	Slope
PQ_1	12
PQ_2	14
PQ_3	16
PQ_4	16

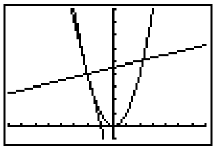
The appropriate units are meters per second.

(b) Approximately 16 m/sec

9. (a)
$$\lim_{h\to 0} \frac{y(-2 + h) - y(-2)}{h} = \lim_{h\to 0} \frac{(-2 + h)^2 - (-2)^2}{h}$$
$$= \lim_{h\to 0} \frac{4 - 4h + h^2 - 4}{h}$$
$$= \lim_{h\to 0} \frac{-4h + h^2}{h}$$
$$= \lim_{h\to 0} (-4 + h)$$
$$= -4$$

(b) The tangent line has slope -4 and passes through $(-2, y(-2)) = (-2, 4)$.
$y = -4[x - (-2)] + 4$
$y = -4x - 4$

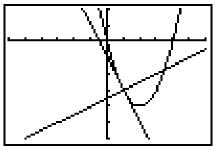
(c) The normal line has slope $-\frac{1}{-4} = \frac{1}{4}$ and passes through $(-2, y(-2)) = (-2, 4)$.
$y = \frac{1}{4}[x - (-2)] + 4$
$y = \frac{1}{4}x + \frac{9}{2}$

(d)

$[-8, 7]$ by $[-1, 9]$

10. (a) $\lim_{h\to 0} \frac{y(1 + h) - y(1)}{h}$
$$= \lim_{h\to 0} \frac{[(1 + h)^2 - 4(1 + h)] - [1^2 - 4(1)]}{h}$$
$$= \lim_{h\to 0} \frac{1 + 2h + h^2 - 4 - 4h + 3}{h}$$
$$= \lim_{h\to 0} \frac{h^2 - 2h}{h}$$
$$= \lim_{h\to 0} (h - 2)$$
$$= -2$$

(b) The tangent line has slope -2 and passes through $(1, y(1)) = (1, -3)$.
$y = -2(x - 1) - 3$
$y = -2x - 1$

(c) The normal line has slope $-\frac{1}{-2} = \frac{1}{2}$ and passes through $(1, y(1)) = (1, -3)$.
$y = \frac{1}{2}(x - 1) - 3$
$y = \frac{1}{2}x - \frac{7}{2}$

(d)

$[-6, 6]$ by $[-6, 2]$

11. (a) $\lim_{h\to 0}\frac{y(2+h)-y(2)}{h}=\lim_{h\to 0}\frac{\frac{1}{(2+h)-1}-\frac{1}{2-1}}{h}$

$=\lim_{h\to 0}\frac{\frac{1}{h+1}-1}{h}$

$=\lim_{h\to 0}\frac{1-(h+1)}{h(h+1)}$

$=\lim_{h\to 0}\left(-\frac{1}{h+1}\right)$

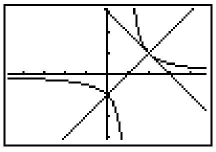
$=-1$

(b) The tangent line has slope -1 and passes through $(2, y(2)) = (2, 1)$.
$y=-(x-2)+1$
$y=-x+3$

(c) The normal line has slope $-\frac{1}{-1}=1$ and passes through $(2, y(2)) = (2, 1)$.

$y=1(x-2)+1$

$y=x-1$

(d) 

$[-4.7, 4.7]$ by $[-3.1, 3.1]$

12. (a) $\lim_{h\to 0}\frac{y(0+h)-y(0)}{h}=\lim_{h\to 0}\frac{(h^2-3h-1)-(-1)}{h}$

$=\lim_{h\to 0}\frac{h^2-3h}{h}$

$=\lim_{h\to 0}(h-3)$

$=-3$

(b) The tangent line has slope -3 and passes through $(0, y(0)) = (0, -1)$.

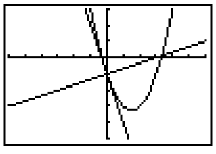
$y=-3(x-0)-1$

$y=-3x-1$

(c) The normal line has slope $-\frac{1}{-3}=\frac{1}{3}$ and passes through $(0, y(0)) = (0, -1)$.

$y=\frac{1}{3}(x-0)-1$

$y=\frac{1}{3}x-1$

(d) 

$[-6, 6]$ by $[-5, 3]$

13. (a) Near $x=2$, $f(x)=|x|=x$.

$\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0}\frac{(2+h)-2}{h}=\lim_{h\to 0}1=1$

(b) Near $x=-3$, $f(x)=|x|=-x$.

$\lim_{h\to 0}\frac{f(-3+h)-f(-3)}{h}=\lim_{h\to 0}\frac{(3-h)-3}{h}$

$=\lim_{h\to 0}-1=-1$

14. Near $x=1$, $f(x)=|x-2|=-(x-2)=2-x$.

$\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0}\frac{[2-(1+h)]-(2-1)}{h}$

$=\lim_{h\to 0}\frac{1-h-1}{h}=\lim_{h\to 0}-1=-1$

15. First, note that $f(0)=2$.

$\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^-}\frac{(2-2h-h^2)-2}{h}$

$=\lim_{h\to 0^-}\frac{-2h-h^2}{h}$

$=\lim_{h\to 0^-}(-2-h)$

$=-2$

$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{(2h+2)-2}{h}$

$=\lim_{h\to 0^+}2$

$=2$

No, the slope from the left is -2 and the slope from the right is 2. The two-sided limit of the difference quotient does not exist.

16. First, note that $f(0)=0$.

$\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^-}\frac{-h-0}{h}=-1$

$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{(h^2-h)-0}{h}$

$=\lim_{h\to 0^+}(h-1)=-1$

Yes. The slope is -1.

17. First, note that $f(2)=\frac{1}{2}$

$\lim_{h\to 0^-}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0^-}\frac{\frac{1}{2+h}-\frac{1}{2}}{h}$

$=\lim_{h\to 0^-}\frac{2-(2+h)}{2h(2+h)}$

$=\lim_{h\to 0^-}\frac{-h}{2h(2+h)}$

$=\lim_{h\to 0^-}-\frac{1}{2(2+h)}$

$=-\frac{1}{4}$

17. continued

$$\lim_{h\to 0^+}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0^+}\frac{\frac{4-(2+h)}{4}-\frac{1}{2}}{h}$$
$$=\lim_{h\to 0^+}\frac{[4-(2+h)]-2}{4h}$$
$$=\lim_{h\to 0^+}\frac{-h}{4h}$$
$$=-\frac{1}{4}$$

Yes. The slope is $-\frac{1}{4}$.

18. No. The function is discontinuous at $x=\frac{3\pi}{4}$

because $\lim_{x\to(3\pi/4)^-}f(x)=\lim_{x\to(3\pi/4)^-}\sin x=\sin\frac{3\pi}{4}=\frac{\sqrt{2}}{2}$

but $f\left(\frac{3\pi}{4}\right)=\cos\frac{3\pi}{4}=-\frac{\sqrt{2}}{2}$.

19. (a) $$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{[(a+h)^2+2]-(a^2+2)}{h}$$
$$=\lim_{h\to 0}\frac{a^2+2ah+h^2+2-a^2-2}{h}$$
$$=\lim_{h\to 0}\frac{2ah+h^2}{h}$$
$$=\lim_{h\to 0}(2a+h)$$
$$=2a$$

(b) The slope of the tangent steadily increases as a increases.

20. (a) $$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{\frac{2}{a+h}-\frac{2}{a}}{h}$$
$$=\lim_{h\to 0}\frac{2a-2(a+h)}{ah(a+h)}$$
$$=\lim_{h\to 0}\frac{-2}{a(a+h)}$$
$$=-\frac{2}{a^2}$$

(b) The slope of the tangent is always negative. The tangents are very steep near $x=0$ and nearly horizontal as a moves away from the origin.

21. (a) $$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{\frac{1}{a+h-1}-\frac{1}{a-1}}{h}$$
$$=\lim_{h\to 0}\frac{(a-1)-(a+h-1)}{h(a-1)(a+h-1)}$$
$$=\lim_{h\to 0}-\frac{1}{(a-1)(a+h-1)}$$
$$=-\frac{1}{(a-1)^2}$$

(b) The slope of the tangent is always negative. The tangents are very steep near $x=1$ and nearly horizontal as a moves away from the origin.

22. (a) $$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{[9-(a+h)^2]-(9-a^2)}{h}$$
$$=\lim_{h\to 0}\frac{9-a^2-2ah-h^2-9+a^2}{h}$$
$$=\lim_{h\to 0}\frac{-2ah-h^2}{h}$$
$$=\lim_{h\to 0}(-2a-h)$$
$$=-2a$$

(b) The slope of the tangent steadily decreases as a increases.

23. Let $f(t)=100-4.9t^2$.

$$\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$
$$=\lim_{h\to 0}\frac{[100-4.9(2+h)^2]-[100-4.9(2)^2]}{h}$$
$$=\lim_{h\to 0}\frac{100-19.6-19.6h-4.9h^2-100+19.6}{h}$$
$$=\lim_{h\to 0}(-19.6-4.9h)$$
$$=-19.6$$

The object is falling at a speed of 19.6 m/sec.

24. Let $f(t)=3t^2$.

$$\lim_{h\to 0}\frac{f(10+h)-f(10)}{h}=\lim_{h\to 0}\frac{3(10+h)^2-300}{h}$$
$$=\lim_{h\to 0}\frac{300+60h+3h^2-300}{h}$$
$$=\lim_{h\to 0}(60+3h)$$
$$=60$$

The rocket's speed is 60 ft/sec.

25. Let $f(r)=\pi r^2$, the area of a circle of radius r.

$$\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}=\lim_{h\to 0}\frac{\pi(3+h)^2-\pi(3)^2}{h}$$
$$=\lim_{h\to 0}\frac{9\pi+6\pi h+\pi h^2-9\pi}{h}$$
$$=\lim_{h\to 0}(6\pi+\pi h)$$
$$=6\pi$$

The area is changing at a rate of 6π in^2/in., that is, 6π square inches of area per inch of radius.

26. Let $f(r)=\frac{4}{3}\pi r^3$.

$$\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0}\frac{\frac{4}{3}\pi(2+h)^3-\frac{4}{3}\pi(2)^3}{h}$$
$$=\frac{4}{3}\pi\lim_{h\to 0}\frac{(2+h)^3-2^3}{h}$$
$$=\frac{4}{3}\pi\lim_{h\to 0}\frac{8+12h+6h^2+h^3-8}{h}$$
$$=\frac{4}{3}\pi\lim_{h\to 0}(12+6h+h^2)$$
$$=\frac{4}{3}\pi\cdot 12$$
$$=16\pi$$

The volume is changing at a rate of 16π in^3/in., that is, 16π cubic inches of volume per inch of radius.

27. $$\lim_{h\to 0}\frac{s(1+h)-s(1)}{h}=\lim_{h\to 0}\frac{1.86(1+h)^2-1.86(1)^2}{h}$$
$$=\lim_{h\to 0}\frac{1.86+3.72h+1.86h^2-1.86}{h}$$
$$=\lim_{h\to 0}(3.72+1.86h)$$
$$=3.72$$

The speed of the rock is 3.72 m/sec.

28. $$\lim_{h\to 0}\frac{s(2+h)-s(2)}{h}=\lim_{h\to 0}\frac{11.44(2+h)^2-11.44(2)^2}{h}$$
$$=\lim_{h\to 0}\frac{45.76+45.76h+11.44h^2-45.76}{h}$$
$$=\lim_{h\to 0}(45.76+11.44h)$$
$$=45.76$$

The speed of the rock is 45.76 m/sec.

29. First, find the slope of the tangent at $x=a$.

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$
$$=\lim_{h\to 0}\frac{[(a+h)^2+4(a+h)-1]-(a^2+4a-1)}{h}$$
$$=\lim_{h\to 0}\frac{a^2+2ah+h^2+4a+4h-1-a^2-4a+1}{h}$$
$$=\lim_{h\to 0}\frac{2ah+h^2+4h}{h}$$
$$=\lim_{h\to 0}(2a+h+4)$$
$$=2a+4$$

The tangent at $x=a$ is horizontal when $2a+4=0$, or $a=-2$. The tangent line is horizontal at $(-2, f(-2))=(-2,-5)$.

30. First, find the slope of the tangent at $x=a$.

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$
$$=\lim_{h\to 0}\frac{[3-4(a+h)-(a+h)^2]-(3-4a-a^2)}{h}$$
$$=\lim_{h\to 0}\frac{3-4a-4h-a^2-2ah-h^2-3+4a+a^2}{h}$$
$$=\lim_{h\to 0}\frac{-4h-2ah-h^2}{h}$$
$$=\lim_{h\to 0}(-4-2a-h)$$
$$=-4-2\text{a}$$

The tangent at $x=a$ is horizontal when $-4-2a=0$, or $a=-2$. The tangent line is horizontal at $(-2, f(-2))=(-2,7)$.

31. (a) From Exercise 21, the slope of the curve at $x=a$, is $-\frac{1}{(a-1)^2}$. The tangent has slope -1 when $-\frac{1}{(a-1)^2}=-1$, which gives $(a-1)^2=1$, so $a=0$ or $a=2$. Note that $y(0)=\frac{1}{0-1}=-1$ and $y(2)=\frac{1}{2-1}=1$, so we need to find the equations of lines of slope -1 passing through $(0,-1)$ and $(2,1)$, respectively.

At $x=0$: $y=-1(x-0)-1$
$y=-x-1$

At $x=2$: $y=-1(x-2)+1$
$y=-x+3$

(b) The normal has slope 1 when the tangent has slope $\frac{-1}{1}=-1$, so we again need to find lines through $(0,-1)$ and $(2,1)$, this time using slope 1.

At $x=0$: $y=1(x-0)-1$
$y=x-1$

At $x=2$: $y=1(x-2)+1$
$y=x-1$

There is only one such line. It is normal to the curve at two points and its equation is $y=x-1$.

32. Consider a line that passes through $(1, 12)$ and a point $(a, 9-a^2)$ on the curve. Using the result of Exercise 22, this line will be tangent to the curve at a if its slope is $-2a$.

$$\begin{aligned}\frac{(9-a^2)-12}{a-1}&=-2a\\ 9-a^2-12&=-2a(a-1)\\ -a^2-3&=-2a^2+2a\\ a^2-2a-3&=0\\ (a+1)(a-3)&=0\\ a&=-1 \text{ or } a=3\end{aligned}$$

At $a=-1$ (or $x=-1$), the slope is $-2(-1)=2$.
$y=2(x-1)+12$
$y=2x+10$

At $a=3$ (or $x=3$), the slope is $-2(3)=-6$.
$y=-6(x-1)+12$
$y=-6x+18$

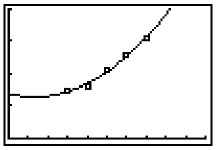
33. (a) $\frac{2.1-1.5}{1995-1993}=0.3$

The rate of change was 0.3 billion dollars per year.

(b) $\frac{3.1-2.1}{1997-1995}=0.5$

The rate of change was 0.5 billion dollars per year.

(c) $y=0.0571x^2-0.1514x+1.3943$



[0, 10] by [0, 4]

33. continued

(d) $\dfrac{y(5) - y(3)}{5 - 3} \approx 0.31$

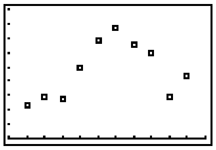
$\dfrac{y(7) - y(5)}{7 - 5} \approx 0.53$

According to the regression equation, the rates were 0.31 billion dollars per year and 0.53 billion dollars per year.

(e)
$$\lim_{h\to 0} \frac{y(7+h) - y(7)}{h} = \lim_{h\to 0} \frac{[0.0571(7+h)^2 - 0.1514(7+h) + 1.3943] - [0.0571(7)^2 - 0.1514(7) + 13943]}{h}$$
$$= \lim_{h\to 0} \frac{0.0571(14h + h^2) - 0.1514h}{h}$$
$$= \lim_{h\to 0} [0.0571(14) - 0.1514 + 0.0571h]$$
$$\approx 0.65$$

The funding was growing at a rate of about 0.65 billion dollars per year.

34. (a)



[7, 18] by [0, 900]

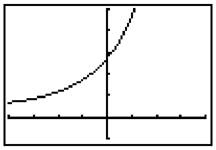
(b)

Q from year	Slope
1988	$\dfrac{440-225}{17-8} \approx 23.9$
1989	$\dfrac{440-289}{17-9} \approx 18.9$
1990	$\dfrac{440-270}{17-10} \approx 24.3$
1991	$\dfrac{440-493}{17-11} \approx -8.8$
1992	$\dfrac{440-684}{17-12} = -48.8$
1993	$\dfrac{440-763}{17-13} \approx -80.8$
1994	$\dfrac{440-651}{17-14} \approx -70.3$
1995	$\dfrac{440-600}{17-15} = -80.0$
1996	$\dfrac{440-296}{17-16} = 144.0$

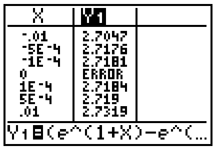
(c) As Q gets closer to 1997, the slopes do not seem to be approaching a limit value. The years 1995–97 seem to be very unusual and unpredictable.

35. (a) $\dfrac{f(1+h) - f(1)}{h} = \dfrac{e^{1+h} - e}{h}$

(b)



[−4, 4] by [−1, 5]



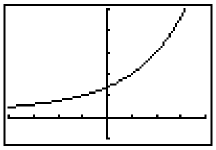
Limit ≈ 2.718

(c) They're about the same.

(d) Yes, it has a tangent whose slope is about e.

36. **(a)** $\dfrac{f(1+h)-f(1)}{h} = \dfrac{2^{1+h}-2}{h}$

(b)



$[-4, 4]$ by $[-1, 5]$

X	Y_1
-.01	1.3815
-5E-4	1.3861
-1E-4	1.3862
0	ERROR
1E-4	1.3863
5E-4	1.3865
.01	1.3911

X=-.01

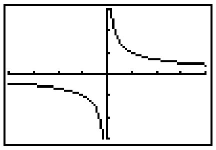
Limit ≈ 1.386

(c) They're about the same.

(d) Yes, it has a tangent whose slope is about ln 4.

37. Let $f(x) = x^{2/5}$. The graph of $y = \dfrac{f(0+h)-f(0)}{h} = \dfrac{f(h)}{h}$

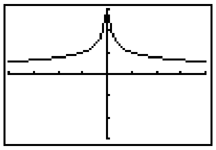
is shown.



$[-4, 4]$ by $[-3, 3]$

The left- and right-hand limits are $-\infty$ and ∞, respectively. Since they are not the same, the curve does not have a vertical tangent at $x = 0$. No.

38. Let $f(x) = x^{3/5}$. The graph of $y = \dfrac{f(0+h)-f(0)}{h} = \dfrac{f(h)}{h}$ is shown.

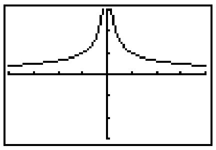


$[-4, 4]$ by $[-3, 3]$

Yes, the curve has a vertical tangent at $x = 0$ because

$\lim_{h\to 0} \dfrac{f(0+h)-f(0)}{h} = \infty$.

39. Let $f(x) = x^{1/3}$. The graph of $y = \dfrac{f(0+h)-f(0)}{h} = \dfrac{f(h)}{h}$ is shown.

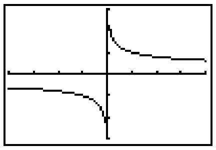


$[-4, 4]$ by $[-3, 3]$

Yes, the curve has a vertical tangent at $x = 0$ because

$\lim_{h\to 0} \dfrac{f(0+h)-f(0)}{h} = \infty$.

40. Let $f(x) = x^{2/3}$. The graph of $y = \dfrac{f(0+h)-f(0)}{h} = \dfrac{f(h)}{h}$ is shown.



$[-4, 4]$ by $[-3, 3]$

The left- and right-hand limits are $-\infty$ and ∞, respectively. Since they are not the same, the curve does not have a vertical tangent at $x = 0$. No.

41. This function has a tangent with slope zero at the origin. It is sandwiched between two functions, $y = x^2$ and $y = -x^2$, both of which have slope zero at the origin.

Looking at the difference quotient,

$-h \le \dfrac{f(0+h)-f(0)}{h} \le h,$

so the Sandwich Theorem tells us the limit is 0.

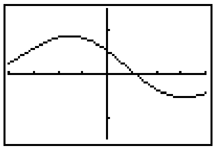
42. This function does not have a tangent line at the origin. As the function oscillates between $y = x$ and $y = -x$ infinitely often near the origin, there are an infinite number of difference quotients (secant line slopes) with a value of 1 and with a value of -1. Thus the limit of the difference quotient doesn't exist.

The difference quotient is $\dfrac{f(0+h)-f(0)}{h} = \sin\dfrac{1}{h}$ which oscillates between 1 and -1 infinitely often near zero.

43. Let $f(x) = \sin x$. The difference quotient is

$\dfrac{f(1+h)-f(1)}{h} = \dfrac{\sin(1+h)-\sin(1)}{h}.$

A graph and table for the difference quotient are shown.



$[-4, 4]$ by $[-1.5, 1.5]$

X	Y_1
-.005	.5424
-.002	.54114
-.001	.54072
0	ERROR
.001	.53988
.002	.53946
.005	.5382

Y1=(sin(1+X)-si…

Since the limit as $h\to 0$ is about 0.540, the slope of $y = \sin x$ at $x = 1$ is about 0.540.

■ Chapter 2 Review Exercises

(pp. 91–93)

1. $\lim_{x\to -2}(x^3 - 2x^2 + 1) = (-2)^3 - 2(-2)^2 + 1 = -15$

2. $\lim_{x\to -2}\frac{x^2+1}{3x^2-2x+5} = \frac{(-2)^2+1}{3(-2)^2-2(-2)+5} = \frac{5}{21}$

3. No limit, because the expression $\sqrt{1-2x}$ is undefined for values of x near 4.

4. No limit, because the expression $\sqrt[4]{9-x^2}$ is undefined for values of x near 5.

5. $\lim_{x\to 0}\frac{\frac{1}{2+x}-\frac{1}{2}}{x} = \lim_{x\to 0}\frac{2-(2+x)}{2x(2+x)} = \lim_{x\to 0}\frac{-x}{2x(2+x)}$

$= \lim_{x\to 0}\left(-\frac{1}{2(2+x)}\right) = -\frac{1}{2(2+0)} = -\frac{1}{4}$

6. $\lim_{x\to\pm\infty}\frac{2x^2+3}{5x^2+7} = \lim_{x\to\pm\infty}\frac{2x^2}{5x^2} = \frac{2}{5}$

7. An end behavior model for $\frac{x^4+x^3}{12x^3+128}$ is $\frac{x^4}{12x^3} = \frac{1}{12}x$.

Therefore:

$\lim_{x\to\infty}\frac{x^4+x^3}{12x^3+128} = \lim_{x\to\infty}\frac{1}{12}x = \infty$

$\lim_{x\to-\infty}\frac{x^4+x^3}{12x^3+128} = \lim_{x\to-\infty}\frac{1}{12}x = -\infty$

8. $\lim_{x\to 0}\frac{\sin 2x}{4x} = \frac{1}{2}\lim_{x\to 0}\frac{\sin 2x}{2x} = \frac{1}{2}(1) = \frac{1}{2}$

9. Multiply the numerator and denominator by $\sin x$.

$\lim_{x\to 0}\frac{x\csc x + 1}{x\csc x} = \lim_{x\to 0}\frac{x+\sin x}{x} = \lim_{x\to 0}\left(1+\frac{\sin x}{x}\right)$

$= \left(\lim_{x\to 0} 1\right) + \left(\lim_{x\to 0}\frac{\sin x}{x}\right) = 1 + 1 = 2$

10. $\lim_{x\to 0} e^x \sin x = e^0 \sin 0 = 1 \cdot 0 = 0$

11. Let $x = \frac{7}{2} + h$, where h is in $\left(0, \frac{1}{2}\right)$. Then

$\text{int}\,(2x-1) = \text{int}\left[2\left(\frac{7}{2}\right)+2h-1\right] = \text{int}\,(6+2h) = 6,$

because $6 + 2h$ is in $(6, 7)$.

Therefore, $\lim_{x\to 7/2^+}\text{int}\,(2x-1) = \lim_{x\to 7/2^+} 6 = 6.$

12. Let $x = \frac{7}{2} + h$, where h is in $\left(-\frac{1}{2}, 0\right)$. Then

$\text{int}\,(2x-1) = \text{int}\left[2\left(\frac{7}{2}\right)+2h-1\right] = \text{int}\,(6+2h) = 5,$

because $6 + 2h$ is in $(5, 6)$.

Therefore, $\lim_{x\to 7/2^-}\text{int}\,(2x-1) = \lim_{x\to 7/2^-} 5 = 5$

13. Since $\lim_{x\to\infty}(-e^{-x}) = \lim_{x\to\infty} e^{-x} = 0$, and $-e^{-x} \le e^{-x}\cos x \le e^{-x}$ for all x, the Sandwich Theorem gives $\lim_{x\to\infty} e^{-x}\cos x = 0$.

14. Since the expression x is an end behavior model for both $x + \sin x$ and $x + \cos x$, $\lim_{x\to\infty}\frac{x+\sin x}{x+\cos x} = \lim_{x\to\infty}\frac{x}{x} = 1$.

15. Limit exists.

16. Limit exists.

17. Limit exists.

18. Limit does not exist.

19. Limit exists.

20. Limit exists.

21. Yes

22. No

23. No

24. Yes

25. **(a)** $\lim_{x\to 3^-} g(x) = 1$

(b) $g(3) = 1.5$

(c) No, since $\lim_{x\to 3^-} g(x) \ne g(3)$.

(d) g is discontinuous at $x = 3$ (and at points not in the domain).

(e) Yes, the discontinuity at $x = 3$ can be removed by assigning the value 1 to $g(3)$.

26. **(a)** $\lim_{x\to 1^-} k(x) = 1.5$

(b) $\lim_{x\to 1^+} k(x) = 0$

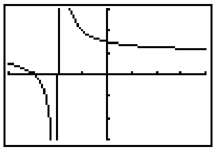
(c) $k(1) = 0$

(d) No, since $\lim_{x\to 1^-} k(x) \ne k(1)$

(e) k is discontinuous at $x = 1$ (and at points not in the domain).

(f) No, the discontinuity at $x = 1$ is not removable because the one-sided limits are different.

27.

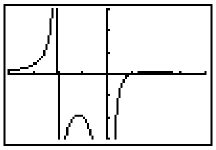


$[-4, 4]$ by $[-3, 3]$

(a) Vertical asymptote: $x = -2$

(b) Left-hand limit $= \lim_{x\to -2^-}\frac{x+3}{x+2} = -\infty$

Right-hand limit: $\lim_{x\to -2^+}\frac{x+3}{x+2} = \infty$

28. 

$[-4, 4]$ by $[-3, 3]$

(a) Vertical asymptotes: $x = 0$, $x = -2$

(b) At $x = 0$:

Left-hand limit $= \lim_{x \to 0^-} \frac{x-1}{x^2(x+2)} = -\infty$

Right-hand limit $= \lim_{x \to 0^+} \frac{x-1}{x^2(x+2)} = -\infty$

At $x = -2$:

Left-hand limit $= \lim_{x \to -2^-} \frac{x-1}{x^2(x+2)} = \infty$

Right-hand limit $= \lim_{x \to -2^+} \frac{x-1}{x^2(x+2)} = -\infty$

29. (a) At $x = -1$:

Left-hand limit $= \lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (1) = 1$

Right-hand limit $= \lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (-x) = 1$

At $x = 0$:

Left-hand limit $= \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x) = 0$

Right-hand limit $= \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (-x) = 0$

At $x = 1$:

Left-hand limit $= \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (-x) = -1$

Right-hand limit $= \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (1) = 1$

(b) At $x = -1$: Yes, the limit is 1.
At $x = 0$: Yes, the limit is 0.
At $x = 1$: No, the limit doesn't exist because the two one-sided limits are different.

(c) At $x = -1$: Continuous because $f(-1) =$ the limit.
At $x = 0$: Discontinuous because $f(0) \neq$ the limit.
At $x = 1$: Discontinuous because the limit does not exist.

30. (a) Left-hand limit $= \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} |x^3 - 4x|$

$= |(1)^3 - 4(1)| = |-3| = 3$

Right-hand limit $= \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^2 - 2x - 2)$

$= (1)^2 - 2(1) - 2 = -3$

(b) No, because the two one-sided limits are different.

(c) Every place except for $x = 1$

(d) At $x = 1$

31. Since $f(x)$ is a quotient of polynomials, it is continuous and its points of discontinuity are the points where it is undefined, namely $x = -2$ and $x = 2$.

32. There are no points of discontinuity, since $g(x)$ is continuous and defined for all real numbers.

33. (a) End behavior model: $\frac{2x}{x^2}$, or $\frac{2}{x}$

(b) Horizontal asymptote: $y = 0$ (the x-axis)

34. (a) End behavior model: $\frac{2x^2}{x^2}$, or 2

(b) Horizontal asymptote: $y = 2$

35. (a) End behavior model: $\frac{x^3}{x}$, or x^2

(b) Since the end behavior model is quadratic, there are no horizontal asymptotes.

36. (a) End behavior model: $\frac{x^4}{x^3}$, or x

(b) Since the end behavior model represents a nonhorizontal line, there are no horizontal asymptotes.

37. (a) Since $\lim_{x \to \infty} \frac{x + e^x}{e^x} = \lim_{x \to \infty} \left(\frac{x}{e^x} + 1\right) = 1$, a right end behavior model is e^x.

(b) Since $\lim_{x \to -\infty} \frac{x + e^x}{x} = \lim_{x \to -\infty} \left(1 + \frac{e^x}{x}\right) = 1$, a left end behavior model is x.

38. (a, b) Note that $\lim_{x \to \pm\infty} \left(-\frac{1}{\ln|x|}\right) = \lim_{x \to \pm\infty} \left(\frac{1}{\ln|x|}\right) = 0$ and

$-\frac{1}{\ln|x|} < \frac{\sin x}{\ln|x|} < \frac{1}{\ln|x|}$ for all $x \neq 0$.

Therefore, the Sandwich Theorem gives

$\lim_{x \to \pm\infty} \frac{\sin x}{\ln|x|} = 0$. Hence

$\lim_{x \to \pm\infty} \frac{\ln|x| + \sin x}{\ln|x|} = \lim_{x \to \pm\infty} \left(1 + \frac{\sin x}{\ln|x|}\right) = 1 + 0 = 1,$

so $\ln|x|$ is both a right end behavior model and a left end behavior model.

39. $\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x^2 + 2x - 15}{x - 3} = \lim_{x \to 3} \frac{(x-3)(x+5)}{x-3}$

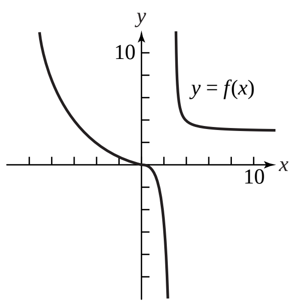
$= \lim_{x \to 3} (x + 5) = 3 + 5 = 8.$

Assign the value $k = 8$.

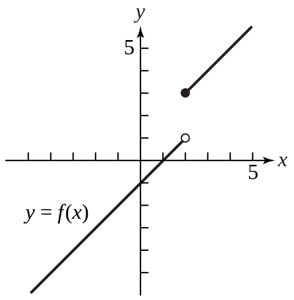
40. $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin x}{2x} = \frac{1}{2} \lim_{x \to 0} \frac{\sin x}{x} = \frac{1}{2}(1) = \frac{1}{2}$

Assign the value $k = \frac{1}{2}$.

41. One possible answer:



42. One possible answer:



43. $\dfrac{f(\pi/2) - f(0)}{\pi/2 - 0} = \dfrac{2 - 1}{\pi/2} = \dfrac{2}{\pi}$

44. $\lim\limits_{h\to0} \dfrac{V(a+h) - V(a)}{h} = \lim\limits_{h\to0} \dfrac{\frac{1}{3}\pi(a+h)^2H - \frac{1}{3}\pi a^2H}{h}$

$= \dfrac{1}{3}\pi H \lim\limits_{h\to0} \dfrac{a^2 + 2ah + h^2 - a^2}{h}$

$= \dfrac{1}{3}\pi H \lim\limits_{h\to0} (2a + h)$

$= \dfrac{1}{3}\pi H(2a)$

$= \dfrac{2}{3}\pi aH$

45. $\lim\limits_{h\to0} \dfrac{S(a+h) - S(a)}{h} = \lim\limits_{h\to0} \dfrac{6(a+h)^2 - 6a^2}{h}$

$= \lim\limits_{h\to0} \dfrac{6a^2 + 12ah + 6h^2 - 6a^2}{h}$

$= \lim\limits_{h\to0} (12a + 6h)$

$= 12a$

46. $\lim\limits_{h\to0} \dfrac{y(a+h) - y(a)}{h}$

$= \lim\limits_{h\to0} \dfrac{[(a+h)^2 - (a+h) - 2] - (a^2 - a - 2)}{h}$

$= \lim\limits_{h\to0} \dfrac{a^2 + 2ah + h^2 - a - h - 2 - a^2 + a + 2}{h}$

$= \lim\limits_{h\to0} \dfrac{2ah + h^2 - h}{h}$

$= \lim\limits_{h\to0} (2a + h - 1)$

$= 2a - 1$

47. (a) $\lim\limits_{h\to0} \dfrac{f(1+h) - f(1)}{h} = \lim\limits_{h\to0} \dfrac{[(1+h)^2 - 3(1+h)] - (-2)}{h}$

$= \lim\limits_{h\to0} \dfrac{1 + 2h + h^2 - 3 - 3h + 2}{h}$

$= \lim\limits_{h\to0} (-1 + h)$

$= -1$

(b) The tangent at P has slope -1 and passes through $(1, -2)$.
$y = -1(x - 1) - 2$
$y = -x - 1$

(c) The normal at P has slope 1 and passes through $(1, -2)$.
$y = 1(x - 1) - 2$
$y = x - 3$

48. At $x = a$, the slope of the curve is

$\lim\limits_{h\to0} \dfrac{f(a+h) - f(a)}{h} = \lim\limits_{h\to0} \dfrac{[(a+h)^2 - 3(a+h)] - (a^2 - 3a)}{h}$

$= \lim\limits_{h\to0} \dfrac{a^2 + 2ah + h^2 - 3a - 3h - a^2 + 3a}{h}$

$= \lim\limits_{h\to0} \dfrac{2ah - 3h + h^2}{h}$

$= \lim\limits_{h\to0} (2a - 3 + h)$

$= 2a - 3$

The tangent is horizontal when $2a - 3 = 0$, at $a = \dfrac{3}{2}$ $\left(\text{or } x = \dfrac{3}{2}\right)$. Since $f\left(\dfrac{3}{2}\right) = -\dfrac{9}{4}$, the point where this occurs is $\left(\dfrac{3}{2}, -\dfrac{9}{4}\right)$.

49. (a) $p(0) = \dfrac{200}{1 + 7e^{-0.1(0)}} = \dfrac{200}{8} = 25$

Perhaps this is the number of bears placed in the reserve when it was established.

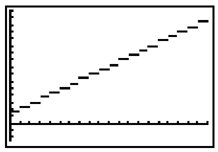
(b) $\lim\limits_{t\to\infty} p(t) = \lim\limits_{t\to\infty} \dfrac{200}{1 + 7e^{-0.1t}} = \dfrac{200}{1} = 200$

(c) Perhaps this is the maximum number of bears which the reserve can support due to limitations of food, space, or other resources. Or, perhaps the number is capped at 200 and excess bears are moved to other locations.

50. (a) $f(x) = \begin{cases} 3.20 - 1.35 \text{ int } (-x + 1), & 0 < x \le 20 \\ 0, & x = 0 \end{cases}$

(Note that we cannot use the formula $f(x) = 3.20 + 1.35 \text{ int } x$, because it gives incorrect results when x is an integer.)

(b)



[0, 20] by [−5, 32]

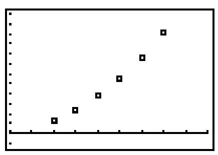
f is discontinuous at integer values of x: 0, 1, 2, ..., 19.

51. (a) Cubic: $y = -1.644x^3 + 42.981x^2 - 254.369x + 300.232$
Quartic: $y = 2.009x^4 - 102.081x^3 + 1884.997x^2 - 14918.180x + 43004.464$

(b) Cubic: $-1.644x^3$, predicts spending will go to 0
Quartic: $2.009x^4$, predicts spending will go to ∞

52. Let $A = \lim\limits_{x\to c} f(x)$ and $B = \lim\limits_{x\to c} g(x)$. Then $A + B = 2$ and $A - B = 1$. Adding, we have $2A = 3$, so $A = \dfrac{3}{2}$, whence $\dfrac{3}{2} + B = 2$, which gives $B = \dfrac{1}{2}$. Therefore, $\lim\limits_{x\to c} f(x) = \dfrac{3}{2}$ and $\lim\limits_{x\to c} g(x) = \dfrac{1}{2}$.

53. (a)



[3, 12] by [−2, 24]

(b)

Year of Q	Slope of PQ
1995	$\dfrac{20.1 - 2.7}{2000 - 1995} = 3.48$
1996	$\dfrac{20.1 - 4.8}{2000 - 1996} = 3.825$
1997	$\dfrac{20.1 - 7.8}{2000 - 1997} = 4.1$
1998	$\dfrac{20.1 - 11.2}{2000 - 1998} = 4.45$
1999	$\dfrac{20.1 - 15.2}{2000 - 1999} = 4.9$

(c) Approximately 5 billion dollars per year.

(d) $y = 0.3214x^2 - 1.3471x + 1.3857$

$$\begin{aligned}
\lim_{h\to 0} \frac{y(10+h) - y(10)}{h} &= \lim_{h\to 0} \frac{[0.3214(10+h)^2 - 1.3471(10+h) + 1.3857] - [0.3214(10)^2 - 1.3471(10) + 1.3857]}{h} \\
&= \lim_{h\to 0} \frac{0.3214(20h + h^2) - 1.3471h}{h} \\
&= 0.3214(20) - 1.3471 \\
&\approx 5.081
\end{aligned}$$

The predicted rate of change in 2000 is about 5.081 billion dollars per year.